RESEARCH ARTICLE

Searching The Least Value Method for Solving First Order Nonlinear Initial Value Problem

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In this paper, Searching least value (SLV) method is employed for the numerical solution of non-linear first order initial value problem. The argument is based on the reproducing kernel space $W^2_2[0,1]$. For this a representation of the exact solution in the reproducing kernel space is constructed. The results show that the present method is an applicable technique and approximates the solution very well.

Keywords: Searching least value (SLV) method; Exact solution; Approximate solution; Gram-Schmidt orthogonal process; Reproducing kernel; initial value problem.

AMS Subject Classification: 34K28, 65L05.

1. Introduction

A lot of methods were applied to solve ordinary initial value problems such as spline, variation iteration method and finite difference method. Sallam and Anwar [1] used a cubic spline collocation method for solving the initial value problem. Sallam [2] constructed a global method, based on quintic $C^1$ spline, for the integration of first order ordinary initial value problems (IVPs) including stiff equations and those possessing oscillatory solutions as well. Venkatesulu and Srinivasu [3] solved nonstandard first order initial value problem. Ghazala and Hamood [4] discussed computational method for solving singularly perturbed first order initial value problem with initial conditions at any end point. Reproducing kernel space has been used for many classical problems in probability and statistics [5, 6].

Consider the following nonlinear first order initial value problem:

\[
\begin{cases}
    u^{(1)} + a(x)u = f(x, u), & 0 < x \leq 1, \\
    u(0) = 0
\end{cases}
\]

where $a(x)$ is a bounded function.

The paper is organized into four sections including the introduction. In section 2, construction of reproducing kernel is presented for solving Eq. (1). The solution of the problem in reproducing kernel Hilbert space is presented in section 3. In section 4, numerical example is shown to demonstrate the accuracy of present method method.

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2. Reproducing Kernal Spaces

2.1 Reproducing kernel space $W^2_2[0,1]$

The inner product space $W^2_2[0,1]$ is defined by $W^2_2[0,1] = \{u(x)\mid u^{(1)} \text{ is absolutely continuous real valued functions in } [0,1], u^{(2)} \in L^2[0,1]\}$. The inner product and norm in $W^2_2[0,1]$ are defined as

$$\langle u(x), v(x) \rangle = u(0)v(0) + u(1)v(1) + \int_0^1 u^{(2)}(x)v^{(2)}(x)dx, \quad u(x), v(x) \in W^2_2[0,1].$$  \hspace{1cm} (2)

$$\|u\| = \sqrt{\langle u(x), u(x) \rangle}, \quad u(x) \in W^2_2[0,1].$$  \hspace{1cm} (3)

2.2 Reproducing kernel space $W^1_2[0,1]$

The inner product space $W^1_2[0,1]$ is defined by $W^1_2[0,1] = \{u(x)\mid u \text{ is absolutely continuous real valued function in } [0,1], u^{(1)} \in L^2[0,1]\}$. The linear operator $L : W^2_2[0,1] \rightarrow W^2_2[0,1]$ is bounded and defined as

$$L(u)(x) = u^{(1)} + a(x)u$$  \hspace{1cm} (6)

and equivalent form of Eq. (6) is

$$\begin{cases} (Lu)(x) = f(x, u) & 0 < x \leq 1 \\ u(0) = 0. \end{cases}$$  \hspace{1cm} (7)

For a fix countable dense subset $D = \{x_i\}_{i=1}^\infty$ of the domain $[0,1]$, let

$$\varphi_i(x) = Q_{x_i}(y), \quad i \in N$$  \hspace{1cm} (8)

Proof See the proof in appendix.

3. The Exact and Approximate Solution

The solution of Eq. (1) is given in the reproducing kernel Hilbert space $W^2_2[0,1]$ and the linear operator $L : W^2_2[0,1] \rightarrow W^2_2[0,1]$ is bounded and defined as

$$\langle u(x), v(x) \rangle = \int_0^1 u(x)v(x)dx, \quad u(x), v(x) \in W^2_2[0,1].$$

In [7], it was proved that $W^1_2[0,1]$ is reproducing kernel Hilbert space.

Theorem 2.1. The space $W^2_2[0,1]$ is a reproducing kernel Hilbert space. That is, for all $u(x) \in W^2_2[0,1]$ and each fixed $x \in [0,1]$, $y \in [0,1]$, there exists $R_x(y) \in W^2_2[0,1]$ such that $\langle u(x), R_x(y) \rangle = u(x)$ and $R_x(y)$ is called the reproducing kernel function of space $W^2_2[0,1]$. The reproducing kernel function $R_x(y)$ is given by

$$R_x(y) = \begin{cases} c_1 + c_2y + c_3y^2 + c_4y^3, & y \leq x \\ d_1 + d_2y + d_3y^2 + d_4y^3, & y > x. \end{cases}$$
where $Q_x(y) \in W_2^3 [0,1]$ is reproducing kernel of $W_2^3 [0,1]$. Further assume that $\psi_i(x) = (L^* \varphi_i)(x)$, where $L^*$ denotes the adjoint operator of $L$.

