RESEARCH ARTICLE

A Unique Common Fixed Point Theorem for Four Maps With Asymptotic Regularity Condition in Cone Metric Spaces

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In this paper, we obtain a unique common fixed point theorem for four self maps with asymptotic regularity condition and satisfying Ciric type weak contractive condition in cone metric spaces. Our result generalizes and improves some recent results in cone metric spaces.

Keywords: Cone metric; weakly compatible maps; asymptotic regularity; complete space

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1. Introduction and Preliminaries

In 2007, Huang and Zhang [1] introduced the concept of cone metric spaces by using ordered Banach space instead of the set of real numbers as a codomain and established Banach contraction principle. Later several authors proved fixed and common fixed point theorems in cone metric spaces. Some interesting references are [2–17]. Recently, Karapinar, Abdeljawad, Tas and Turkoglu etc. obtained fixed and common fixed point theorems in cone Banach spaces(See [18–25]).

In 2011, Jankovic, Kadelburg and Radevovic [26] has shown that all fixed point results in cone metric spaces obtained recently, in which the underlying cone is assumed to be normal, can be reduced to the corresponding results in metric spaces. They have also shown that when the underlying cone is non-normal and solid this is not possible.

In this paper, we prove a unique common fixed point theorem for four self mappings satisfying Ciric type contraction condition in cone metric spaces, in which the underlying cone is neither regular nor normal, using asymptotically regular condition.

Throughout this paper, let $\mathbb{Z}^+$ be the set of all positive integers.

Definition 1.1 [1] Let $E$ be a real Banach space and $P$ be a subset of $E$. $P$ is called a cone if and only if

1. $P$ is closed, non-empty and $P \neq \{0\}$,
(2) $a, b \in R$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$,
(3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subseteq E$, we define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $x - y \in P$.
We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, $\text{int}P$ denotes the interior of $P$.

The cone $P$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow ||x|| \leq K||y||.$$ 

The least positive number satisfying above is called the normal constant of $P$. There are no normal cones with normal constant $K \leq 1$ (See [27]).

The cone $P$ is called regular if every increasing sequence which is bounded from above is convergent.
That is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n \leq \ldots \leq y,$$

for some $y \in E$, then there is $x \in E$ such that $||x_n - x|| \to 0$ as $n \to \infty$.

Equivalently the cone $P$ is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that every regular cone is a normal but the converse need not be true [27].

A cone $P \subseteq E$ is said to be solid if $\text{int}P \neq \phi$. There are ordered Banach spaces with cone $P$ which is not normal but solid.

Example 1.2 [27] Let $E = C^f_{R_+}[0, 1]$ with $||f|| = ||f||_\infty + ||f'||_\infty$ and $P = \{f \in E / f \geq 0\}$. Then $P$ is a non-normal cone with $\text{int}P \neq \phi$.

In the following we always suppose that $E$ is a Banach space, $P$ is a solid cone in $E$ and $\leq$ is partial ordering with respect to $P$.

Definition 1.3 [1] Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space.

Definition 1.4 [1] Let $(X, d)$ be a cone metric space. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$. If for every $c \in E$ with $0 \ll c$, there is $n_0 \in Z^+$ such that $d(x_n, x) \ll c$ for all $n \geq n_0$, then $\{x_n\}$ is said to be convergent to $x$ and $x$ is called the limit of $\{x_n\}$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

If for every $c \in E$ with $0 \ll c$, there is $n_0 \in Z^+$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$, then $\{x_n\}$ is called Cauchy sequence in $X$. If every Cauchy sequence is convergent in $X$ then $X$ is called a complete cone metric space.

Remark 1.5 Let $E$ be an ordered Banach space with cone $P$. Then

(1) if $u \leq v$ and $v \ll w$ then $u \ll w$,
(2) if $u \ll v$ and $v \ll w$ then $u \ll w$,
(3) if $0 \leq u \ll c$ for each $c \in \text{int}P$, then $u = 0$,
(4) $c \in \text{int}P$ if and only if $[-c, c]$ is a neighborhood of $0$,
(5) if $P$ is a solid cone and if a sequence $\{x_n\}$ is convergent in a cone metric space $(X, d)$, then the limit of $\{x_n\}$ is unique.
We note that (2) holds when $n = 0$, since the pair $(S, T)$ commute at their coincidence points, that is, if $fu = gu$ for some $u \in X$ then $fgu = gfu$.

