Positive Radial Solutions of A Singular Quasi-Linear Elliptic Equation with Sign Changing Nonlinearities

Junli Yuan *,† and Zuodong Yang ‡,‡

* School of Science, Nantong University, Jiangsu Nantong 226006, China.
† Institute of Mathematics, School of Mathematics Science, Nanjing Normal University, Jiangsu Nanjing 210046, China.
‡ College of Zhongbei, Nanjing Normal University, Jiangsu Nanjing 210046, China.

(Received: 28 September 2011, Accepted: 21 January 2012)

Existence of positive radial solutions of a class of quasi-linear elliptic equation

\[
\begin{align*}
  -\text{div}(|\nabla u|^{p-2}\nabla u) &= f(|x|,u), \quad x \in B \\
  u > 0, \quad x \in B \\
  u = 0, \quad x \in \partial B
\end{align*}
\]

is obtained. The existence is obtained by constructing upper and lower solutions and employing an approximation procedure. Throughout, our nonlinearity is allowed to change sign. The singularity may occur at \( u = 0 \) and \( |x| = 1 \).

Keywords: Quasi-linear elliptic equation; radial solutions; Existence; Upper and lower solution.

AMS Subject Classification: 35J65, 35B25.

1. Introduction

In this paper, we study the existence of positive radial solutions of a class of quasi-linear elliptic equation

\[
\begin{align*}
  -\text{div}(|\nabla u|^{p-2}\nabla u) &= f(|x|,u), \quad x \in B \\
  u > 0, \quad x \in B \\
  u = 0, \quad x \in \partial B
\end{align*}
\]

where \( N \geq 3, \ 1 < p < N \), \( B \) is the unit open ball centered at the origin in \( \mathbb{R}^N \), i.e., \( B = \{ x \in \mathbb{R}^N : |x| < 1 \} \). The function \( f \) is allowed to change sign. In addition \( f \) may not be a Caratheodory function because of the singular behavior of the \( u \) variable, i.e., \( f \) may be singular at \( u = 0 \). Model examples are

\[
\begin{align*}
  f(|x|,u) &= |x|^{1/u} - (1 - |x|)^{-1} \quad \text{or} \quad f(|x|,u) = \frac{g(|x|)}{u^\sigma} - h(|x|), \quad \sigma > 0
\end{align*}
\]

‡ Corresponding author.
Email address: zdyang_jun@263.net (Z. Yang).
Project Supported by the National Natural Science Foundation of China(Grant No.11171092). Project Supported by the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No.08KJB110005).
where $g(|x|) > 0$ for $x \in B$ and $h(|x|)$ may change sign.

Equations of the above form are mathematical models occurring in studies of the $p$-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory [1, 2], non-Newtonian filtration [3] and the turbulent flow of gas in a porous medium [4]. In the non-Newtonian fluid theory, the quantity $p$ is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudo-plastics. If $p = 2$, they are Newtonian fluids.

For radial solutions the problem (1) can be reduced to the following equivalent problem which involves an ordinary differential equation:

$$
\begin{cases}
- (t^{N-1} |u'|^{p-2}u')' = t^{N-1} f(t, u), & t \in (0, 1), \\
u > 0, & t \in (0, 1), \\
u'(0) = 0, & u(1) = 0.
\end{cases}
$$

(2)

For $p > 1$, the existence, uniqueness and non-existence of positive solutions for the quasilinear elliptic equations with eigenvalue problems

$$
\begin{cases}
\text{div}(|\nabla u|^{p-2} \nabla u) + \lambda f(|x|, u) = 0, & x \in \Omega, \\
u > 0 \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}
$$

(3)

where $\lambda > 0$, $\Omega \subset \mathbb{R}^N$, $N \geq 2$ have been studied by many authors, see [5–15] and the references therein.

When $\Omega$ is an annulus, assume $f \in C(0, \infty)$ and $\lim_{s \to 0^+} f(s) = \infty$. $f$ also satisfies

$$
\int_0^\delta f(s) ds < \infty \quad \text{for any} \quad \delta \in [0, \infty).
$$

It has been shown in [8] for all $\lambda > 0$ and $1 < p < N$ that there exists at least one positive solution for Eqns (3).

