Solving Integro-differential equations systems by Galerkin method with orthogonal polynomials

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Abstract

In this paper, we shall use Galerkin method with ultraspherical orthogonal polynomials base to approximate Integro- Differential Equations Systems. Leap Frog method for unconstrained optimization with least squares approximation method is used to solve the resulting algebraic equation system. Error estimates for the approximations are discussed. Numerical results are included to confirm the efficiency and accuracy of the method.

Keywords: Spectral methods, Approximation by ultraspherical polynomials, Leap frog method, Integral equations, Integro- differential equations.


1. Introduction

Sezer and Gulsu \textsuperscript{13} gave Taylor polynomial approximation for the solution of high-order linear Fredholm difference equations with variable coefficients and mixed argument under the mixed conditions about any point. Arikoglu and Ozkol \textsuperscript{2} extended the differential transform method (DTM) to solve the integro-differential equations. They introduced new theorems for the transformation of integrals and proved. Dascoglu \textsuperscript{5} presented an approximation method for high-order linear Fredholm-Volterra integro-differential equations (FVIDE) in the most general form under the mixed conditions in terms of Chebyshev polynomials. This method transforms FVIDE and the conditions into the matrix equations which correspond to a system of linear algebraic equations with unknown Chebyshev coefficients.

Wang \textsuperscript{16} established a reliable algorithm for solving high-order nonlinear Volterra-Fredholm integro-differential equations, by using the theories and methods of mathematics analysis and computer algebra. Jia \textsuperscript{10} derived an asymptotic error expansions for mixed finite element approximations of the integro-differential

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equation, and Richardson extrapolation is applied to improve the accuracy of the approximations by two different schemes with the help of an interpolation post-processing technique.


Spectral methods using expansion in orthogonal polynomials such as Chebyshev or Ultraspherical polynomials is successful in the numerical approximation of various mathematical problems, see for instance, Gottlieb and Orszag [8], Ahues et al. [1] and Canuto et al. [4]. If these polynomials are used as basis functions, then the rate of decay of the expansion coefficients is determined by the smoothness properties of the function being expanded. This choice of trial functions is responsible for the superior approximation properties of spectral methods compared with finite difference and finite element methods.

The purpose of this paper is to solve integral and integro-differential systems. We shall combine ultraspherical integral approximation with Galerkin method to reformulate the integral and integro-differential problem to unconstrained optimization problem. The resulting unconstrained optimization problem is solved by penalty partial quadratic interpolation technique. The error bound for the used approximation is discussed to ensure the efficiency of the proposed method. We include enough numerical examples and comparisons to confirm the accuracy of the method.

The outlines of this paper is as follows: in section 2 we present the form of the problem discussed in this paper. In section 3 we discussed the spectral approximation method that depends on ultraspherical orthogonal polynomials. in section 4 we state Galerkin method that transforms the systems of integro-differential equations into systems of algebraic equations. In section 5 we apply the proposed method depend on Galerkin ultraspherical integral approximation for solving systems of integro-differential equations. Error estimates for the approximation is discussed in section 6. In section 7, we present Numerical results. We conclude the results of this work in section 8.

2. Setting of the problem

A system of integro-differential equations can be considered, in general form, as

\[
\frac{dy_\ell}{dt} = a_\ell + \sum_{k=1}^{m} b_{\ell k} y_k + \int_{0}^{t} \sum_{k=1}^{m} K_{\ell k} (t, s, y_k(s), y'_k(s)) ds
\]

\[
y_\ell (0) = D_\ell, \ \ell = 1, 2, \ldots, m.
\]  

Where \(a_\ell\) and \(b_{\ell k}\) depend only on \(t\).

We suppose that system [1] has a unique solution. However, the necessary and sufficient conditions for existence and uniqueness of the solution of it could be found in (Yusufoglu [17]).

3. Ultraspherical approximation of integrals

We consider any function \(f(x) \in C^\infty[-1, 1]\) and let \(S = \{x_k : \mathbb{B}_N^{(a)} (x_k) = 0, k = 0, 1, \ldots, N\}\), then we have the following theorems and corollary (El-Hawary et al. [6]):
Theorem 3.1. Let \( f(x) \) be approximated by ultraspherical polynomials, then there exists a matrix \( Q = [q_{ij}] \), \( i, j = 0, 1, \ldots, N \) satisfying:

\[
\int_{-1}^{x_i} f(x) dx \approx \sum_{k=0}^{N} q_{ik} (\alpha) f(x_k),
\]

where

\[
q_{ik} (\alpha) = \sum_{j=0}^{N} \left( \lambda_j^{[\alpha]} \right)^{-1} \omega_k^{[\alpha]} C_j^{[\alpha]}(x_k) \int_{-1}^{x_i} C_j^{[\alpha]}(x) dx
\]

