RESEARCH ARTICLE

Existence and Uniqueness Results for First Order Difference Equation

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(Received: 7 May 2011, Accepted: 5 Jun 2011)

This paper is devoted to obtain existence of solutions of first order difference initial value problem

\[ \Delta x(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t_0 \geq 0, \]

where \( f : J \times \mathbb{R} \rightarrow \mathbb{R}, J \) being the set of nonnegative integers. To prove the existence result Tychnoff’s fixed point theorem is used. We also discussed the uniqueness of the solution of above initial value problem.

Keywords: Difference Equation; Existence of solution; Fixed Point Theorem.

AMS Subject Classification: 39A05, 39A10, 54E50.

1. Introduction

Agarwal [1], Kelley and Peterson [2] developed the theory of difference equations and difference inequalities. Existence of solutions to second order boundary problems using Schauders fixed point theorem is obtained by K. L. Bondar [3]. Existence of maximal and minimal solution of difference first order nonlinear initial value problem is obtained by K. L. Bondar, V. C. Borkar and S. T. Patil [4–6]. He also discussed Comparison results regarding maximal and minimal solutions as well as minmax solutions. Some more results on this are obtained by Eloe [7]. Some differential and integral inequalities are given in [8].

In this paper, we shall use Tychnoff’s fixed point theorem for locally convex linear spaces to prove existence of solutions of the difference equation

\[ \Delta x(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t_0 \geq 0, \]  \tag{1}

where \( f : J \times \mathbb{R} \rightarrow \mathbb{R}, J \) being the set of nonnegative integers.

2. Preliminary Notes

Let \( J = \{t_0, t_0 + 1, ..., t_0 + a\}, \ t_0 \in \mathbb{R} \) and \( E \) be an open subset of \( \mathbb{R} \). Consider the difference equations with an initial condition,

\[ \Delta u(t) = g(t, u(t)), \quad u(t_0) = u_0. \]  \tag{2}

where \( u_0 \in E, \ u : J \rightarrow E, \ g : J \times E \rightarrow R. \)

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The function $\phi : J \rightarrow R$ is said to be a solution of initial value problem (2), if it satisfies

$$\Delta \phi(t) = g(t, \phi(t)); \quad \phi(t_0) = u_0.$$  

The initial value problem (2) is equivalent to the problem

$$u(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, u(s)).$$

By summation convection $\sum_{s=t_0}^{t-1} g(s, u(s)) = 0$ and so $u(t)$ given above is the solution of eq. (2).

Now we define the maximal and minimal solution of eq. (2).

Definition 2.1 Let $r(t)$ be any solution of eq. (2) on $J$. Then $r(t)$ is said to be maximal solution of eq. (2), if every solution $u(t)$ of eq. (2) existing on $J$, the inequality $u(t) \leq r(t)$ holds for $t \in J$. A solution $\rho(t)$ of eq. (2) is said to be minimal solution of eq. (2), if $\rho(t) \leq u(t)$ for $t \in J$.

Following results are proved in [4].

Theorem 2.2 [4] Let $E$ be an open subset of $R$, $g : R_0 \rightarrow R$, where $R_0 = \{(t, u) \in J \times E \text{ with } |u - u_0| \leq b\}$; $|g(t, u)| \leq M$ on $R_0$ and $g(t, u)$ is nondecreasing in $u$ for all $t \in J$. Let $m : J \rightarrow R$ such that

1. $(t, m(t)) \in R_0$,
2. $m(t_0) \leq u_0$,
3. $\Delta m(t) \leq g(t, m(t))$ for $t \in [t_0, t_0 + \alpha]$, $\alpha = min\{a, b/2M + b\}$.

If $r(t)$ is maximal solution of eq. (2) on $[t_0, t_0 + \alpha]$, then $m(t) \leq r(t)$ on $[t_0, t_0 + \alpha]$.

