Population growth models via He-Laplace method

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Abstract

In this article, Homotopy perturbation method using Laplace transform (He-laplace method) is used as a basis to solve volterra’s model for population growth of a species within a closed system. This model is a nonlinear integro-differential equation where the integral term represents the effect of toxin. Using the prescribed method the solution of this problem is reducing into a series solution. Few terms of this series approximates the whole solution very well. It is observed that after applying He-Laplace method, results are easily obtained.

Keywords: Population Dynamics, Homotopy perturbation method, Laplace transform method, Volterra integral equation, Non-linear differential equation.

1. Introduction

Population of human being tends to increase as far as their environment will allow. Therefore, we can say that most of populations are in a dynamic state of equilibrium. Their numbers increase in a delicate balance that is influenced by limiting factors. Population dynamics of a specific species are determined by these underlying factors. These factors, in general are nutritional components competition and waste concentration increase. Population growth models have a great deal of application in different branches of science and engineering. It arises naturally in biological science, applied mathematics, physics and other disciplines such as theory of elasticity, forecasting human population, torsion of a wire, electric potential distribution.

Population dynamics express itself in different forms depending on what kind of species to which member of a population belongs. Sometimes, a population will grow suddenly in a population explosion that creates adverse environmental conditions. As a result, great numbers die suddenly, called a population crash. Taking the microbial growth curve, as our example to be studied, we find by analyzing it, when microorganisms are cultivated in a batch culture or closed system, that the resulting curve has four distinct phases: lag phase, exponential phase, stationary phase and death phase. For more details see [16].

The study of volterra integral equations originated with the work of volterra in population dynamics [7, 8].

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In [12], Integro differential equations find useful in a wide range of application fields such as Computer graphics, Image processing, Biological problems and financial problems. Therefore, their numerical solutions are very useful to analyse the problems related to various fields. Nonlinear integro differential equations are solved by some techniques such as perturbation methods, non-perturbation methods, Adomian decomposition, homotopy analysis method and numerical methods.

Scudo [13] indicates that volterra proposed a model for a population $u(t)$ of identical individuals which are crowding and sensitivity to “total metabolism” as follows:

$$\frac{du}{dt} = au - bu^2 - cu \int_0^t u(\tau) d\tau, \quad u(0) = \alpha. \quad (1)$$

Where $u = u(t)$ and $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient and $c > 0$ is the toxicity coefficient. The coefficient $c$ indicates the essential behavior of the population evolution before its level falls to zero in the long run. We know that if $c = 0$, then we have the well-known logistic equation. The last term contains the integral that indicates the “total metabolism” or total amount of toxins produced since time zero. The individual death rate is proportional to this integral and so the population death rate due to toxicity must include a factor $u$. The presence of the toxic term due to the system being closed always causes the population level to fall to zero in the long run, as will be seen later. The relative size of the sensitivity to toxins $c$ determines the manner in which the population evolves before its fated decay. Eq. (1) can be solved in quasi-closed form, the result of which is very insightful. If we set $v = \int u(\tau) d\tau$, then we can eliminate $t$ in favour of $u$ and $v$ and separate variables obtaining an inverse integral form for $v$.

A much more informative approach R. D. Small, K. Yukio and M. Masayasu [14, 17] uses singular perturbation techniques to find a closed form approximations to the solutions of Eq. (1).

In [14], the author showed that if $c/ab$ is large, where the populations are strongly sensitive to toxins, that the solution is proportional to $\text{sec} h^2(t)$. In this case, the solution $u(t)$ has a smaller amplitude. Furthermore, for $c/ab$ is small, where population are weakly sensitivity to toxins, the author showed that a rapid rise occurs along the logistic curve that will reach a peak and then followed by a slow exponential decay. The efficient study for solving Eq. (1) has been carried out for several decades. Recently, we have seen the appearance of several studies dealing with the solution of population growth in a closed system see [4, 2, 18].

In this paper, we will use He-Laplace methods to find an approximate solution of Eq. (1). This provides an analytical solution in the form of an infinite power series. However, there is a practical need to evaluate this solution and try to obtain numerical values from the infinite power series. The main aim of this paper is to compare the performance of the He-Laplace method with Sinc methods [8, 15] and Adomain decomposition method [1]. As a result, for the same number of terms, He-Laplace method yields relatively more accurate results with rapid convergence than other methods.

This paper is organized as follows: In section 2, we describe the Homotopy perturbation method. In section 3, we use the He-Laplace method is used to solve the model equation (1). In section 4, Sinc method is explained. In section 5, numerical examples are introduced. In section 6, concluding remarks are given.