When $\Omega = \mathbb{R}^N$, suppose $p = N \geq 2$, $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $|f(r, \cdot)|$ is nondecreasing for all $r \geq 0$ and

$$
\int_c^\infty r^{N-1} |f(r, 3\theta \log r)| dr < \infty,
$$

for some large positive constant $\theta > 0$. Then there exists a constant $\tilde{\lambda} > 0$ such that $0 < \lambda \leq \tilde{\lambda}$, it has been shown in [9] that Eqn.(3) has at least one positive entire solution satisfying $\theta \leq u(|x|) \leq 3\theta \log |x|, \quad |x| \geq e$.

On the other hand, it was shown in [9] that problem

$$
\begin{cases}
\text{div}(|\nabla u|^{p-2} \nabla u) + \lambda f(u) = 0, & x \in B_R(0), \\
u = 0, & x \in \partial B_R(0),
\end{cases}
$$

has at least two positive radially symmetric solutions when $f \in C^1(R)$ and $f(u) > 0$ for $u \geq 0$, and satisfies $\lim_{u \to \infty} (f(u)/u^\theta) = \infty$, where $\theta > p - 1$.

Motivated by the results of the above cited papers, we further study the existence of positive radial solutions for (1), the results of the semilinear equations are extended to the quasi-linear ones. We can find the related results for $p = 2$ in [16–20]. The main differences between $p = 2$ and $p \neq 2$ are known in [6, 7]. When $p = 2$, it is well known that all the positive solutions in $C^2(B_R)$ of the problem

$$
\begin{cases}
\Delta u + f(u) = 0 \quad \text{in} \quad B_R \\
u(x) = 0 \quad \text{on} \quad \partial B_R
\end{cases}
$$

are unique and non-existence for Eqns (3).
are radially symmetric solutions for very general \( f \) (see [21]). Unfortunately, this result does not apply to the case \( p \neq 2 \). Kichenassamy and Smoller showed that there exist many positive nonradial solutions of the above problem for some \( f \) (see [12]). The major stumbling block in the case of \( p \neq 2 \) is that certain nice features inherent to the case \( p = 2 \) seem to be lost or at least difficult to verify. The following results obtained complement corresponding results in [8–11], and extended to results in [20].

By a modification of the method given in [8, 9, 11, 20], we obtain the following results:

Theorem 1.1 Assume the following conditions:

1. There exists a constant \( L > 0 \) such that for any compact set \( e \subset [0, 1) \), there is a \( \varepsilon = \varepsilon_e > 0 \) with \( f(t, u) > L \) for \( t \in e, \ u \in (0, \varepsilon] \),
2. for any \( \delta > 0 \) there is an \( h_\delta \in L^1(0, 1) \) such that \( |f(t, u)| \leq h_\delta(t) \) for \( t \in [0, 1) \) and \( u \geq \delta \).

Then problem (1) has at least one positive radial solution \( u \in C[0, 1] \cap C^1[0, 1) \).

The organization of this paper is as follows. In section 2, we prove Theorem 2.4 relative to upper and lower solutions. In section 3, we firstly construct upper and lower solutions, then using approximation procedure to prove our main result Theorem 1.1.

2. Upper and Lower Solutions

Consider the boundary value problem

\[
\begin{cases}
-(t^{N-1}\Phi_p(u'))' = t^{N-1}f(t, u), & t \in (0, 1), \\
u'(0) = 0, & u(1) = a \geq 0.
\end{cases}
\]

where \( \Phi_p(s) = |s|^{p-2}s, \ f : [0, 1) \times (0, \infty) \to \mathbb{R} \) is a continuous function.

