RESEARCH ARTICLE

Generalized $\alpha$-Bernoulli Polynomials through Weighted Pochhammer Symbols

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In this paper, we define generalized $\alpha$-difference operators $\Delta_{\pm\ell}$ and generalized $\alpha$-Bernoulli polynomial $B_{\alpha(n+1)}(k, -\ell)$ which is a solution of generalized $\alpha$-difference equation $u(k, -\ell) - \alpha u(k) = (n+1)k^n$, $n \in \mathbb{N}(1)$, to obtain a formula for sum of $n^{th}$ power arithmetic-geometric series of the form

$$\sum_{r=1}^{[\frac{k}{\ell}]} \left(\frac{1}{r}\right)^r (k + \ell - rl)^n = -\frac{1}{n+1} B_{\alpha(n+1)}(k, -\ell) - \alpha [\frac{k}{\ell}] c_j.$$

Suitable examples are given to illustrate the main results.

Keywords: Generalized $\alpha$-difference operator; generalized $\alpha$-Bernoulli polynomials.

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1. Introduction

The theory of difference equations is based on the operator $\Delta$ defined as

$$\Delta x(n) = x(n+1) - x(n), \quad n \in \mathbb{N}$$

where $\mathbb{N} = \{0, 1, 2, 3, \ldots, \}$. Eventhough many authors [1, 2] have suggested the definition of $\Delta$ as

$$\Delta x(n) = x(n + \ell) - x(n), \quad \ell, n \in \mathbb{N},$$

no significant progress took place on this line. But recently, when we took up the definition of $\Delta$ as given in (2), the theory of difference equations are developed in a different direction. We obtained some interesting results in Number Theory. For convenience, we labelled the operator $\Delta$ defined by (2) as $\Delta_{\ell}$ and by defining its inverse $\Delta^{-1}_{\ell}$, many interesting results on Number Theory were obtained. By extending the study for complex function and $\ell$ to be real, some new qualitative properties like rotatory, expanding and shrinking, spiral and weblike were studied for the solutions of difference equations involving $\Delta_{\ell}$. The results obtained can be found in [3–7]. Also, the authors extend the theory of $\Delta_{\ell}$ to the more generalized difference operator $\Delta_{\alpha(\ell)}$ and obtain some significant results, relations, theorems and formulae in Number Theory using the inverse of $\Delta_{\alpha(\ell)}$.

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With this background, in this paper, we find a formula for sum of \( n^{th} \) power of arithmetico-geometric progression using the generalized \( \alpha \)-Bernoulli polynomial \( B_{\alpha(n+1)}(k, -\ell) \), and a solution of the generalized \( \alpha \)-difference equation

\[
u(k - \ell) - \alpha \nu(k) = (n + 1)k^n, \quad n \in \mathbb{N}(1).
\]

Throughout this paper, we make use of the following assumptions:

(i) \( \mathbb{N}(0) = \{0, 1, 2, 3, \ldots\} \).
(ii) \( \mathbb{N}_\ell(a) = \{a, a + \ell, a + 2\ell, \ldots\} \).
(iii) \( \alpha, \ell \) are positive reals and \( n, r \) are positive integers.
(iv) \( \lfloor x \rfloor \) is the integer part of \( x \), and \( \Delta_{\alpha(-\ell)}^{-1} u(k) \big|_j = \Delta_{\alpha(\pm \ell)}^{-1} u(k) - \alpha \left[ \frac{j\ell}{r} \right] \Delta_{\alpha(\pm \ell)}^{-1} u(j) \).
(v) \( c, c_0, c_1, c_2, \ldots \) are constants.
(vi) \( rC_i = \frac{r!}{(r-i)!i!} \) where \( 0! = 1, r! = 1.2.3\ldots r \).
(vii) \( [0, \infty) \) is the set of all nonnegative reals.

2. Preliminaries

In this section, we present some definitions and preliminary results which are used in our subsequent discussion.