2. Basic concept of homotopy perturbation method

Consider the following nonlinear differential equation:

$$A(y) - f(r) = 0, \quad r \in \Omega, \quad (2)$$

with the boundary conditions of

$$B \left( y, \frac{\partial y}{\partial n} \right) = 0, \quad r \in \Gamma, \quad (3)$$

where $A, B, f(r)$ and $\Gamma$ are a general differential operator, a boundary operator, a known analytic function and the boundary of the domain $\Omega$, respectively.
The operator $A$ can generally be divided into a linear part $L$ and a nonlinear part $N$. Eq. (2) may therefore be written as:

$$L(y) + N(y) - f(r) = 0,$$

By the homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \rightarrow R$ which satisfies:

$$H(v,p) = (1-p) [L(v) - L(y_0)] + p [A(v) - f(r)] = 0,$$

or

$$H(v,p) = L(v) - L(y_0) + pL(y_0) + p[N(v) - f(r)] = 0,$$

where $p \in [0,1]$ is an embedding parameter, while $y_0$ is an initial approximation of Eq. (2), which satisfies the boundary conditions. Obviously, from Eqs. (5) and (6) we will have:

$$H(v,0) = L(v) - L(y_0) = 0.$$

The changing process of $p$ from zero to unity is just that of $v(r,p)$ from $y_0$ to $y(r)$. In topology, this is called deformation, while $L(v) - L(y_0)$ and $A(v) - f(r)$ are called homotopy. If the embedding parameter $p$ is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of Eqs. (5) and (6) can be written as a power series in $p$:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots \infty.$$  

(8)

Setting $p = 1$ in Eq. (8), we have

$$y = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \ldots$$

(9)

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The series (9) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$. Moreover, He [5] made the following suggestions:

1. The second derivative of $N(v)$ with respect to $v$ must be small because the parameter may be relatively large, i.e., $p \rightarrow 1$.
2. The norm of $L^{-1}\left( \frac{\partial N}{\partial v} \right)$ must be smaller than one so that the series converges.

3. He-Laplace method

In this section, He-Laplace method is applied to the following class of non-linear integro-differential equation:

$$\frac{du}{dt} = au - bu^2 - cu \int_0^t u(\tau)d\tau,$$

(10)

$$u(0) = \alpha.$$

(11)

The method consists of first applying the Laplace transform to both sides of (10) as follows:

$$L[u'(t)] = L\left\{ [f(u)] - c \int_0^t u(\tau)d\tau \right\},$$

(12)

where $f(u) = au - bu^2$ and $L$ is the Laplacian operator. Using the initial condition (11), we have:

$$sL[u] - u(0) = L[f(u)] - cL\left[ \int_0^t u(\tau)d\tau \right],$$

(13)

$$L[u] = \frac{\alpha}{s} + \frac{1}{s}L[f(u)] - \frac{c}{s}L\left[ \int_0^t u(\tau)d\tau \right].$$

(14)
Taking inverse Laplace transform, we have:

\[ u(t) = g(t) + L^{-1}\left(\frac{1}{s}L[f(u)]\right) - cL^{-1}\left(\frac{1}{s}L\left[\int_0^t u(\tau)d\tau\right]\right). \] \hspace{1cm} (15)

Where \( g(t) \) represents the prescribed initial condition. Now we apply homotopy perturbation method (see [6, 10, 9] and their reference in) as follows:

\[ u(t) = \sum_{n=0}^{\infty} p^n u_n(t), \] \hspace{1cm} (16)

where the term \( u_n(t) \) are to recursively calculated and the nonlinear term \( f(u) \) and \( \int_0^t u(\tau)d\tau \) can be decomposed as:

\[ f(u) = a \sum_{n=0}^{\infty} p^n u_n(t) - b \sum_{n=0}^{\infty} p^n H_n(u), \] \hspace{1cm} (17)

and

\[ u(t)u(\tau) = \sum_{n=0}^{\infty} B_n(t, \tau). \] \hspace{1cm} (18)

Here, \( H_n(u) \) are called He’s polynomial (see [3, 11]) that are given by:

\[ H_n(u_0, u_1, u_2, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ f\left(\sum_{i=0}^{\infty} p^i u_i\right)\right] \] \hspace{1cm} (19)

Few First terms of polynomials \( H_n(u), B_n(t, \tau) \) can be calculated as follows:

\[
\begin{align*}
H_0(u) &= u_0(t) \\
H_1(u) &= 2u_0(t) u_1(t) \\
H_2(u) &= u_1^2(t) + 2u_0(t)u_2(t) \\
H_3(u) &= 2u_1(t)u_2(t) + 2u_0(t)u_3(t)
\end{align*}
\]\hspace{1cm} (20)

And

\[
\begin{align*}
B_0(t, \tau) &= u_0(t) u_0(\tau) \\
B_1(t, \tau) &= u_0(\tau) u_1(t) + u_1(\tau) u_0(t) \\
B_2(t, \tau) &= u_0(\tau) u_2(t) + u_1(\tau) u_1(t) + u_2(\tau) u_0(t) \\
B_3(t, \tau) &= u_0(\tau) u_3(t) + u_1(\tau) u_2(t) + u_2(\tau) u_1(t) + u_3(\tau) u_0(t)
\end{align*}
\]\hspace{1cm} (20)

And so on, the other polynomials can be constructed in a similar way. We assign the zeroth component by \( u_0 = u(0) = \alpha \). The remaining components \( u_n(t), n \geq 1 \) can be completely determined such that each term is computed by using the previous term.