Definition 2.1 \( \alpha(t), \beta(t) \) are called lower and upper solutions relative to (4). If \( \alpha(t), \beta(t) \in C[0, 1] \cap C^1[0, 1), \Phi_p(\alpha') \in C^1(0, 1), \Phi_p(\beta') \in C^1(0, 1) \) such that

\[
\begin{cases}
-(t^{N-2}\Phi_p(\alpha'))' \leq t^{N-1}f(t, \alpha), & t \in (0, 1), \\
\alpha'(0) \leq 0, & \alpha(1) \leq a.
\end{cases}
\]

\[
\begin{cases}
-(t^{N-1}\Phi_p(\beta'))' \geq t^{N-1}f(t, \beta), & t \in (0, 1), \\
\beta'(0) \geq 0, & \beta(1) \geq a.
\end{cases}
\]

Let \( E = \{(t, u) | 0 \leq t < 1, \ \alpha(t) \leq u \leq \beta(t), \ \alpha(t), \beta(t) \in C[0, 1] \cap C^1[0, 1]\} \).

Lemma 2.2 Suppose that \( f : [0, 1) \times \mathbb{R} \to \mathbb{R} \) is continuous and there is a function \( h \in C([0, 1), (0, \infty)) \) such that \( |f(t, u)| \leq h(t) \) for \( (t, u) \in [0, 1) \times \mathbb{R} \) and \( h \in L^1(0, 1) \). Then, boundary value problem (4) has a solution.

Proof Solving (4) is equivalent to finding a \( u \in C[0, 1] \) which satisfies

\[
u(t) = a - \int_t^1 \Phi_p^{-1}(-\int_0^s z^{N-1}s^{1-N}f(z, u)dz)ds
\]

where \( \Phi_p^{-1} \) is an inverse of \( \Phi_p \).

1. When \( 1 < p \leq 2 \), define the operator \( \Psi : C[0, 1] \to C[0, 1] \) by

\[
\Psi u = a - \int_t^1 \Phi_p^{-1}(-\int_0^s z^{N-1}s^{1-N}f(z, u)dz)ds.
\]
Then, solving (4) is equivalent to finding \( u \in C[0,1] \) which satisfies \( \Psi u = u \). We claim that \( \Psi : C[0,1] \to C[0,1] \) is continuous. Let \( u_n \to u \) uniformly on \([0,1]\). We need to show that \( \Psi u_n \to \Psi u \) uniformly on \([0,1]\). We have

\[
\Psi u_n - \Psi u = \int_t^1 \Phi_p^{-1}(-\int_0^s z^{N-1}s^{1-N} f(z,u)dz)ds - \Phi_p^{-1}(-\int_0^s z^{N-1}s^{1-N} f(z,u_n)dz)ds,
\]

Pay attention to \( \Phi_p^{-1} \) is continuous, then \( \Psi : C[0,1] \to C[0,1] \) is continuous.

Let \( U = \{ u \in C[0,1], |u| \leq |a| + M^{1/(p-1)} + 1 \} \) where \( M := \int_0^1 h(z)dz \). In the following we prove \( \Psi \) is a mapping from \( U \) to \( U \).

\[
|\Psi u| \leq |a| + \int_0^1 |\Phi_p^{-1}(-\int_0^s z^{N-1}s^{1-N} f(z,u)dz)ds| \leq |a| + \int_0^1 (\int_0^1 h(z)dz)^{1/(p-1)}ds
\]

\[
\leq |a| + M^{1/(p-1)} + 1
\]

We next show the equicontinuity of \( \Psi(U) \) on \([0,1]\).

\[
|(\Psi u)'(t)| = |\Phi_p^{-1}(-\int_0^t z^{N-1}t^{1-N} f(z,u)dz)| = (\int_0^t z^{N-1}t^{1-N} f(z,u)dz)^{1/(p-1)}
\]

\[
\leq (\int_0^1 h(z)dz)^{1/(p-1)} = M^{1/(p-1)}.
\]

Thus the Arzela-Ascoli theorem implies that \( \Psi(U) \) is relatively compact. By Schauder-Tychonoff fixed point theorem we conclude that \( \Psi \) has a fixed point \( u \) in \( C[0,1] \).