Definition 2.1 Let \( u(k), k \in [0, \infty) \) be a real or complex valued function. Then the generalized \( \alpha \)-difference operator for \( u(k) \) is defined as

\[
\Delta_{\alpha(-\ell)} u(k) = u(k - \ell) - \alpha u(k), \quad \ell \in (0, \infty).
\] (3)

Lemma 2.2 If \( E \) is the usual shift operator, then:

(i)

\[
E^{-\ell} = \Delta_{\alpha(-\ell)} + \alpha.
\] (4)

(ii)

\[
\alpha + \Delta_{\alpha(-\ell)} = (\Delta_{\alpha(-1)})^\ell.
\] (5)

(iii)

\[
\Delta_{\alpha(-\ell)}^n = \sum_{r=0}^{n} nC_r(-\alpha)^r E^{-\ell(n-r)}.
\] (6)

(iv)

\[
\Delta_{\alpha(\ell)} \Delta_{\alpha(-\ell)} + \alpha \Delta_{\alpha(\ell)} + \alpha \Delta_{\alpha(-\ell)} + \alpha = 1.
\] (7)

(v)

\[
\Delta_{\alpha(\ell)} - \Delta_{\alpha(-\ell)} = E^\ell - E^{-\ell}.
\] (8)

(vi)

\[
\Delta_{\alpha(\ell)} + \Delta_{\alpha(-\ell)} + 2\alpha = E^\ell + E^{-\ell}.
\] (9)
Lemma 2.3 If \( n \) is an even positive integer, then:

\[
(\Delta_{\alpha(\ell)} - \Delta_{\alpha(-\ell)})^n = (\Delta_{\alpha(n\ell)} - \Delta_{\alpha(-n\ell)} + 2\alpha) - nC_1(\Delta_{\alpha((n-2)\ell)} \\
- \Delta_{\alpha((2-n)\ell)} + 2\alpha) + nC_2(\Delta_{\alpha((n-4)\ell)} - \Delta_{\alpha((4-n)\ell)}) \\
+ 2\alpha + \cdots + (-1)^{\frac{n}{2}} nC_{n/2}.
\]

If \( n \) is an odd positive integer, then:

\[
(\Delta_{\alpha(\ell)} - \Delta_{\alpha(-\ell)})^n = (\Delta_{\alpha(n\ell)} - \Delta_{\alpha(-n\ell)}) - nC_1(\Delta_{\alpha((n-2)\ell)} \\
- \Delta_{\alpha((2-n)\ell)}) + nC_2(\Delta_{\alpha((n-4)\ell)} - \Delta_{\alpha((4-n)\ell)}) \\
- \cdots + (-1)^{\frac{n+1}{2}} nC_{n+1}(\Delta_{\alpha(\ell)} - \Delta_{\alpha(-\ell)}).
\]

**Proof** The proof follows from (8) and (9). □

Lemma 2.4 The relation between \( \Delta_{\alpha(\ell)} \), \( \Delta_{-\ell} \) and \( E^{-\ell} \) is

\[
\Delta_{\alpha(\ell)} - \Delta_{-\ell} = \Delta_{\alpha(2\ell)} E^{-\ell}.
\]

Definition 2.5 Let \( \ell \) be a positive real. Then, the inverse of generalized \( \alpha \)-difference operator \( \Delta_{\alpha(\ell)} \) is defined as if \( \Delta_{\alpha(\ell)} v(k) = u(k) \), then

\[
\Delta_{\alpha(\ell)}^{-1} u(k) = v(k) - \alpha \left[ \frac{k}{\ell} \right] v(j).
\]

Definition 2.6 Let \( \ell \) be a positive real. Then, the inverse of generalized \( \alpha \)-difference operator \( \Delta_{\alpha(-\ell)} \) is defined as if \( \Delta_{-\alpha(\ell)} v(k) = u(k) \), then:

\[
\Delta_{\alpha(-\ell)}^{-1} u(k) = v(k) - \left( \frac{1}{\alpha} \right) \left[ \frac{k}{\ell} \right] v(j).
\]