where

\[
\left( \omega_k^{[\alpha]} \right)^{-1} = \sum_{j=0}^{N} \left( \lambda_j^{[\alpha]} \right)^{-1} \left( C_j^{[\alpha]}(x_k) \right)^2,
\]

and

\[
\lambda_j^{[\alpha]} = 2^{j+2\alpha+\tau_j} j! \Gamma\left[\frac{\alpha+\frac{1}{2}}{2} + \frac{j+1}{2}\right] \hat{K}_j^{[\alpha]},
\]

and

\[
\tau = \begin{cases} 
1 & \text{if } \alpha = j = 0 \\
0 & \text{otherwise}
\end{cases}
\]

\[
\left[ \int_{-1}^{x_i} \int_{-1}^{x_i} \ldots \int_{-1}^{x_i} f(x) dx \ldots dx \right] \approx Q^{[M]}(\alpha) [f]
\]

\[
q_{ij}^{[M]} = \frac{(x_i-x_j)^{M-1}}{(M-1)!} q_{ij}, \quad i, j = 0, 1, \ldots,
\]

where

\[
\hat{K}_j^{[\alpha]} = 2^{j+\alpha} \Gamma\left[\frac{\alpha+\frac{1}{2}}{2} + j+1\right] \Gamma\left(\alpha+1\right).
\]

Theorem 3.2. Let \( f(x) \) be approximated by ultraspherical polynomials, then there exists a number \( \xi = \xi(x) \) in \([-1,1]\) such that

\[
\int_{-1}^{x_i} f(x) dx = \sum_{k=0}^{N} q_{ik} f(x_k) + \left[ E_N(\alpha, x_i, \xi) \right], \quad x_k \in S, \quad i = 0, 1, \ldots, N,
\]

where

\[
E_N(\alpha, x_i, \xi) = \frac{f^{(N+1)}(\xi)}{(N+1)! \hat{K}_{N+1}^{[\alpha]}} \int_{-1}^{x_i} C_{N+1}^{[\alpha]}(x) dx.
\]

Note: For \( x \in [0,1] \), the elements of the matrix \( Q \) are to be halved.

4. Galerkin method

Consider the spectral approximation \( u_N(x) \) is used to solve the integro- differential problem \( L(u) = f \) with orthogonal base function \( \psi_k(x) \), \( k = 1, 2, \ldots, N \). The Galerkin idea is to require the residual \( [12] \).

\[
R_N(x) = L(u_N(x)) - f(x).
\]

To be orthogonal to each of the base functions. That is the inner product

\[
\langle R_N, \psi_j \rangle = 0, \quad j = 1, 2, \ldots, N,
\]

using the inner product definition (Golberg [7]).

\[
\int_{-1}^{1} w(x) R_N(x) \psi_j(x) dx = 0, \quad j = 1, 2, \ldots, N.
\]

With \( w \) is the weight function corresponds to the orthogonal base functions \( \psi_k(x) \), \( k = 1, 2, \ldots, N \).
5. Application of the method

To solve this problem using our ultraspherical spectral method, we give the ultraspherical approximation for the highest order derivative in (1), namely

$$\frac{dy_\ell(t)}{dt} = \Phi_\ell(t),$$

and by successive integration and applying the conditions (1), we obtain:

$$y_\ell(t) = \int_0^t \Phi_\ell(s) \, ds + C.$$  \hspace{1cm} (12)

With $C = D_\ell, \ell = 1, 2, \ldots, m$ using Eq. (1). Now we apply our ultraspherical spectral approximation, equation (12) in view of (3)-(4) leads to:

$$y_\ell(t_i) = \sum_{j=0}^N q_{ij}(\alpha) \Phi_{kj} + D_\ell, \ i = 0, 1, \ldots N.$$  \hspace{1cm} (13)

Substituting in (1), we obtain the following system of equations

$$I(\Phi_\ell(x)) = \int_{-1}^1 C_\Omega^{[\alpha]}(x) (1 - x^2)^{\alpha - \frac{1}{2}} [\Phi_\ell(x) + a_\ell(x) + \sum_{k=1}^m b_{\ell k}(x) \left( \sum_{j=0}^N q_{ij} \Phi_{kj} + D_k \right)$$

$$+ \sum_{j=0}^N q_{ij} \sum_{k=1}^m K_{\ell k} \left(x, s_j \left[ \sum_{l=0}^N q_{jl} \Phi_{kl} + D_k \right], \Phi_k(x) \right)] \, dx.$$  \hspace{1cm} (14)