(2) \( p > 2 \), since \( \Phi_p^{-1} \) does not belong to \( C^1(R) \), thus we consider the equation:

\[
\begin{aligned}
(t^{N-1}g_\varepsilon(u'))' &= -t^{N-1}f(t,u), \quad t \in (0,1), \\
\varepsilon u'(0) &= 0, \quad \varepsilon u(1) = a \geq 0,
\end{aligned}
\]

where \( g_\varepsilon(u) = \varepsilon u + \Phi_p(u) \). The same ideas as in (1). Define \( \Psi : C[0,1] \to C[0,1] \)

\[
\Psi u = a - \int_t^1 g_\varepsilon^{-1}(-\int_0^s z^{N-1}s^{1-N} f(z,u)dz)ds.
\]

We can prove that \( \Psi \) has a fixed point for all \( \varepsilon > 0 \). Then the problem (5) has at least one solution \( u_\varepsilon \in C^1 \) for all \( \varepsilon > 0 \) and

\[
|u_\varepsilon|_1 = \max\{|u_\varepsilon|_0, |u_\varepsilon'|_0\} \leq M^* \quad \text{(here } M^* \text{ is independent of } \varepsilon).\]

Thus

\[
(g_\varepsilon(u_\varepsilon')(t))' \leq M \quad \text{(here } M \text{ is independent of } \varepsilon).\]

From the Arzela-Ascoli theorem implies that \( g_\varepsilon(u_\varepsilon') \to v \), as \( \varepsilon \to 0 \), here \( v \in C[0,1] \). The boundedness of \(|u_\varepsilon'|_0\) implies that \( \Phi_p(u_\varepsilon') \to v \) as \( \varepsilon \to 0 \). As \( \Phi_p : R \to R \) is strictly increasing and continuous, we have \( u_\varepsilon' \to \Phi_p^{-1}(v) \) as \( \varepsilon \to 0 \). Then we obtain

\[
u_\varepsilon = a - \int_t^1 u_\varepsilon'(s)ds \to a - \int_t^1 \Phi_p^{-1}(v)ds =: u \quad \text{(} \varepsilon \to 0).\]
From (5), \( u_\varepsilon \) satisfies
\[
\begin{align*}
\varepsilon t^{N-1}u'_\varepsilon + t^{N-1}\Phi_p(u'_\varepsilon) &= -\int_0^t s^{N-1}f(s,u)ds, \\
u'_\varepsilon(0) &= 0, \quad u_\varepsilon(1) = a \geq 0,
\end{align*}
\]
(6)

Let \( \varepsilon \to 0 \) for (6), then we obtain that \( u \in C^1[0,1] \) is a solution of (4).

Lemma 2.3 (Weak comparison principle) Let \( u_1, u_2 \in C[r,s] \cap C^1(r,s) \) satisfy
\[
\int_r^s t^{N-1}|u'_1|^{p-2}u'_1\psi'dt \leq \int_r^s t^{N-1}|u'_2|^{p-2}u'_2\psi'dt
\]
for all non-negative \( \psi \in C_0^\infty \), that is, \( -(t^{N-1}|u'_1|^{p-2}u'_1)' \leq -(t^{N-1}|u'_2|^{p-2}u'_2)' \) in \( (r,s) \), in the weak sense. Then, the inequalities \( u_1(r) \leq u_2(r), u_1(s) \leq u_2(s) \) imply that \( u_1 \leq u_2 \) in \( (r,s) \).

Theorem 2.4 Let \( \alpha, \beta \) be, respectively, a lower solution and an upper solution for problem (4) such that:

(a1) \( \alpha(t) \leq \beta(t) \) for \( t \in [0,1] \) and suppose that

(a2) \( E \subset [0,1) \times (0,\infty) \).

Assume also that there is a function \( h \in C([0,1),(0,\infty)) \) such that:

(a3) \( |f(t,u)| \leq h(t) \) for \( (t,u) \in E \) and

(a4) \( h \in L^1(0,1) \).

Then problem (5) has at least one solution \( u \in C[0,1] \cap C^1[0,1] \) such that \( \alpha(t) \leq u(t) \leq \beta(t) \) for \( t \in [0,1] \).