Theorem 2.7 If \( k \in [0, \infty) \), \( \ell \in (0, \infty) \) and \( j = k - \left[ \frac{k}{\ell} \right] \ell \), then:

\[
\Delta_{\alpha(\ell)}^{-1} u(k)^k_j = \sum_{r=1}^{\left[ \frac{k}{\ell} \right]} \alpha^{r-1} u(k-r\ell).
\]

**Proof** The proof follows from (11). □

Theorem 2.8 If \( -k \in [-0, -\infty) \) and \( \ell \in (0, \infty) \), then:

\[
\Delta_{\alpha(\ell)}^{-1} u(-k)^k_j = \sum_{r=1}^{\left[ \frac{k}{\ell} \right]} \alpha^{r-1} u(-k+r\ell).
\]

**Proof** The proof follows from (11). □

Corollary 2.1 If \( -k \in [-0, -\infty) \) and \( \ell \in (0, \infty) \), then

\[
\sum_{r=1}^{\left[ \frac{k}{\ell} \right]} \alpha^{r-1}(-k+r\ell)^2 = \frac{(-k)^2}{(1-\alpha)} - \frac{2\ell k}{(1-\alpha)^2} + \frac{2\ell^2}{(1-\alpha)^3} - \frac{\ell^2}{(1-\alpha)^2}
\]
\[-\alpha \left[ \frac{j}{\pi} \right] \left\{ \frac{(-j)^2}{(1 - \alpha)} - \frac{2\ell j}{(1 - \alpha)^2} + \frac{2\ell^2}{(1 - \alpha)^3} - \frac{\ell^2}{(1 - \alpha)^2} \right\}. \tag{15} \]

**Proof** The proof follows by taking \( u(-k) = (-k)^2 \) in (14) and (11).

**Example 2.9** By taking \( k = 35, \ell = 3, \alpha = 2 \) and \( j = 2 \) in (15), we get

\[
\sum_{r=1}^{[\frac{j}{\pi}]} 2^{r-1}(-35 + 4r)^2 = \left\{ \frac{(-35)^2}{(1 - 2)} - \frac{(2)(3)(35)}{(1 - 2)^2} + \frac{(2)(3)^2}{(1 - 2)^3} \right\} - \frac{3^2}{(1 - 2)^2}
\]

\[
= 86602.
\]

**Theorem 2.10** If \( k \in [0, \infty) \) and \( \ell \in (-0, -\infty) \), then

\[
\Delta_{\frac{1}{\alpha(-\ell)}}^{-1} u(\pm k)^k_j = \sum_{r=1}^{[\frac{j}{\pi}]} \left( \frac{1}{\alpha} \right)^r u(k + \ell - r\ell). \tag{16} \]

**Proof** The proof follows from the relation

\[
\Delta_{\frac{1}{\alpha(-\ell)}} u(k) = -\frac{1}{\alpha} (\Delta_{\frac{1}{\alpha}(2\ell)})^{-1} u(k - \ell)
\]

and (12).

**Corollary 2.2** If \( k \in [0, \infty) \) and \( \ell \in (0, \infty) \), then

\[
\sum_{r=1}^{[\frac{j}{\pi}]} \left( \frac{1}{\alpha} \right)^r (k + \ell - r\ell) = -\frac{k}{1 - \alpha} - \frac{\ell}{(1 - \alpha)^2} - \left( \frac{1}{\alpha} \right)^{[\frac{j}{\pi}]} \left\{ \frac{-j}{1 - \alpha} - \frac{\ell}{(1 - \alpha)^2} \right\} \tag{17} \]

**Proof** The proof follows by taking \( u(k) = k \) in (16) and (12).

**Example 2.11** By taking \( k = 25, \ell = 4, \alpha = 3 \) and \( j = 1 \) in (17), we find

\[
\sum_{r=1}^{[\frac{j}{\pi}]} \left( \frac{1}{3} \right)^r (29 - 4r) = -\frac{25}{1 - 3} - \frac{4}{(1 - 3)^2} - \left( \frac{1}{3} \right)^{[\frac{j}{\pi}]} \left\{ \frac{-1}{1 - 3} - \frac{4}{(1 - 3)^2} \right\}
\]

\[
= 11.500686.
\]