This integration is singular, so we approximate it by Stenger [15]:

$$\int_{-1}^1 f(\tau) d\tau \approx h \sum_{j=-N}^N f(\Theta(jh)) (\Theta)'(jh).$$ \hspace{1cm} (15)

For a specified value of $h$, where

$$\Theta(x) = \frac{e^x - 1}{e^x + 1},$$ \hspace{1cm} (16)

$$I_\Omega(\Phi_\ell) = h \sum_{i=1}^m \frac{2e^{\xi_i}}{(e^{\xi_i} + 1)^2} C_\Omega^{[\alpha]}(\Theta(x_i)) (1 - \Theta^2(x_i))^{\alpha - \frac{1}{2}} [\Phi_\ell(\Theta(x_i)) + a_\ell(\Theta(x_i)) + \sum_{k=1}^m b_{\ell k}(x_i) \left( \sum_{j=0}^N q_{ij} \Phi_{kj} + D_k \right)$$

$$+ \sum_{j=0}^N q_{ij} \sum_{k=1}^m K_{\ell k} \left(\Theta(x_i), s_j \left[ \sum_{l=0}^N q_{jl} \Phi_{kl} + D_k \right], \Phi_k(\Theta(x_i)) \right)],$$ \hspace{1cm} (17)

with $\Omega = 1, 2, \ldots, n$. This system of algebraic equation (17) in the $m \times n + 1$ unknowns: $\Phi_\ell(t_i), \ell = 1, 2, \ldots, m, \ i = 0, 1, \ldots n$ and $\alpha$. Can be reformulated making use of least squares approximation to be an unconstrained optimization problems of the form: Minimize

$$J = \sum_{\Omega=0}^N I_\Omega^2.$$ \hspace{1cm} (18)

This problem can be solved via Leap Frog Algorithm (Snyman [14]).

Remark 5.1. To satisfy the constraint $\alpha > -\frac{1}{2}$, we make the following change of variable:

$$\alpha = e^{(\beta^2 + \varepsilon)} - \frac{3}{2}, \quad 0 < \varepsilon << 1,$$

and then the problem (17) depends on $\beta$ which has no constraints.
6. Error estimates

**Theorem 6.1.** Assume that the system of integro-differential equations (1) is approximated by ultraspherical integral method indicated by (2)-(3) and assuming that $y^{(N+2)}_\ell(t)$ is bounded, i.e.,

\[ \left\| y^{(N+2)}_\ell(t) \right\| \leq M, \]  

(19)

then there exists a number $\xi = \xi(x)$ in $[-1,1]$ such that

\[ E[y](x_i, \xi) \leq M \frac{(N+1)!}{(N+1)! R^{2N+1}_N} C_{N+1} \int_{-1}^{t_i-1} C[\alpha] N+1 \left( \psi \right) ds. \]  

(20)

For the approximation of the integration (15) we have the error estimation (Stenger [15]):

\[ \left| \int_{-1}^{1} f(\tau) d\tau - h \sum_{j=-N}^{N} f(\psi(jh)) (\psi)'(jh) \right| \leq \Lambda e^{-\delta\sqrt{N}}, \]  

(21)

for some positive value $\Lambda$, $\delta$.

7. Numerical results

**Example 7.1.** Our first example is following system of linear Volterra integro-differential equation ( Yusufoglu [17]).

\[
\begin{align*}
y_1'(t) &= -3t^2 y_1(t) + (\pi - 2t^3) y_2(t) + 6 \int_0^t ((t-s)y_1(s) + (t-s)^2 y_2(s)) ds, \\
y_2'(t) &= -(\pi + 3t^3) y_1(t) - 6t^2 y_2(t) + 12 \int_0^t ((t-s)^2 y_1(s) + (t-s)y_2(s)) ds,
\end{align*}
\]

with the exact solutions

\[
y_1(t) = \sin(\pi t), \quad y_2(t) = \cos(\pi t).
\]

The numerical results are tabulated as Table 1 and Figure 1. The table introduces maximum absolute error (MAE) in the constraints, the objective function value $J$ and MAE in the approximate solutions $y_1$ and $y_2$.

| N   | $|J|$          | MAE($y_1$) | MAE($y_2$) |
|-----|---------------|------------|------------|
| 4   | 0.17216E-26   | 3.05005E-02| 5.98493E-02|
| 8   | 0.16149E-26   | 1.77206E-03| 2.99004E-03|
| 12  | 0.17871E-25   | 1.98085E-04| 4.27645E-04|
| 16  | 0.98143E-25   | 4.19236E-05| 6.15135E-05|
Example 7.2. Consider the integro-differential equation systems (Maleknejad and Tavassoli Kajani [12]).

