A correction note on “Three-step iterative methods for nonlinear equations” and generalization of method

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ABSTRACT. In the paper [Muhammad Aslam Noor, Khalida Inayat Noor, Three-step iterative methods for nonlinear equations, Applied Mathematics and Computation, 183 (2006), pp. 322-327], Authors presented an algorithm (Algorithm 2.3) and stated a theorem (Theorem 2.3) to prove the cubic order of convergence but the given proof does not show cubic order of convergence. Actually, the mathematical derivation steps to develop the Algorithm 2.3 are wrong. In this note, we present the correct mathematical developments and finally provide computational order of convergence in the favor of our claim and provide the generalization of the method.

KEYWORDS. Nonlinear equations; Iterative methods; Convergence order; Decomposition methods.

1. Introduction

Nonlinear algebraic equations are very important in nonlinear science. However, to find exact analytical solutions of these nonlinear equations are not always possible. Alternative tools to solve them are iterative methods [1, 2, 3, 4, 5, 6]. Iterative methods start from some initial guess in the neighborhood of a root of nonlinear equation and refine it to meet the convergence conditions. The Newton’s method (NM) to solve nonlinear equations is very famous which is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots.$$  (1)

It is quadratically convergent in some neighborhood of the root \(\alpha\) of \(f(x) = 0\). In [7], authors used decomposition method [8] to develop the “Three-step iterative methods for nonlinear equations”. In the following section we discuss all the necessary steps to provide correct derivation.

2. Convergence analysis

Let \(\alpha\) be a simple root of nonlinear equation \(f(\alpha) = 0\) and \(\gamma\) be an initial guess in the vicinity of \(\alpha\). The Taylor’s series expansion of \(f\) around \(\gamma\) is

$$f(x) = 0, \quad f(x - \gamma + \gamma) = 0, \quad f(\gamma) + f'(\gamma)(x - \gamma) + g(x) = 0, \quad g(x) = f(x) - f(\gamma) - f'(\gamma)(x - \gamma).$$  (2) (3) (4) (5)
If we rearrange (5), we obtain
\[ x = c + N(x), \quad (6) \]
where
\[ c = \gamma - \frac{f(\gamma)}{f'(\gamma)}, \quad (7) \]
\[ N(x) = -\frac{g(x)}{f'(\gamma)}. \quad (8) \]

Note that if
\[ f(x_0) = g(x_0) \quad (9) \]
then (5) implies that
\[ x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)} = c. \quad (10) \]

By using the idea of [8], we can express \( x \) as
\[ x = \sum_{i=0}^{\infty} x_i. \quad (11) \]

By substituting \( x \) in (6), we obtain
\[ \sum_{i=0}^{\infty} x_i = c + N\left( \sum_{i=0}^{\infty} x_i \right), \quad (12) \]
where \( N\left( \sum_{i=0}^{n} x_i \right) \) can be written as
\[ N(x_0 + x_1 + x_2 + \cdots + x_n) = N(x_0) + \left[ N(x_0 + x_1) - N(x_0) \right] + \left[ N(x_0 + x_1 + x_2) - N(x_0 + x_1) \right] + \cdots + \left[ N(x_0 + x_1 + x_2 + \cdots + x_n) - N(x_0 + x_1 + x_2 + \cdots + x_{n-1}) \right]. \quad (13) \]

(12) can be expresses as (by using the idea of (13))
\[ \sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left( \sum_{j=0}^{i} x_j \right) - N\left( \sum_{j=0}^{i-1} x_j \right) \right\}. \quad (14) \]

By equating terms on both sides of (14), we get iterative schemes:
\[
\begin{align*}
  x_0 & = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}, \\
  x_1 & = N(x_0), \\
  x_2 & = N(x_0 + x_1) - N(x_0), \\
  & \vdots \\
  x_{n+1} & = N(x_0 + x_1 + \cdots + x_n) - N(x_0 + x_1 + \cdots + x_{n-1}), \quad n = 1, 2, \ldots.
\end{align*}
\]
In order to get iterative scheme, we evaluate $N(x_0), N(x_0 + x_1), N(x_0 + x_1 + x_2)$.