**Lemma 2.12** If \( \ell \in (0, \infty) \), then

\[
(\Delta_{\alpha(\ell)} - \Delta_{-\ell})^{-1} = \Delta_{\frac{1}{\alpha(2\ell)}}^{-1} E^{\ell}. \tag{18} \]

**Proof** From (13), we find

\[
(\Delta_{\frac{1}{\alpha(2\ell)}}^{-1} E^{\ell}) u(k) = E^{\ell} \left\{ \sum_{r=1}^{[\frac{j}{\pi}]} u(k - 2r\ell) \right\}.
\]
But
\[
\Delta_{\alpha(2\ell)}^{-1} E^\ell = (\Delta_{\alpha(2\ell)} E^{-\ell})^{-1}. \tag{19}
\]

The proof follows from (19) and (10).

3. Relation between $\Delta_{\alpha(\ell)}$ and generalized Pochhammer symbols

Definition 3.1 [8] For $n \in \mathbb{N}$, the weighted Pochhammer symbols are defined as
\[
(k)_{n(\pm \ell)} \equiv \prod_{m=0}^{n-1} (k \pm m\ell), \quad (k)_{0(\pm \ell)} \equiv 1, \quad (k)_{-n(\pm \ell)} \equiv 0. \tag{20}
\]

Theorem 3.2 Let $p(k), k \in [0, \infty)$ be a function and $k = j + n\ell$ with $n \in \mathbb{N}$. Then
\[
u(k + 2\ell) - \alpha p(k + \ell) u(k + \ell) = q(k + \ell), k \in [0, \infty) \tag{21}
\]
has a solution
\[
u(j + n\ell + \ell) = \alpha \left[ \frac{j}{n} \right] + 2 u(j) \prod_{i=0}^{n-1} p(j + i\ell),
\]
where $n = \frac{k-j}{\ell}$, which yields, by taking $u(j) = c_j$,
\[
u(k + \ell) = c_j \alpha \left[ \frac{j}{n} \right] [p(j)]_{(n+1)(\ell)}, j \in [0, \ell). \tag{24}
\]

Dividing (21) by $[p(j)]_{(n+1)(\ell)} = \prod_{i=0}^{n} p(j + i\ell)$, we have
\[
\Delta_{\alpha(\ell)} \left\{ \frac{u(k + \ell)}{\prod_{i=0}^{n} p(j + i\ell)} \right\} = \frac{q(k + \ell)}{\prod_{i=0}^{n+1} p(j + i\ell)}. \tag{25}
\]

Using (11) in (25) and applying (13), we get proof.
4. Generalized $\alpha$-Bernoulli Polynomials and its Applications

In this section, by using $s_i^n$, $S_i^n$ and $B_n(k, -\ell)$, the Stirling numbers of first, second kinds and generalized Bernoulli polynomials respectively, we develop a method to find generalized $\alpha$-Bernoulli polynomial which is a solution to the generalized $\alpha$-difference equation

$$u(k - \ell) - \alpha u(k) = (n + 1)k^n,$$

for $\ell \in (0, \infty)$ and $n \in \mathbb{N}(1)$.

**Definition 4.1** Let $B_n(k) = B_n(0) + b_{n-1}k + b_{n-2}k^2 + \ldots + b_1k^{n-1} + k^n$ be $n^{th}$ degree Bernoulli polynomials with Bernoulli numbers $B_n(0)$ for $n \in \mathbb{N}(0)$. The generalized Bernoulli polynomials in $k$ for $-\ell$ are defined as

$$B_n(k, -\ell) = B_n(0)(-\ell)^n + b_{n-1}(-\ell)^{n-2}k + \ldots + b_1k^{n-1} + k^n(-\ell)^{-1}. \quad (26)$$