Substituting Eqs. (16), (17), (18) in (15), we get:

\[
\sum_{n=0}^{\infty} p^n u_n(t) = g(t) - p \left( L^{-1}\left\{\frac{a}{s}L\left[\sum_{n=0}^{\infty} p^n u_n(t) - \frac{c}{s}L\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right\} - cL^{-1}\left\{\frac{a}{s}L\left[\int_0^t \sum_{n=0}^{\infty} p^n B_n(t, \tau)\right]\right\}\right) \] \hspace{1cm} (21)
Comparing the like powers of \( p \), the following approximations are obtained:

\[
p^0 : u_0(t) = g(t) = u(0) = \alpha \\
\frac{\partial u}{\partial t} + c u = f(u) - c u \int_0^t u(\tau) d\tau, \quad u(0) = \alpha.
\]

The series solutions thus entirely determined, however, in many cases the exact solution in a closed form may be obtained. For numerical comparison purposes, we constructed the solution \( u(t) \) as

\[
\lim_{n \to \infty} \phi_n = u(t),
\]

where \( \phi_n(t) = \sum_{k=0}^{n-1} u_k(t) \). It is also clear that a better approximation can be obtained by evaluating more components of the series solutions \((16)\) of \( u(t) \).

4. Sinc method

We again write equation \((1)\) in the following form:

\[
L u = f(u) - c u \int_0^t u(\tau) d\tau, \quad u(0) = \alpha.
\]

Where \( f(u) = au - bu^2 \), and \( L \) is the first-order derivative with respect to \( t \). A sinc approximate solution of Eq. \((24)\) takes the form:

\[
u_m(x) = \sum_{j=-M}^{M-1} u_j S_j \phi(x), \quad m = 2M.
\]

Where the basis functions for the half-line are defined by the composition:

\[
S_j \phi(x) = \sin[(\pi/h)(\phi(x) - jh)] / [(\pi/h)(\phi(x) - jh)],
\]

and \( \phi(x) = \ln x \) and \( h \) is the step size. Substitute \( u_m(x) \) into Eq. \((24)\) and evaluate at the \( m = 2M \) sinc nodes \( \phi^{-1}(t_k) = x_k = \exp(kh), k = -M, \ldots, M - 1 \). At the discrete system:

\[
-\frac{1}{h} \zeta(\phi') I_m^{(-1)} \mathbf{u} = f(\mathbf{u}) - c \mathbf{u} (h I_m^{(-1)} \zeta(1/\phi')) \mathbf{u}.
\]

Where the notation \( \zeta(\phi') \) denote a \( 2M \times 2M \) diagonal matrix with \( k^{th} \) diagonal entry given by \( \phi'(x_k) \). The entries of the matrix \( I_m^{(-1)} \) are defined by the integrals:

\[
\delta_k^{(-1)} = \frac{1}{h} \int_{-\infty}^{x_k} S_j(s) ds.
\]
The coefficients matrix $I_m^{(1)}$ is obtained from the explicit values for the derivative of the sinc basis functions at the nodes:

$$\frac{dS_j(x)}{dx} = \frac{1}{h} \delta_{jk}^{(1)} = \frac{1}{h} \begin{cases} 0, & j = k \\ (-1)^{k-j} \frac{\sin((k-j)\pi/2)}{k-j}, & j \neq k \end{cases}$$

Collecting the numbers $\delta_{jk}^{(1)} - M \leq j, k \leq M - 1$ leads to the definition of the $m \times m$ skew-symmetric matrix $I_m^{(1)}$. The notation $o$ denotes the hadamard matrix multiplication. The collection system (25) is satisfied by the solution $u(t)$ of (24) at grid points $t_k$ with an error of exponential order (see [1]).