Proof Define function
\[
F(t, u) = \begin{cases}
    f(t, \beta(t)) + (u - \beta(t))/(1 + u - \beta(t)), & u \geq \beta(t), \\
    f(t, u), & \alpha(t) \leq u(t) \leq \beta(t), \\
    f(t, \alpha(t)) + (\alpha(t) - u)/(1 + \alpha(t) - u), & u \leq \alpha(t).
\end{cases}
\]

By (a2), \( F(t, u) \) is continuous in \([0,1) \times \mathbb{R}\), and noting that \( |F(t,u)| \leq h(t) + 1 \) for \((t,u) \in [0,1) \times \mathbb{R}\). Combining the condition (a4) and the Lemma 2.2, there exists at least one \( u \in C[0,1] \cap C^1[0,1] \) which satisfies
\[
\begin{align*}
-\Phi_p(u'(t)) &= t^{N-1}F(t,u), \quad t \in (0,1), \\
u'(0) &= 0, \quad u(1) = a \geq 0.
\end{align*}
\]
(7)

If we can prove \( u(t) \) satisfies \( \alpha(t) \leq u(t) \leq \beta(t) \), therefore \( u(t) \) is a solution of (4). Thus theorem obtained proved.

Indeed, suppose the first inequality is not true. Then there exists a \( t^* \in [0,1) \) with \( u(t^*) < \alpha(t^*) \). There are two cases to consider, namely \( u(0) < \alpha(0) \) and \( u(0) \geq \alpha(0) \).

Case 1. \( u(0) < \alpha(0) \). Then by \( u(1) \geq \alpha(1) \), there exists a \( t_1 \in (0,1) \) such that
\[
\begin{align*}
    u(t) &< \alpha(t), \quad t \in [0,t_1), \\
    u(t_1) &= \alpha(t_1), \\
    u'(t_1) &\geq \alpha'(t_1).
\end{align*}
\]
(8)
Because for \( t \in (0, t_1) \), we have
\[
\frac{1}{t^{N-1}}(t^{N-1}\Phi_p(u'))' - \frac{1}{t^{N-1}}(t^{N-1}\Phi_p(\alpha'))' \leq -F(t, u) + f(t, \alpha) \\
= -(\alpha(t) - u)/(1 + \alpha(t) - u).
\]
That is
\[
(t^{N-1}\Phi_p(u'))' + t^{N+1}(\alpha(t) - u)/(1 + \alpha(t) - u) \leq (t^{N-1}\Phi_p(\alpha'))' \quad t \in (0, t_1),
\tag{9}
\]
Integrating (9) from 0 to \( t_1 \), we have
\[
t_1^{N-1}\Phi_p(u'(t_1)) < t_1^{N-1}\Phi_p(\alpha'(t_1)).
\]
That is \( u'(t_1) < \alpha'(t_1) \). This is a contradiction with (8).

**Case 2.** \( u(0) \geq \alpha(0) \). Then by \( u(1) \geq \alpha(1) \), there exists a maximal open interval \( (r, s) \subset [0, 1] \) such that
\[
\begin{cases}
\alpha'(0) \leq 0, \\
\alpha(1) = 0, \\
\alpha(t) > 0, \
t \in (0, 1), \\
\alpha(t) \leq \epsilon_1, \
t \in [e_n \setminus e_{n-1}], \
\end{cases}
\tag{10}
\]
For \( t \in (r, s) \), we have
\[
-\frac{1}{t^{N-1}}(t^{N-1}\Phi_p(u'))' = f(t, \alpha(t)) + (\alpha(t) - u)/(1 + \alpha(t) - u).
\]
On the other hand, as \( \alpha \) is a lower solution for (4), we also have
\[
-\frac{1}{t^{N-1}}(t^{N-1}\Phi_p(\alpha'))' \leq f(t, \alpha(t)).
\]
That is
\[
-(t^{N-1}\Phi_p(\alpha'))' \leq -(t^{N-1}\Phi_p(u'))'.
\]
Combining (10) and Lemma 2.3, for \( t \in (r, s) \) we have \( \alpha(t) \leq u(t) \), a contradiction with (10). ■