Remark 2.3 [4] Let $E$ be an open subset of $R$, $g : R_0 \rightarrow R$, where $R_0 = \{(t, u) \in J \times E \text{ with } |u - u_0| \leq b\}$; $|g(t, u)| \leq M$ on $R_0$ and $g(t, u)$ is nondecreasing in $u$ for all $t \in J$. Let $m : [t_0 - a, t_0] \rightarrow R$ such that

1. $(t, m(t)) \in R_0$,
2. $m(t_0) \geq u_0$,
3. $\Delta m(t) \leq g(t, m(t))$ for $t \in [t_0 - a, t_0]$, $\alpha = min\{a, b/2M + b\}$.

If $\rho(t)$ is minimal solution of (2) on $[t_0 - a, t_0]$, then $\rho(t) \leq m(t)$ on $[t_0 - a, t_0]$.

3. Main Results

To prove the main result we require following theorem.

Theorem 3.1 (Tychonoff’s Theorem) [8] Let $B$ be a complete, locally convex, linear space and $B_0$ is closed, convex subset of $B$. Let the mapping $T : B \rightarrow B$ continuous and $T(B_0) \subset B_0$. If $T(B_0)$ is compact, then $T$ has a fixed point in $B_0$.

Theorem 3.2 Let $f : J \times R \rightarrow R$ is continuous, and, for $(t, x) \in J \times R$,

$$|f(t, x)| \leq g(t, |x|),$$  

where $J$ be set of nonnegative integers, $g : J \times R_+ \rightarrow R_+$ is continuous and $g(t, u)$ is monotonic nondecreasing in $u$ for each $t \in J$. Assume that, for every $u_0 > 0$, the difference equation

$$\Delta u(t) = g(t, u(t)), \quad u(t_0) = u_0 \quad t_0 \geq 0$$

is true.
has a solution \( u(t) = u(t, t_0, u_0) \) existing for \( t \geq t_0 \). Then for every \( x_0 \in \mathbb{R} \) such that \( |x_0| \leq u_0 \), there exists a solution \( x(t) = x(t, t_0, x_0) \) of eq. (1) for \( t \geq t_0 \), satisfying
\[
|x(t)| \leq u(t), \ t \geq t_0.
\]

**Proof** Let us consider the space \( B \) of all real valued continuous functions from \([t_0, t_0 + a]\) into \( \mathbb{R} \). Define a norm on \( B \) by \( ||x|| = \sup \{|x(t)| : t_0 \leq t \leq t_0 + a\} \) for \( x \in B \), then \( B \) is complete, locally convex, linear space. Let us now define a subset \( B_0 \) of \( B \) as follows
\[
B_0 = \{x \in B : |x(t)| \leq u(t), t \geq t_0\},
\]
where \( u(t) \) is a solution of eq. (4) existing for \( t \geq t_0 \). It is clear that the set \( B_0 \) is closed convex and bounded. Consider the mapping
\[
T(x)(t) = x_0 + \sum_{s=t_0}^{t-1} f(s, x(s)),
\]
whose fixed point corresponds to a solution of eq. (1). Evidently the operator \( T \) is compact and therefore \( \bar{T}(B_0) \) is compact in view of the boundedness of \( B_0 \).

Now we observe that for any \( x \in B_0 \),
\[
|T(x)(t)| \leq |x_0| + \sum_{s=t_0}^{t-1} g(s, |x(s)|),
\]
because of eqs. (6) and (3). Using the monotonic character of \( g(t, u) \) in \( u \), the definition of \( B_0 \), and the fact that \( u(t) \) is a solution of eq. (4) such that \( |x_0| \leq u_0 \), it follows from eq. (7) that
\[
|T(x)(t)| \leq u(t).
\]
This implies \( T(B_0) \subset B_0 \). Hence Theorem 3.1 proves the result. \( \blacksquare \)

**Theorem 3.3** Assume that

1. the function \( g(t, u) \) is continuous and nonnegative for \( t_0 \leq t \leq t_0 + a \), \( 0 \leq u \leq 2b \), and, for every \( t^* \), \( t_0 < t^* < t_0 + a \), \( u(t) \equiv 0 \) is the only function on \( t_0 \leq t < t^* \), which satisfies
\[
\Delta u(t) = g(t, u(t)), \ u(t_0) = 0
\]
for \( t_0 \leq t < t^* \);
2. \( f : \mathbb{R}_0 \to \mathbb{R} \), where \( \mathbb{R}_0 = \{t \in [t_0, t_0 + a] : |x - x_0| \leq b\} \), and for \( (t, x), (t, y) \in \mathbb{R}_0 \),
\[
|f(t, x) - f(t, y)| \leq g(t, |x - y|).
\]