\[
\begin{align*}
    y_1'(t) + \int_0^1 e^{-s}y_1(s)ds - \int_0^1 e^{(t+2)s}y_2'(s)ds &= 2e^t + \frac{e^{t+1} - 1}{t+1}, \\
    -y_2'(t) + \int_0^1 e^{ts}y_1'(s)ds + \int_0^1 e^{(t+s)}y_2(s)ds &= e^t + e^{-t} + \frac{e^{t+1} - 1}{t+1},
\end{align*}
\]

with the exact solutions

\[
y_1(0) = 1, \quad y_2(0) = 1.
\]

Table 2 is the numerical results for Example 7.2.

| $N$ | $|I|$ | MAE($y_1$) | MAE($y_2$) |
|-----|------|------------|------------|
| 4   | 8.76543E-27 | 1.23452E-02 | 1.54678E-02 |
| 8   | 6.5432E-27  | 8.65981E-04 | 8.23400E-04 |
| 12  | 7.6543E-26  | 2.36809E-05 | 2.27645E-05 |
| 16  | 3.4567E-26  | 9.45678E-07 | 9.42096E-07 |
Example 7.3. Consider the following system of linear Volterra integro-differential equations (Yusufog [17]):

\[ y'_1(t) = 1 + t + t^2 - y_2(t) - \int_0^t (y_1(s) + y_2(s)) \, ds, \]

\[ y'_2(t) = -1 - t + y_2(t) - \int_0^t (y_1(s) - y_2(s)) \, ds, \]

with the exact solutions

\[ y_1(t) = t + e^t, \quad y_2(t) = t - e^t. \]

Table 3: MAE and the objective function $|I|$ of Example 7.3

| $N$ | $|J|$     | MAE($y_1$) | MAE($y_2$) |
|-----|----------|------------|------------|
| 4   | 0.34623E-27 | 3.29236E-03 | 1.28785E-03 |
| 8   | 0.76236E-27 | 7.68493E-04 | 5.98893E-04 |
| 12  | 0.58714E-26 | 2.68085E-05 | 4.47655E-05 |
| 16  | 0.79843E-26 | 1.49236E-06 | 6.35635E-06 |
Example 7.4. Our last example is following system of integro-differential equation (Maleknejad and Tavassoli Kajani(2004)):

\[
2\pi y_1'(t) - \int_0^1 \cos(2\pi s) \sin(4\pi t) y_1'(s) \, ds + \int_0^1 \sin(4\pi t + 2\pi s) y_2'(s) \, ds = 2\pi \cos(2\pi t)(1 + \sin(2\pi t)),
\]

\[
y_2'(t) + \int_0^1 \cos(4\pi t) \sin(2\pi s) y_1(s) \, ds + \int_0^1 \cos(4\pi t + 2\pi s) y_2(s) \, ds = \cos(2\pi t)(2\pi - \sin(2\pi t)),
\]

with the exact solutions \(y_1(t) = \cos(2\pi t), \ y_2(t) = \sin(2\pi t)\).

Table 4: MAE and the objective function \(|J|\) of Example 7.4

| N  | \(|J|\) | MAE\((y_1)\) | MAE\((y_2)\) |
|----|--------|--------------|--------------|
| 4  | 0. 54876E-25 | 3.25234E-02  | 3.57846E-02  |
| 8  | 0. 32654E-25 | 5. 41659E-04  | 4.40340E-04  |
| 12 | 0. 54763E-25 | 2.46098E-05  | 1.14576E-05  |
| 16 | 0.13567E-26  | 4.17856E-07  | 3.39620E-07  |
Figure 4: Plot of numerical solutions of Example 7.4.

Table 5: MAE compared to error estimates of the approximations, N=16, α = 0.55.

<table>
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<tr>
<th>Example</th>
<th>MAE($y_1$)</th>
<th>E($y_1$)</th>
<th>MAE($y_2$)</th>
<th>E($y_2$)</th>
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</table>

8. Conclusion

The basic idea of our present method is transformation integral and integro-differential systems to an unconstrained optimization problem, by using ultraspherical approximation and Galerkin method. solving the resulting unconstrained optimization problem is easier than solving the original problem. The convergence of the proposed method depends on the ultraspherical approximation method (El-Hawary et al. [6]) and the PLF technique (Theorem 3.1, 3.2).

The tables given previously show that the suggested technique is quite reliable. It can be successfully applied to both linear and nonlinear integral and integro-differential systems. The method produce an accurate solution at small number of nodes. The error estimate for the approximation is suitable to measure the accuracy and the efficiency of the proposed method.

9. Acknowledgements

This work was supported by Deanship of Scientific Research of University of Dammam (Project No. 2011127).

References