\[
x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)},
\]
\[
N(x_0) = -\frac{g(x_0)}{f'(\gamma)} = -\frac{f(x_0) - f(\gamma) - f'(\gamma)(x_0 - \gamma)}{f'(\gamma)},
\]
\[
N(x_0) = -\frac{f(x_0) - f(\gamma) - f'(\gamma)(\gamma - \frac{f(\gamma)}{f'(\gamma)} - \gamma)}{f'(\gamma)},
\]
\[
N(x_0) = -\frac{f(x_0) - f(\gamma) - f'(\gamma)(\gamma - \frac{f(\gamma)}{f'(\gamma)} - \gamma)}{f'(\gamma)},
\]
\[
x_1 = N(x_0) = -\frac{f(x_0)}{f'(\gamma)},
\]
\[
N(x_0 + x_1) = -\frac{g(x_0 + x_1)}{f'(\gamma)} = -\frac{f(x_0 + x_1) - f(\gamma) - f'(\gamma)(x_0 + x_1 - \gamma)}{f'(\gamma)},
\]
\[
N(x_0 + x_1) = -\frac{f(x_0 + x_1) - f(\gamma) - f'(\gamma)(\gamma - \frac{f(\gamma)}{f'(\gamma)} - \gamma)}{f'(\gamma)},
\]
\[
N(x_0 + x_1) = -\frac{f(x_0 + x_1) + f(x_0)}{f'(\gamma)} = -\frac{f(x_0 + x_1)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)} = N(x_0) - \frac{f(x_0 + x_1)}{f'(\gamma)},
\]
\[
x_2 = N(x_0 + x_1) - N(x_0) = -\frac{f(x_0 + x_1)}{f'(\gamma)}. \tag{16}
\]

If we concise the results, we have

\[
x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)}, \tag{19}
\]
\[
x_1 = -\frac{f(x_0)}{f'(\gamma)}, \tag{20}
\]
\[
x_2 = -\frac{f(x_0 + x_1)}{f'(\gamma)}. \tag{21}
\]

Note that in [7] $x_2$ at page 324 in equation (16) is given as

\[
x_2 = -\frac{f(x_0 + x_1)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)}, \tag{22}
\]

which is actually wrong. If we approximate $x = x_0 + x_1$ then we have the following iterative scheme (which is Algorithm 2.2 in [7].)

\[
\begin{cases}
  x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)}, \\
  x = x_0 + x_1 = x_0 - \frac{f(x_0)}{f'(\gamma)},
\end{cases} \tag{23}
\]

or

\[
\begin{cases}
  y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\
  x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \tag{24}
\end{cases}
\]

(24) is given in [9] and it has cubic order of convergence. If we approximate $x = x_0 + x_1 + x_2$, we get

\[
\begin{cases}
  x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)}, \\
  x_0 + x_1 = x_0 - \frac{f(x_0)}{f'(\gamma)}, \\
  x = x_0 + x_1 + x_2 = x_0 + x_1 - \frac{f(x_0 + x_1)}{f'(\gamma)}, \quad \text{(by using (21))} \tag{25}
\end{cases}
\]

or

\[
\begin{cases}
  x_0 + x_1 + x_2 = x_0 + x_1 - \frac{f(x_0 + x_1)}{f'(\gamma)} = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{f(x_0)}{f'(\gamma)} - \frac{f(x_0 + x_1)}{f'(\gamma)}. \\
\end{cases}
\]
Note that in [7] the expression for \( x = x_0 + x_1 + x_2 \) is \( x = \gamma - \frac{f(\gamma)}{f'(\gamma)} \), which is wrong. (25) can be written as

\[
\begin{align*}
    y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
    z_n &= y_n - \frac{f(y_n)}{f'(y_n)}, \\
    x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}.
\end{align*}
\] (26)

The convergence order of iterative scheme (26) is four. A short proof of (26) is discussed as follows. We denote the error at nth-step by \( e_n = x_n - \alpha \), where \( \alpha \) is a simple root of nonlinear algebraic equation \( f(x) = 0 \) and \( c_1 = f'(\alpha), c_k = \frac{e_k}{e_{k-1}} f^{(k)}(\alpha) \) for \( k = 2, 3, \ldots \).

\[
\begin{align*}
    f(x_n) &= c_1(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)), \\
    f'(x_n) &= c_1(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6)), \\
    y_n - \alpha &= c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^2)e_n^4 + O(e_n^5), \\
    f(y_n) &= c_1c_2e_n^2 - 2c_1(-c_3 + c_2^2)e_n^3 + c_1(3c_4 - 7c_2c_3 + 5c_2^2)e_n^4 + O(e_n^5), \\
    z_n - \alpha &= 2c_2c_3e_n^3 + (7c_2c_3 - 9c_2^2)e_n^4 + O(e_n^5), \\
    e_{n+1} &= 4c_2^3e_n^5 + O(e_n^5)\] (32)

Noor et al. [7] stated the following algorithm (Algorithm 2.3).

**Algorithm 2.3.** For a given \( x_0 \), compute the approximate solution \( x_{n+1} \) by the iterative schemes

**Predictor-steps:**

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \] (17)

\[
z_n = \frac{f(y_n)}{f'(y_n)}. \] (18)

**Corrector-step:**

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n + z_n)}{f'(y_n)}, \quad n = 0, 1, 2, \ldots \] (19)

Figure 1. Algorithm 3 from [7].