Then the difference equation
\[
\Delta x(t) = f(t, x), \ x(t_0) = x_0
\]
has at most one solution on \( t_0 \leq t \leq t_0 + a \).
Proof Suppose there are two solutions \( x_1(t) \) and \( x_2(t) \) of the eq. (4) on \( t_0 \leq t \leq t_0 + a \). Define \( m(t) = |x_1(t) - x_2(t)| \). Then using eq. (9) we get,

\[
\Delta m(t) = |\Delta x_1(t) - \Delta x_2(t)| \\
= |f(t, x_1(t)) - f(t, x_2(t))| \\
\leq g(t, |x_1(t) - x_2(t)|) \\
= g(t, m(t)).
\]

Also \( m(t_0) = x_1(t_0) - x_2(t_0) = x_0 - x_0 = 0 \). For any \( t^* \) such that \( t_0 < t^* < t_0 + a \), we obtain from an application of Theorem 3.2 the inequality

\[
m(t) \leq r(t), \quad t_0 \leq t < t^*,
\]

where \( r(t) \) is the maximal solution of eq. (8). The assumption (1) now assures that \( m(t) \equiv 0 \) on \( t_0 \leq t < t^* \), proving the theorem.

Theorem 3.4 Let the function \( g(t, u) \) satisfies assumption (1) in Theorem 3.3. Assume further that the function \( g_1(t, u) \) is continuous and nonnegative for \( t_0 \leq t \leq t_0 + a \), \( 0 \leq u \leq 2b \), \( g_1(t, 0) \equiv 0 \), and

\[
g_1(t, u) \leq g(t, u), \quad t \neq t_0. \tag{11}
\]

Then, for every \( t^* \), \( t_0 < t^* < t_0 + a \), \( u(t) \equiv 0 \) is the only function on \( t_0 \leq t < t^* \) which satisfies

\[
\Delta u(t) = g_1(t, u(t)), \quad u(t_0) = 0 \tag{12}
\]

for \( t_0 \leq t < t^* \).

Proof Let us show that the maximal solution \( r(t) \) of eq. (12) is identically zero. Suppose, on contrary, that there exists a \( \sigma \), \( t_0 < \sigma < t_0 + a \), such that \( r(\sigma) > 0 \). Because of eq. (11), we have

\[
\Delta r(t) \leq g(t, r(t)), \quad t_0 < t \leq \sigma.
\]

If \( \rho(t) \) is the minimal solution of

\[
\Delta u(t) = g(t, u(t)), \quad u(\sigma) = r(\sigma),
\]

an application of Remark 2.3, shows that

\[
\rho(t) \leq r(t) \quad t_0 < t \leq \sigma. \tag{13}
\]

The solution \( \rho(t) \) can be continued to \( t = t_0 \). If \( \rho(\tau) = 0 \), for some \( \tau \), \( t_0 < \tau < \sigma \), we can effect the continuation by defining \( \rho(t) = 0 \) for \( t_0 < t < \tau \).

Otherwise eq. (13) ensures the possibility of continuation. Since \( r(t_0) = 0 \), \( \lim_{t \to t_0} \rho(t) = 0 \), and we define \( \rho(t_0) = 0 \). Furthermore, since \( g_1(t, u) \) is continuous at \( (t_0, 0) \) and \( g_1(t_0, 0) = 0 \), \( \Delta r(t_0) = 0 \). This, because of eq. (13), implies that \( \Delta \rho(t_0) = 0 \). But we have assumed that \( g(t, u) \) satisfies assumption (1) of Theorem 3.3. Hence, \( \rho(t) \equiv 0 \). This contradicts the fact that \( \rho(\sigma) = r(\sigma) > 0 \). Therefore \( r(t) \equiv 0 \), and the proof is complete.

References