We now give the following definitions.

Definition 1.7 Let $(X, d)$ be a cone metric space with solid cone $P$ and $f, g, S, T$ be self maps on $X$. The pair $(S, T)$ is said to be asymptotically regular at $x_0 \in X$ with respect to the pair $(f, g)$ if there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $y_{2n+1} = g x_{2n+1} = S x_{2n}$, $y_{2n+2} = f x_{2n+2} = T x_{2n+1}$, $n = 0, 1, 2, \ldots$ for each $c \in \text{int} P$, there exists $n_0 \in \mathbb{Z}^+$ such that $d(y_n, y_{n+1}) \ll c$ for all $n \geq n_0$.

The pair $(S, T)$ is said to be asymptotically regular at $x_0 \in X$ if there exists sequence $\{x_n\}$ in $X$ such that $x_{2n+1} = S x_{2n}$, $x_{2n+2} = T x_{2n+1}$, $n = 0, 1, 2, \ldots$ and for each $c \in \text{int} P$, there exists $n_0 \in \mathbb{Z}^+$ such that $d(x_n, x_{n+1}) \ll c$ for all $n \geq n_0$.

The map $S$ is said to be asymptotically regular at $x_0 \in X$ with respect to $f$ if there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $y_n = S x_n = f x_{n+1}$, $n = 0, 1, 2, \ldots$ and for each $c \in \text{int} P$, there exists $n_0 \in \mathbb{Z}^+$ such that $d(y_n, y_{n+1}) \ll c$ for all $n \geq n_0$.

Now we prove our main theorem.

2. Main Result

Theorem 2.1 Let $(X, d)$ be a cone metric space with solid cone $P$ and let $S, T, f, g : X \to X$ be mappings such that

1. $d(Sx, Ty) \leq M(x, y) - \varphi(M(x, y))$, for all $x, y \in X$, where $\varphi : P \to P$ is continuous, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t \in P - \{0\}$ and

$$M(x, y) \in \{d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)\},$$

2. $S(X) \subseteq g(X), T(X) \subseteq f(X)$,

3. the pair $(S, T)$ is asymptotically regular at some point $x_0 \in X$ with respect to the pair $(f, g)$,

4. either $f(X)$ or $g(X)$ is a complete subspace of $X$ and,

5. the pairs $(S, f)$ and $(T, g)$ are weakly compatible.

Then $S$, $T$, $f$ and $g$ have a unique common fixed point in $X$.

Proof Let $x_0 \in X$. Since $S(X) \subseteq g(X), T(X) \subseteq f(X)$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $y_{2n+1} = g x_{2n+1} = S x_{2n}$, $y_{2n+2} = f x_{2n+2} = T x_{2n+1}$, $n = 0, 1, 2, \ldots$.

Since the pair $(S, T)$ is asymptotically regular with respect to $(f, g)$ at $x_0$, for each $c \in \text{int} P$ there exists $n_0 \in \mathbb{Z}^+$ such that

$$d(y_n, y_{n+1}) \ll c, \quad \text{for all} \quad n \geq n_0. \quad (1)$$

To prove $\{y_n\}$ is a Cauchy sequence it is sufficient to verify that

$$d(y_{2m}, y_{2n+1}) \ll c \quad \text{for all} \quad n \geq m. \quad (2)$$

We note that (2) holds when $n = m$. We assume that (2) holds for some $n = k \geq m$. Thus have

$$d(y_{2m}, y_{2k+1}) \ll c. \quad (3)$$
Now we prove (2) for $n = k + 1$.

d (y_{2m}, y_{2k+3}) \leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2m+1}, y_{2k+2})
\leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (x_{2m}, x_{2k+1})
\leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3})
+ M (x_{2m}, x_{2k+1}) - \varphi (M (x_{2m}, x_{2k+1})).

\tag{4}

We have the following five cases.