3. **Proof of The Main Theorem**

**Proof** Let for any \( n \in N, n \geq 1 \) and \( e_n \) be the compact subinterval of \([0, 1]\) defined by
\[
e_n := [0, 1 - \frac{1}{2^n+1}],
\]
By assumption (1), there is \( \epsilon_n > 0 \) such that \( f(t, u) > L, (t, u) \in e_n \times (0, \epsilon_n] \). Without loss of generality (taking if we need a smaller \( \epsilon_n \)), we can assume that \( \{\epsilon_n\} \) is a decreasing sequence and \( \lim_{n \to +\infty} \epsilon_n = 0 \).
We can choose a function \( \alpha \in C^1[0, 1] \) and \( t^{n-1}\Phi_p(\alpha') \in C^1[0, 1] \) such that
\[
\begin{cases}
\alpha'(0) \leq 0, \\
\alpha(1) = 0, \\
\alpha(t) > 0, \
t \in (0, 1), \\
\alpha(t) \leq \epsilon_1, \
t \in [e_n \setminus e_{n-1}], \\
n \geq 2.
\end{cases}
\tag{11}
\]
To show how a $C^1$-function $\alpha$ with these properties can be constructed, consider first the step function $\gamma : [0, 1] \to (0, \infty)$ given by

$$
\gamma(t) = \begin{cases} 
\varepsilon_1, & t \in \varepsilon_1, \\
\varepsilon_n, & t \in \varepsilon_n \setminus \varepsilon_{n-1}, \ n \geq 2, \\
0, & t = 1.
\end{cases}
$$

Since $\gamma$ is nonincreasing, we obtain that $\gamma(t) = \int_1^t \gamma(s) \, ds \leq \gamma(t)$, $t \in [0, 1]$, and, moreover, $\gamma$ is continuous and decreasing. Repeating this argument two further times, we find a strictly convex $C^2$-function

$$
\gamma_3(t) = \int_1^t (\int_s^1 \gamma_1(\tau) \, d\tau) \, ds \leq \gamma(t) \quad t \in [0, 1].
$$

Now we can define $\alpha$ as a $C^1$-function with $\alpha(t) = \gamma_3(t)$ for $t \in [0, 1]$. Then $\alpha$ satisfies the assumption of (11) and $f(t, u) \geq L$ for all $(t, u) \in [0, 1) \times \{0 < u \leq \alpha(t)\}$. Set

$$
k_0 := \min\{1, (L/(|m|_{\infty} + 1))^{1/(p-1)}\},
$$

where $m(t) = \frac{1}{t^{N-1}}(t^{N-1}P_p(\alpha'))'$.

Now we give some claims which yield the proof of the theorem:

**Claim 1.** Let $h(t, u) \geq f(t, u)$ for $(t, u) \in [0, 1) \times (0, \infty)$ with $h : [0, 1) \times (0, \infty) \to (0, \infty)$ a continuous function and let $v \in C[0, 1] \cap C^1[0, 1)$, $v(t) > 0$ for $t \in (0, 1)$, be any solution of

$$
\begin{cases}
-\frac{1}{t^{N-1}}(t^{N-1}P_p(v'))' = h(t, v), & t \in (0, 1), \\
v'(0) \geq 0, & v(1) > 0.
\end{cases}
$$

Then

$$
v(t) \geq k_0 \alpha(t) \quad t \in [0, 1]. \quad (12)
$$

**Proof** Suppose (12) is not true. There are two cases to consider, namely $v(0) < k_0 \alpha(0)$ and $v(0) \geq k_0 \alpha(0)$.

**Case 1.** $v(0) < k_0 \alpha(0)$. Then by $v(1) > k_0 \alpha(1) = 0$, there exists a $t_1 \in (0, 1)$ such that

$$
\begin{cases}
v(t) < k_0 \alpha(t), & t \in [0, t_1), \\
v(t_1) = k_0 \alpha(t_1), \\
v'(t_1) \geq k_0 \alpha'(t_1).
\end{cases}
$$

Then for $t \in (0, t_1)$, we have

$$
-\frac{1}{t^{N-1}}(t^{N-1}P_p(v'))' = h(t, v) \geq f(t, v) \geq L \geq k_0^{p-1}(|m|_{\infty} + 1) > -\frac{1}{t^{N-1}}(t^{N-1}P_p(k_0 \alpha'))'.
$$