The system (25) can be efficiently solved by means of closely related to a nonlinear Gauss-Seidel scheme which amounts to computing:

$$\frac{1}{h} \zeta(\phi') I_m^{(1)} \pi^{(n+1)} = f(\pi^{(n)}) - c \pi^{(n)} o \left( h I_m^{(1)} \zeta(1/\phi') \right) \pi^{(n)}.$$

With replacement of $\pi^{(n)}$ by $\pi^{(n+1)}$ as soon as that value is computed. To prove the convergence of the solution for the discrete system by fixed point iteration. The idea is to produce a sequence of iterations that converges to the solution. We proceed as follows:

We know that the solution in Eq. (26) can be written in a more compact form:

$$\pi^{(n+1)} = H(\pi^{(n)}), \quad n = 0, 1, 2, \ldots$$

Where the $m \times m$ matrix $H(\pi)$ is given by:

$$H(\pi^{(n)}) = \left( -\frac{1}{h} \zeta(\phi') I_m^{(1)} \right)^{-1} \left[ f(\pi^{(n)}) - c \pi^{(n)} o \left( h I_m^{(1)} \zeta(1/\phi') \right) \pi^{(n)} \right].$$

We should note that the matrix $I_m^{(1)}$ is a real skew symmetric matrix and if $m$ is even then it is always invertible. By selecting an initial approximate $\pi^{(0)}$, we iterate the continuous map $H$ repeatedly via the formula in (27). We can choose $M$ sufficiently large such that $\|H(\pi)\| < r < 1$, for $\pi$ in any given fixed ball $B_r$ about the origin, where $r$ is a constant with $0 < r = \theta \left( M^2 \exp((-\sqrt{\pi} d \alpha M)) \right)$. Using the Mean value theorem and the condition $\|H(\pi)\| < r$, we see that the distance between two consecutive iterations $\pi^{(n)}$ and $\pi^{(n+1)}$ in the ball $B_r$ is given by:

$$\|\pi^{(n+1)} - \pi^{(n)}\| \leq r^n \|\pi^{(1)} - \pi^{(0)}\|$$

and so it is easy to show that the sequence \{\pi^{(n)}\} is Cauchy and hence converges to some $\pi$. For details about the Sinc method and its applications for solving differential equation, we refer the reader to [8, 15].

5. Numerical examples

The He-Laplace method provides an analytical solution in the form of an infinite power series. However, there is a practical need to evaluate this solution and try to obtain numerical values from the infinite power series. In order to investigate the accuracy of the He-Laplace method with a finite number of terms, Eq. (10) is also solved numerically, and the corresponding results are compared. For numerical purpose, Sinc method is used and the parameter $M$ is taken to be 16. First, we solve Eq. (10) by He-Laplace Method with initial condition $u_0(t) = \alpha$. Then other components $u_1(t), u_2(t), u_3(t), \ldots, u_n(t)$ are calculated using Eq. (22) in as follows:
\[ u_1(t) = (a\alpha - b\alpha^2) t - c\alpha^2 \frac{t^2}{2!}, \]
\[ u_2(t) = (2b^2\alpha^3 - 3ab\alpha^2 + a^2\alpha) \frac{t^2}{2!} + (5bc\alpha^3 - 4ac\alpha^2) \frac{t^3}{3!} + 4c^2\alpha^3 \frac{t^4}{4!}, \]
\[
\vdots
\]

and so on. In order to compute more components \( u_3(t), u_4(t), u_5(t), u_n(t) \) for the perturbation series Eq. (16), one can use Mathematica. Now \( u(t) \) was evaluated with the following expansion:

\[ u(t) = \alpha + (a\alpha - b\alpha^2) t - c\alpha^2 \frac{t^2}{2!} + (2b^2\alpha^3 - 3ab\alpha^2 + a^2\alpha) \frac{t^2}{2!} + (5bc\alpha^3 - 4ac\alpha^2) \frac{t^3}{3!} + 4c^2\alpha^3 \frac{t^4}{4!} + \ldots \] (28)

Case (i): we consider \( a = b = 1, c = 0.1, \alpha = 0.05, 0.1, 0.5 \) and solve Eq. (28). Its solution is called \( u_{HLM} \). In Table 1, we compare the numerical solution obtained by Sinc method, \( u_{Sinc} \) using five iterations with \( u_{HLM} \). In this case where \( c/ab \) is considered to be small, where the populations are strongly sensitive to toxins and the solution \( u(t) \) goes a larger amplitude.

Case (ii): we consider \( a = b = c = 1 \) (\( c/ab \) is large) and \( \alpha = 0.05, 0.1, 0.5 \). In Table 2, we find that the solution of Eq. (10) decay more faster than the previous one.

<table>
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<th>( t_i )</th>
<th>( u_{HLM}(t_i) )</th>
<th>( u_{Sinc}(t_i) )</th>
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Table 1: Numerical values when \( a = b = 1, c = 0.1, \alpha = 0.05, 0.1, 0.5 \).
6. Discussions

In this paper, we have applied He-laplace method for solving non-linear integro-differential equations and compared our results with the Sinc method. It is shown that the He-laplace method is simple and easy to use. It also minimizes the computational results. But the Sinc method has a complicated computational calculus and is not easy to use. More accuracy can be achieved using He-laplace method by evaluating more components of the solution.

References