The statement of related theorem (Theorem 2.1) is given below in figure 2. If we subtract (27) from (23) in figure 2, we get the following error equation

\[
e_{n+1} = c_2e_n^2 + (2c_3 - 4c_2^2)e_n^3 + O(e_n^4),\] (33)

which shows that the order of convergence is two.

**3. Generalization of Iterative scheme**

By adding the values of \( x_i \)'s in (15), we obtain

\[
x_1 + x_2 + x_3 + \cdots + x_n = N(x_0 + x_1 + x_2 + \cdots + x_{n-1}). \] (34)

By using (10), (8) can be expressed as

\[
N(X) = X - \frac{f(X)}{f'(\gamma)} - x_0, \quad \quad N(X) + x_0 = X - \frac{f(X)}{f'(\gamma)}. \] (35)
If \( x \) is approximated by

\[
\begin{align*}
x &= x_0 + x_1 + x_2 + x_3 + \cdots + x_n, \\
x &= x_0 + N(x_0 + x_1 + x_2 + \cdots + x_{n-1}), \\
x &= x_0 + x_1 + x_2 + \cdots + x_{n-1} - \frac{f(x_0 + x_1 + x_2 + \cdots + x_{n-1})}{f'(\gamma)}, \\
x_0 + x_1 + x_2 + \cdots + x_{n-1} + x_n &= x_0 + x_1 + x_2 + \cdots + x_{n-1} - \frac{f(x_0 + x_1 + x_2 + \cdots + x_{n-1})}{f'(\gamma)}, \\
x_n &= -\frac{f(x_0 + x_1 + x_2 + \cdots + x_{n-1})}{f'(\gamma)}, \quad n = 1, 2, 3, \ldots
\end{align*}
\]

(39) gives the following iterative scheme.

\[
\begin{align*}
x_0 &= \gamma - \frac{f(x_0)}{f'(\gamma)}, \\
x_0 + x_1 &= x_0 - \frac{f(x_0)}{f'(\gamma)}, \\
x_0 + x_1 + x_2 &= x_0 + x_1 - \frac{\frac{f(x_0 + x_1)}{f'(\gamma)}}{f'(\gamma)} \\
&\quad \vdots \\
x_0 + x_1 + x_2 + \cdots + x_{n-1} + x_n &= x_0 + x_1 + x_2 + \cdots + x_{n-1} - \frac{f(x_0 + x_1 + x_2 + \cdots + x_{n-1})}{f'(\gamma)}.
\end{align*}
\]

Alternative way to write (41) is

\[
\begin{align*}
y_0 &= x_n \\
y_1 &= y_0 - \frac{f(y_0)}{f'(y_0)} \\
y_2 &= y_1 - \frac{f(y_2)}{f'(y_2)} \\
y_3 &= y_2 - \frac{f(y_3)}{f'(y_3)} \\
&\quad \vdots \\
y_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)} \\
x_{n+1} &= y_{n+1}
\end{align*}
\]

The order of convergence of iterative scheme is \( n + 2 \) for \( n = 0, 1, 2, 3, \ldots \) and also noted that the number of functional evaluations are also \( n + 2 \). Proof of (42) is simple and straightforward.

4. Numerical computations

**Definition 4.1.** Let \( x_{n-1}, x_n \) and \( x_{n+1} \) be successive iterations around the root \( \alpha \) of \( f(x) = 0 \), the computational order of convergence (COC) [10], can be approximated by

\[
COC \approx \frac{\ln |(x_{n+1} - \alpha) (x_{n} - \alpha)^{-1}|}{\ln |(x_{n} - \alpha) (x_{n-1} - \alpha)^{-1}|}.
\]

In order to verify the computational order of convergence a set of seven functions [7] is listed in Table 1.

The absolute error comparison between iterative schemes (24), (26) and Algorithm 2.3, is depicted in Table 2. Total number of function evaluations per iteration are three, four, four for (24), (26), Algorithm 2.3 respectively.

Table 2 also shows total number of iterations and computational order of convergence for each iterative scheme. The given different number of iterations are reported in [7].
Table 1. List of test functions

<table>
<thead>
<tr>
<th>Functions</th>
<th>Roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1(x) = \sin(x)^2 - x^2 + 1$</td>
<td>$\alpha = -1.404491648215341226035086818$</td>
</tr>
<tr>
<td>$f_2(x) = x^2 - \exp(x) - 3 \times x + 2$</td>
<td>$\alpha = 0.2575302854398607604553673049$</td>
</tr>
<tr>
<td>$f_3(x) = \cos(x) - x$</td>
<td>$\alpha = 0.7390851332151606416553120877$</td>
</tr>
<tr>
<td>$f_4(x) = (x-1)^3 - 1$</td>
<td>$\alpha = 2$</td>
</tr>
<tr>
<td>$f_5(x) = x^3 - 10$</td>
<td>$\alpha = 2.154434690031883721759293567$</td>
</tr>
<tr>
<td>$f_6(x) = x \exp(x^2) - \sin(x)^2 + 3 \cos(x) + 5$</td>
<td>$\alpha = -1.207647827130918927009416758$</td>
</tr>
<tr>
<td>$f_7(x) = \exp(x^2 + 7x - 30) - 1$</td>
<td>$\alpha = 3$</td>
</tr>
</tbody>
</table>