So when $t \in (0, t_1)$,

$$
(t^{N-1}P_p(v'))' < (t^{N-1}P_p(k_0 \alpha'))'.
$$

Integrating (13) from 0 to $t_1$, we have $\Phi_p(v'(t_1)) < \Phi_p(k_0 \alpha'(t_1))$. Thus $v'(t_1) < k_0 \alpha'(t_1)$. This is a contradiction.
**Case 2.** \( v(0) \geq k_0 \alpha(0) \). Then by \( v(1) \geq k_0 \alpha(1) \), there exists a maximal open interval \((r, s) \subset [0, 1]\) such that

\[
\begin{align*}
\{ v(r) &= k_0 \alpha(r), \ v(s) = k_0 \alpha(s), \\
v(t) &< k_0 \alpha(t), \ t \in (r, s).
\end{align*}
\tag{14}
\]

For \( t \in (r, s) \), in the similar way as Case 1 we have

\[-(t^{N-1} \Phi_p(v'))' > -(t^{N-1} \Phi_p(k_0 \alpha'))'.\]

Combining (14), by Lemma 2.3, for \( t \in (r, s) \) we have \( v(t) > k_0 \alpha(t) \). This is a contradiction. So we have \( v(t) \geq k_0 \alpha(t), \ t \in [0, 1] \). We define now, for each \( n \in \mathbb{N}, \ n \geq 1 \),

\[
\eta_n(t) := \min\{t, 1 - \frac{1}{2^{n+1}}\}, \ t \in [0, 1]
\]

and set

\[
\tilde{f}_n(t, u) := \max\{f(\eta_n(t), u), f(t, u)\}.
\]

We have that, for each index \( n \), \( \tilde{f}_n : [0, 1) \times (0, \infty) \to (-\infty, +\infty) \) is continuous and

\[
\begin{align*}
\tilde{f}_n(t, u) &\geq f(t, u), \ (t, u) \in [0, 1) \times (0, \infty), \\
\tilde{f}_n(t, u) &\neq f(t, u), \ (t, u) \in \epsilon_n \times (0, \infty).
\end{align*}
\]

Hence the sequence of functions \( \{\tilde{f}_n\} \) converges to \( f \) uniformly on any set of the form \( K \times (0, 1) \), where \( K \) is an arbitrary compact subset of \( [0, 1) \).

Next we define, by induction,

\[
\begin{align*}
f_1(t, u) &:= \tilde{f}_1(t, u), \\
f_2(t, u) &:= \min\{f_1(t, u), \tilde{f}_2(t, u)\}, \\
&\vdots \\
f_{n+1}(t, u) &:= \min\{f_n(t, u), \tilde{f}_{n+1}(t, u)\}, \\
&\vdots
\end{align*}
\]

Each of the \( f_n \) is a continuous function defined on \( [0, 1) \times (0, \infty) \); moreover,

\[
f_1(t, u) \geq f_2(t, u) \geq \cdots \geq f_{n}(t, u) \geq f_{n+1}(t, u) \geq \cdots \geq f(t, u),
\]

and the sequence \( \{f_n\} \) converges to \( f \) uniformly on the compact subsets of \( [0, 1) \times (0, \infty) \). We also note that \( f_n(t, u) = f(t, u), (t, u) \in \epsilon_n \times (0, \infty) \). We consider now the sequence of boundary value problems

\[
\begin{align*}
\left\{ \begin{array}{ll}
-t^{N-1} \Phi_p(u')' = f_n(t, u), & t \in (0, 1), \\
\alpha_n(t) & \equiv c \text{ is a (strict) lower solution for problem (15)n.}
\end{array} \right.
\tag{15}_n
\]

**Claim 2.** For any \( c \in (0, \epsilon_n) \), the constant function \( \alpha_n(t) \equiv c \) is a (strict) lower solution for problem (15)_n.

**Claim 3.** Any solution \( u_n(t) \) of (15)_n is an upper solution for (15)_{n+1}. 

Claim 4. Problem (15) has at least one solution.

Remark 3.1 The proof of the above three Claims is similar to [20]. So we omit it.