Theorem 3.1 \{ $\psi_i(x) \}_{i=1}^{\infty}$ is a complete system of $W_2^3 [0,1]$ and $\psi_i(x) = L_y R_x(y)|_{y=x}$.

Proof For each fixed $u(x) \in W_2^3 [0,1]$, let $\langle u(x), \psi_i(x) \rangle = 0$ ($i = 1, 2, \ldots$), which implies

$$\langle u(x), (L^* \varphi_i)(x) \rangle = \langle L u(x), \varphi_i(x) \rangle = (L u)(x_i) = 0.$$  

(9)

Since $\{ x_i \}_{i=1}^{\infty}$ is dense in $[0,1]$, $(L u)(x) = 0$, which implies $u \equiv 0$ from the existence of $L^{-1}$.

From Eq. (20), it can be written as

$$\psi_i(x) = \langle \psi_i(y), R_x(y) \rangle = \langle (L^* \varphi_i)(x), R_x(y) \rangle = \langle \varphi_i(y), L R_x(y) \rangle = L_y R_x(y)|_{y=x_i}.$$  

The subscript $y$ by the operator $L$ indicates that $L$ applies to the function of $y$.

To orthonormalize the sequence $\{ \psi_i \}_{i=1}^{\infty}$ in the reproducing kernel space $W_2^3 [0,1]$ Gram-Schmidt process can be used as

$$\overline{\psi_i(x)} = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \quad i = 1, 2, \ldots$$  

(8)

Theorem 3.2 If $\{ x_i \}_{i=1}^{\infty}$ is dense in $[0,1]$ and the solution of Eq. (1) is unique, for all $u(x) \in W_2^3 [0,1]$, the series is convergent in the norm of $\| \cdot \|_{W_2^3}$. If $u(x)$ is exact solution then the solution of Eq. (1) has the form

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \alpha_k \overline{\psi_i}(x).$$

where $\alpha_k = f(x_k, u(x_k))$, ($k = 1, 2, \ldots$), $\{ x_i \}_{i=1}^{\infty}$ is dense in $[a, b]$.

Proof Since $u(x) \in W_2^3 [0,1]$ and can be expanded in the form of Fourier series about normal orthogonal system $\{ \psi_i \}_{i=1}^{\infty}$ as

$$u(x) = \sum_{i=1}^{\infty} \langle (u(x), \overline{\psi_i}(x)) \rangle \overline{\psi_i}(x).$$  

(9)

Since the space $W_2^3 [0,1]$ is Hilbert space so the series $\sum_{i=1}^{\infty} \langle (u(x), \overline{\psi_i}(x)) \rangle \overline{\psi_i}(x)$ is convergent in the norm of $\| \cdot \|_{W_2^3}$. From Eqs. (8) and (9), it can be written as

$$u(x) = \sum_{i=1}^{\infty} \langle (u(x), \overline{\psi_i}(x)) \rangle \overline{\psi_i}(x) = \sum_{i=1}^{\infty} \langle u(x), \sum_{k=1}^{i} \beta_{ik} \psi_k(x) \rangle \overline{\psi_i}(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \psi_k(x) \rangle \overline{\psi_i}(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle L u(x), \varphi_k(x) \rangle \overline{\psi_i}(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle L u(x), \varphi_k(x) \rangle \overline{\psi_i}(x)$$
If \( u(x) \) is the exact solution of Eq. (1) and \( Lu = f(x, u(x)) \), then

\[
\begin{align*}
    u(x) &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x, u(x)) \varphi_k(x) \\
    &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u(x_k)) \psi_i(x)
\end{align*}
\]

The approximate solution of \( u(x) \) is given by

\[
    u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_n(x_k)) \psi_i(x)
\]

(10)

where \( \alpha_k = f(x_k, u(x_k)) \), \( (k = 1, 2, \ldots) \).

Eq. (1) is nonlinear so the approximate solution can be obtained using the following method. In order to obtain \( \alpha_k \) in Eq. (10), substitute \( \sum_{k=1}^{i} [f(x_k, u_n(x_k)) - \alpha_k]^2 = 0 \).

\[
    J(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^{n} [f(x_k, u_n(x_k)) - \alpha_k]^2,
\]

(11)

if \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are least value points then the approximate solutions \( u_n(x) \) are obtained. \( \blacksquare \)

Next we will give the following algorithm to obtain \( u_n(x) \) in Eq. (10).