**Theorem 4.2** For $n \in \mathbb{N}(1)$, the generalized $\alpha$-Bernoulli polynomial $B_{\alpha(n+1)}(k, -\ell)$ can be expressed as

$$B_{\alpha(n+1)}(k, -\ell) = B_{n+1}(k, -\ell) + \left[\frac{\ell}{\alpha}\right] \sum_{r=1}^{\left[\frac{\ell}{\alpha}\right]} \left(\alpha^r - \alpha^{r-1}\right) B_{n+1}(k - r\ell, -\ell) \quad (27)$$

which is a solution to the generalized $\alpha$-difference equation

$$u(k - \ell) - \alpha u(k) = (n + 1)k^n, \quad (28)$$

where $B_n(0)$ is obtained by the recurrence relations

$$B_0(0) = 1; B_n(0) = -\frac{1}{n+1} \sum_{i=0}^{n-1} (n+1)C_iB_i(0). \quad (29)$$

**Proof** From (13), we have

$$\Delta_{\alpha(\ell)}^{-1}(n+1)k^n = \Delta_\ell^{-1}(n+1)k^n + \left[\frac{\ell}{\alpha}\right] \sum_{r=1}^{\left[\frac{\ell}{\alpha}\right]} (\alpha^r - \alpha^{r-1}) \Delta_\ell^{-1}E^{-r\ell}(n+1)k^n.$$

But $B_{n+1}(k, -\ell)$ is a solution to the generalized difference equation $u(k - \ell) - u(k) = (n + 1)k^n$. Then, we obtain

$$B_{\alpha(n+1)}(k, -\ell) = B_{n+1}(k, -\ell) + \left[\frac{\ell}{\alpha}\right] \sum_{r=1}^{\left[\frac{\ell}{\alpha}\right]} (\alpha^r - \alpha^{r-1}) B_{n+1}(k - r\ell, -\ell) \quad (30)$$

which yields (27).

**Theorem 4.3** Let $n \in \mathbb{N}(0)$, $\ell \in (0, \infty)$, $k \in [n\ell, \infty)$. Then

$$\frac{1}{n+1} B_{\alpha(n+1)}(k, -\ell) = -\left[\frac{\ell}{\alpha}\right] \sum_{r=1}^{\left[\frac{\ell}{\alpha}\right]} \left(\frac{1}{\alpha}\right)^r (k + \ell - r\ell)^n - \left(\frac{1}{\alpha}\right)^{\left[\frac{\ell}{\alpha}\right]} c_j. \quad (31)$$
Proof (31) follows by (3), (16) and (26).

Theorem 4.4 If $B_{\alpha(n+1)}(k, \ell)$ is the generalized $\alpha$-Bernoulli polynomial for $\ell$ in $k$ of degree $n+1$, then

$$B_{\alpha(n+1)}(k, -\ell) = (n+1) \left\{ w_n(k) - \left( \frac{1}{\alpha} \right)^{\frac{1}{2}} w_n \left( k - \left\lceil \frac{k}{\ell} \right\rceil \ell \right) \right\},$$

(32)

where

$$w_n(k) = \frac{k^n}{1-\alpha} + \frac{1}{1-\alpha} \sum_{i=1}^{n} nC_i \Delta_{\alpha(-\ell)}^{-1} k^{n-i} \ell^i.$$

Proof The proof follows by the recurrence relation on $w_n(k)$, for $n = 1, 2, ...$ ■

The following example illustrates Theorems 4.3 and 4.4.

Example 4.5 By taking $k = 22$, $n = 1$, $\alpha = 5$ and $\ell = 4$ in (32), we find the value of arithmetico-geometric series

$$\sum_{r=1}^{22} \left( \frac{1}{5} \right)^r (26 - 4r) = \left\{ B_{5(2)}(22, -4) - \left( \frac{1}{5} \right)^5 B_{5(2)}(2, -4) \right\}$$

$$= \left\{ 22 \left( -\frac{1}{5} \right)^5 + 4 \left( -\frac{1}{4} \right)^2 - \left( \frac{1}{5} \right)^5 \left\{ 2 \left( -\frac{1}{4} \right)^2 + 4 \left( -\frac{1}{4} \right)^2 \right\} \right\}$$

$$= 5.24992.$$

References