By Claim 2 and proceeding by induction using Claim 3, we obtain (via Lemma 2.2) a sequence \( \{u_n(t)\} \) of solutions to (15) such that

\[
\begin{cases}
\varepsilon_n \leq u_n(t) \leq u_{n-1}(t), & t \in [0, 1], \\
k_0\alpha(t) \leq u_n(t), & t \in [0, 1], \\
u_n(0) = 0, \\
u_n(1) = \varepsilon_n.
\end{cases}
\]

We see that the series of functions \( \{u_j(t)\}_{j=1}^{\infty} \) converges pointwise on [0, 1]. Let

\[ u(t) = \lim_{n \to \infty} u_n(t). \]

It is clear that, for any \( n \geq 1 \),

\[ k_0\alpha(t) \leq u(t) \leq u_n(t) \quad t \in [0, 1]. \tag{16} \]

Now let \( K \subset (0, 1) \) be a compact interval.

There is an index \( n^* = n^*(K) \) such that \( K \subset e_n \) for all \( n \geq n^* \) and therefore, for these \( n \geq n^* \) and \( t \in K \),

\[ -\frac{1}{t^{N-1}}(t^{N-1}\Phi_p(u_n'))' = f_n(t, u_n) = f(t, u_n). \]

Hence, the function \( u_n \) is a solution of the equation in (2) for all \( t \in K \) and \( n \geq n^* \). Moreover,

\[ \sup\{|f(t, u)| : \quad t \in K, \quad k_0\alpha(t) \leq u \leq u_n(t)\} < \infty. \]

Thus, by the Ascoli-Arzela theorem it is standard to conclude that \( u \) is a solution of (2) on interval \( K \). Since \( K \) is arbitrary, we find that \( u \in C^1(0, 1) \) and

\[ -\frac{1}{t^{N-1}}(t^{N-1}\Phi_p(u'))' = f_n(t, u), \quad t \in (0, 1). \]

Moreover, \( u(1) = \lim_{n \to \infty} \varepsilon_n = 0 \). We next check the continuity of \( u \) at \( t = 1 \).

Let \( \varepsilon > 0 \) be given. Take \( n_\varepsilon \) such that \( u_{n_\varepsilon}(1) < \varepsilon \). By the continuity of \( u_{n_\varepsilon}(t) \) in \( t = 1 \), we can find a constant \( \delta = \delta_\varepsilon > 0 \) such that for \( t \in (\delta, 1) \), \( 0 < u_{n_\varepsilon} < \varepsilon \). Hence from (16) for \( t \in (\delta, 1) \) we obtain \( 0 < u(t) < \varepsilon \). We next prove \( u'(0) = 0 \). Let \( K = [0, \frac{1}{2}] \). \( \{u_n'\} \) is uniformly bounded and equicontinuous in \( K \). So \( u_n \to u \), \( u_n' \to u' \) as \( n \to \infty \) and \( u'(0) = 0 \). As in the above proof, we have that \( u' \) is continuous at \( t = 0 \).

Example 3.2 Consider the following boundary value problem:

\[
\begin{cases}
-\left(\frac{|u'|^{p-2}u'}{t} - \frac{N-1}{t}|u'|^{p-2}u'\right) = \frac{1}{u^n} - t, & t \in (0, 1) \\
u > 0, & t \in (0, 1) \\
u'(0) = 0, & u(1) = 0,
\end{cases}
\tag{17}
\]

where \( \alpha > 0 \). Then (17) has at least one positive solution \( u \in C[0, 1] \cap C^1[0, 1] \).
Let $L = 1$. For any compact set $e \subset [0, 1)$, there is a $\varepsilon = \frac{1}{2^{1/\alpha}} > 0$. For $t \in e, u \in (0, \varepsilon]$,

$$\frac{1}{u^\alpha} - t \geq \frac{1}{\varepsilon^\alpha} - t \geq 2 - 1 = L.$$  

On the other hand, for any $\delta > 0$, there is an $h_\delta = \frac{1}{\delta^\alpha} + t$ such that $h_\delta \in L^1(0, 1)$ and $|\frac{1}{u^\alpha} - t| \leq h_\delta(t)$ for $t \in [0, 1)$, $u \geq \delta$. So (1) and (2) of the Theorem 1.1 are satisfied. Then (17) has at least one positive solution $u \in C[0, 1] \cap C^1[0, 1)$.

References