(a) If $M (x_{2m}, x_{2k+1}) = d (y_{2m}, y_{2m+1})$, from (4), we have

\[
d (y_{2m}, y_{2k+3}) \leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2m}, y_{2k+1}) - \varphi (d (y_{2m}, y_{2k+1}))
\leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2m}, y_{2k+1})
\leq \frac{c}{3} + \frac{c}{3} + \frac{c}{3} \quad \text{for each } n \geq n_0 = n_0 (c) \text{ from (1), (3)}
= c.
\]

(b) If $M (x_{2m}, x_{2k+1}) = d (y_{2m}, y_{2m+1})$, from (4), we have

\[
d (y_{2m}, y_{2k+3}) \leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2m}, y_{2m+1}) - \varphi (d (y_{2m}, y_{2m+1}))
\leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2m}, y_{2m+1})
\leq \frac{c}{3} + \frac{c}{3} + \frac{c}{3} \quad \text{for each } n \geq n_0 = n_0 (c) \text{ from (1)}
= c.
\]

(c) If $M (x_{2m}, x_{2k+1}) = d (y_{2k+1}, y_{2k+2})$, from (4), we have

\[
d (y_{2m}, y_{2k+3}) \leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2k+1}, y_{2k+2}) - \varphi (d (y_{2k+1}, y_{2k+2}))
\leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2k+1}, y_{2k+2})
\leq \frac{c}{3} + \frac{c}{3} + \frac{c}{3} \quad \text{for each } n \geq n_0 = n_0 (c) \text{ from (1)}
= c.
\]

(d) If $M (x_{2m}, x_{2k+1}) = d (y_{2m}, y_{2k+2})$, from (4), we have

\[
d (y_{2m}, y_{2k+3}) \leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2m}, y_{2k+2}) - \varphi (d (y_{2m}, y_{2k+2}))
\leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2m}, y_{2k+1}) + d (y_{2k+1}, y_{2k+2})
\leq \frac{c}{3} + \frac{c}{3} + \frac{c}{3} + \frac{c}{q} \quad \text{for each } n \geq n_0 = n_0 (c) \text{ from (1), (3)}
= c.
\]

(e) If $M (x_{2m}, x_{2k+1}) = d (y_{2k+1}, y_{2m+1})$, from (4), we have

\[
d (y_{2m}, y_{2k+3}) \leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2k+1}, y_{2m+1}) - \varphi (d (y_{2k+1}, y_{2m+1}))
\leq d (y_{2m}, y_{2m+1}) + d (y_{2k+2}, y_{2k+3}) + d (y_{2k+1}, y_{2m+1}) + d (y_{2m}, y_{2m+1})
\leq \frac{c}{3} + \frac{c}{3} + \frac{c}{3} + \frac{c}{3} \quad \text{for each } n \geq n_0 = n_0 (c) \text{ from (1), (3)}
= c.
\]

From all the above five cases, we conclude that $d (y_{2m}, y_{2k+3}) \ll c$ for each $c \in int P$. Thus (2) holds for $n = k + 1 \geq m$. Hence (2) holds for each $n \geq m$. Therefore $\{y_n\}$ is a Cauchy sequence.
Suppose \( f(X) \) is complete. Since \( \{y_{2n+2}\} \subseteq f(X) \) is a Cauchy sequence in the complete cone metric space \((f(X), d))\), it follows that \( \{y_{2n+2}\} \) converges to some \( v \in f(X) \). There exists \( x \in X \) such that \( v = fx \).

Since \( \{y_n\} \) is Cauchy in \( X \) and \( \{y_{2n+2}\} \to v \), it follows that \( \{y_{2n+1}\} \to v \).

\[
\begin{align*}
d(Sx, v) & \leq d(Sx, y_{2n+2}) + d(y_{2n+2}, v) \\
& = d(Sx, Tx_{2n+1}) + d(y_{2n+2}, v) \\
& \leq M(x, x_{2n+1}) - \varphi(M(x, x_{2n+1})) + d(y_{2n+2}, v). \\
\end{align*}
\]

(5)

\[
M(x, x_{2n+1}) \in \left\{ d(fx, gx_{2n+1}), d(fx, Sx) \right\} \\
\quad + \left\{ d(v, y_{2n+1}), d(v, Sx), d(y_{2n+1}, y_{2n+2}) \right\} \\
= \left\{ d(v, y_{2n+1}), d(v, Sx), d(y_{2n+1}, y_{2n+2}) \right\} \\
\quad + \left\{ d(Sx,v), d(y_{2n+2}, y_{2n+1}, Sx) \right\}.
\]

For infinitely many \( n \), we have the following five cases.