Table 2. Numerical comparison between absolute errors($|x_n - \alpha|$)

<table>
<thead>
<tr>
<th>$f_n(x), x_0$</th>
<th>Iter</th>
<th>(24) COC- (24)</th>
<th>(26) COC- (26)</th>
<th>Algo. 2.3</th>
<th>(COC)-Algo. 2.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1, -1.0$</td>
<td>5</td>
<td>3.6e3</td>
<td>-0.1140</td>
<td>1.7e-161</td>
<td>3.4e-8</td>
</tr>
<tr>
<td>$f_2, 2.0$</td>
<td>4</td>
<td>3.2e-42</td>
<td>3.000</td>
<td>1.4e-117</td>
<td>2.2e-12</td>
</tr>
<tr>
<td>$f_3, 1.7$</td>
<td>4</td>
<td>5.2e-73</td>
<td>3.000</td>
<td>1.1e-220</td>
<td>7.5e-17</td>
</tr>
<tr>
<td>$f_4, 3.5$</td>
<td>5</td>
<td>1.8e-28</td>
<td>3.000</td>
<td>4.7e-101</td>
<td>4.9e-8</td>
</tr>
<tr>
<td>$f_5, 1.5$</td>
<td>5</td>
<td>1.3e-32</td>
<td>3.000</td>
<td>4.6e-186</td>
<td>1.8e-8</td>
</tr>
<tr>
<td>$f_6, -2.0$</td>
<td>6</td>
<td>3.3e-35</td>
<td>3.000</td>
<td>8.3e-153</td>
<td>3.0e-9</td>
</tr>
<tr>
<td>$f_7, 3.5$</td>
<td>8</td>
<td>1.5e-16</td>
<td>2.996</td>
<td>2.5e-99</td>
<td>0.000013</td>
</tr>
</tbody>
</table>

5. Conclusions

We have shown that the proof for order of convergence of Algorithm 2.3 is not cubic. Actually due to wrong calculations, authors ended with wrong algorithm. Computational order of convergence also verify that convergence order is not three. The Maple program are listed to show complete proofs. Finally a generalization of iterative method is given.

Acknowledgements

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References

Theorem 2.1. Let \( r \in I \) be a simple zero of sufficiently differentiable function \( f: I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) for an open interval \( I \). If \( x_0 \) is sufficiently close to \( r \), then the three-step iterative method defined by Algorithm 2.3 has third-order convergence.

Proof. Let \( r \) be a simple zero of \( f \). Since \( f \) is sufficiently differentiable, by expanding \( f(x_n) \) and \( f'(x_n) \) about \( r \), we get

\[
f(x_n) = f'(r)[c_n + c_2e_n^2 + c_3e_n^3 + \cdots], \tag{20}
\]

\[
f'(x_n) = f'(r)[1 + 2c_2e_n^2 + 3c_3e_n^3 + 4c_4e_n^4 + \cdots], \tag{21}
\]

where \( c_k = \frac{f^{(k)}(r)}{f''(r)} \), \( k = 1, 2, 3, \ldots \) and \( e_n = x_n - r \).

Now, from (20) and (21), we have

\[
\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_3 - c_2)e_n^3 + \cdots \tag{22}
\]

From (17) and (22), we have

\[
y_n = r + c_2e_n^2 + 2(c_3 - c_2)e_n^3 + \cdots \tag{23}
\]

Now expanding \( f(y) \) about \( r \) and using (23), we have

\[
f(y_n) = f'(r)[c_2e_n^2 + 2(c_3 - c_2)e_n^3 + \cdots]. \tag{24}
\]

Now from (21) and (24), we have

\[
z_n = -c_2e_n^2 - 2(c_3 - 2c_2)e_n^3 + \cdots \tag{25}
\]

Now again expanding \( f(y_n + z_n) \) about \( r \) and using (23) and (25), we have

\[
f(y_n + z_n) = f'(r)[2c_2^2e_n^4 + \cdots]. \tag{26}
\]

From (21) and (26), we get

\[
\frac{f(y_n + z_n)}{f'(x_n)} = 2c_2^2e_n^4 - 2c_2e_n^4 + \cdots \tag{27}
\]

From (22), (25) and (27), it follows that Algorithm 2.3 has third-order convergence. \( \square \)

Figure 2. Theorem 2.1 from [7].