**Step 1:** Take initial values \( \alpha_k^0 \) \( (k = 1, 2, \ldots, i) \).

**Step 2:** Substitute \( \alpha_k^0 \) \( (k = 1, 2, \ldots, i) \) in Eq. (10) and obtain \( u_n^0(n) \).

**Step 3:** Calculate \( J(\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0) \)

**Step 4:** If \( J(\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0) < 10^{-30} \) then the calculation process terminates; otherwise, substitute \( u_n^0(x) \) into Eq. (10) then \( \alpha_k^1 \) \( (k = 1, 2, \ldots, i) \) is yielded and go to next step.

**Step 5:** Calculate \( J(\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1) \) if \( J(\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1) < J(\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0) \) then replace \( \alpha_k^0 \) by \( \alpha_k^1 \) and return to step 2, otherwise \( \alpha_k^0 \) will be used.

4. Numerical Examples

Numerical example is studied to demonstrate the accuracy of the present method using mathematica 5.2.

**Problem 1** Consider the nonlinear initial value problem

\[
\begin{align*}
    u^{(1)} - u(x)u(x) &= x^2 \cos(x) + 2x \sin(x) - x^4 \sin^2(x), 0 < x \leq 1 \\
    u(0) &= 0.
\end{align*}
\]

(12)

The exact solution is \( u(x) = x^2 \sin x \). The results are summarized in Table 1.

**Problem 2** Consider the initial value problem

\[
\begin{align*}
    u^{(1)} + u \sin u &= f(x), 0 < x \leq 1 \\
    u(0) &= 0.
\end{align*}
\]

(13)

The exact solution is \( u(x) = x + x^2 \). The results are summarized in Table 2.
Table 1.: The numerical results when \((n = 56)\)

<table>
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<tr>
<th>(x)</th>
<th>Absolute error</th>
<th>Relative error</th>
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<tr>
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<td>1.95E-05</td>
<td>2.74E-02</td>
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<tr>
<td>50/56</td>
<td>9.57E-05</td>
<td>1.54E-04</td>
</tr>
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Table 2.: The numerical results when \((n = 56)\)

<table>
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<td>50/56</td>
<td>6.21E-04</td>
<td>3.67E-04</td>
</tr>
</tbody>
</table>

5. Conclusion

In this paper, we have used the reproducing kernel space method to find the solution of nonlinear first order initial value problem. The results showed that the convergence and accuracy of the method for numerically analyzed first order initial value problems is in a good agreement with the analytical solutions. The numerical results are displayed to demonstrate the validity of this method. Mathematica software is used for all computational work.

Appendix

Since \(R_x(y) \in W_2^2[0,1]\), and from Eq. (2), it can be written as

\[
\langle u(y), R_x(y) \rangle = u(0)v(0) + u(1)v(1) + \int_0^1 u^{(2)}(y)R_x^{(2)}(y)dy,
\]

(14)

\[
R_x(0) = 0.
\]

(15)

Since, \(u \in W_2^2[0,1]\) so \(u(0) = 0\). If

\[
R_x(1) - R_x^{(3)}(1) = 0, \quad R_x^{(2)}(1) = 0, \quad R_x^{(2)}(0) = 0
\]

(16)

then Eq. (14) implies that

\[
\langle u(y), R_x(y) \rangle = \int_0^1 u(y)R_x^{(4)}(y)dy
\]
For all \( x \in [0, 1] \), if \( R_x(y) \) also satisfies
\[
R_x^{(4)}(y) = \delta(y - x)
\] (17)
then
\[
\langle u(y), R_x(y) \rangle = u(x)
\] (18)

When \( y \neq x \) characteristic equation of Eq. (17) is given by \( \lambda^4 = 0 \), then the characteristic values \( \lambda \) can be determined whose multiplicity is 4. The reproducing kernel \( R_x(y) \) can be defined as
\[
R_x(y) = \begin{cases}
  c_1 + c_2 y + c_3 y^2 + c_4 y^3, & y \leq x, \\
  d_1 + d_2 y + d_3 y^2 + d_4 y^3, & y > x
\end{cases}
\] (19)
and let \( R_x(y) \) satisfies
\[
R_x^{(k)}(x + 0) = R_x^{(k)}(x - 0), \quad k = 0, 1, 2
\] (20)
and integrating (19) from \( x - \varepsilon \) to \( x + \varepsilon \) with respect to \( y \) and \( \varepsilon \to 0 \), use jump degree of \( R_x^{(3)}(y) \) at \( y = x \)
\[
R_x^{(3)}(x + 0) - R_x^{(3)}(x - 0) = 1
\] (21)
The coefficients \( c_i \) and \( d_i \) \((i = 1, 2, 3, 4)\) can be determined from Eqs. (15), (16), (20) and (21).

References