(f) If \( M(x, x_{2n+1}) = d(v, y_{2n+1}) \), then from (5), we have

\[
\begin{align*}
d(Sx, v) & \leq d(v, y_{2n+1}) - \varphi(d(v, y_{2n+1})) + d(y_{2n+2}, v) \\
& \leq d(v, y_{2n+1}) + d(y_{2n+2}, v) \\
& \ll \frac{c}{2} + \frac{c}{2} \quad \text{for each } n \geq n_0 = n_0(c) \\
& = c.
\end{align*}
\]

Hence \( 0 \leq d(Sx, v) \ll c \), for each \( c \in intP \). Therefore \( d(Sx, v) = 0. \)

(g) If \( M(x, x_{2n+1}) = d(Sx, v) \), then from (5), we have

\[
\begin{align*}
d(Sx, v) & \leq d(Sx, v) - \varphi(d(Sx, v)) + d(y_{2n+2}, v) \\
\varphi(d(Sx, v)) & \leq d(y_{2n+2}, v) \ll c \quad \text{for each } n \geq n_0 = n_0(c).
\end{align*}
\]

Hence \( 0 \leq \varphi(d(Sx, v)) \ll c \), for each \( c \in intP \). Therefore \( \varphi(d(Sx, v)) = 0. \) Thus \( d(Sx, v) = 0. \)

(h) If \( M(x, x_{2n+1}) = d(y_{2n+1}, y_{2n+2}) \), then from (5), we have

\[
\begin{align*}
d(Sx, v) & \leq d(y_{2n+1}, y_{2n+2}) - \varphi(d(y_{2n+1}, y_{2n+2})) + d(y_{2n+2}, v) \\
& \leq d(y_{2n+1}, y_{2n+2}) + d(y_{2n+2}, v) \\
& \ll \frac{c}{2} + \frac{c}{2} \quad \text{for each } n \geq n_0 = n_0(c) \\
& = c.
\end{align*}
\]

Hence \( 0 \leq d(Sx, v) \ll c \), for each \( c \in intP \). Therefore \( d(Sx, v) = 0. \)

(i) If \( M(x, x_{2n+1}) = d(v, y_{2n+2}) \), then as in case(f), we have \( d(Sx, v) = 0. \)

(j) If \( M(x, x_{2n+1}) = d(Sx, y_{2n+1}) \), then from (5), we have

\[
\begin{align*}
d(Sx, v) & \leq d(Sx, y_{2n+1}) - \varphi(d(Sx, y_{2n+1})) + d(y_{2n+2}, v) \\
& \leq d(Sx, v) + d(v, y_{2n+1}) - \varphi(d(Sx, y_{2n+1})) + d(y_{2n+2}, v) \\
\varphi(d(Sx, y_{2n+1})) & \leq d(v, y_{2n+1}) + d(y_{2n+2}, v) \\
& \ll \frac{c}{2} + \frac{c}{2} \quad \text{for each } n \geq n_0 = n_0(c) \\
& = c.
\end{align*}
\]

Therefore for each \( c \in intP \), there exists \( n_0 = n_0(c) \in \mathbb{Z}^+ \) such that \( \varphi(d(Sx, y_{2n+1})) \ll c \) for all \( n \geq n_0, \) i.e., \( \lim_{n \to \infty} \varphi(d(Sx, y_{2n+1})) = 0. \) Since \( \varphi \) is continuous, we get \( \lim_{n \to \infty} d(Sx, y_{2n+1}) = \)
0. Hence $Sx = \lim_{n \to \infty} y_{2n+1}$. Therefore $Sx = v$.

From all above five cases, we can conclude that $Sx = v = fx$. Since the pair $(S,f)$ is weakly compatible, we have $fv = Sv$.

\[
d(Sv,v) \leq d(Sv,y_{2n+2}) + d(y_{2n+2},v) \\
= d(Sv,Tx_{2n+1}) + d(y_{2n+2},v) \\
\leq M(v,x_{2n+1}) - \varphi(M(v,x_{2n+1})) + d(y_{2n+2},v) \\
(6)
\]

\[
M(v,x_{2n+1}) \in \left\{ \begin{array}{l} 
\{d(fv,gx_{2n+1}), d(fv, Sv), d(gx_{2n+1}, Tx_{2n+1}), \\
\quad d(fv, Tx_{2n+1}), d(gx_{2n+1}, Sv) \} \\
\{d(Sv,y_{2n+1}), 0, d(y_{2n+1}, y_{2n+2}) \} \\
\{d(Sv,y_{2n+2}), d(y_{2n+1}, Sv) \} \\
\{d(Sv,y_{2n+1}), 0, d(y_{2n+1}, y_{2n+2}), d(Sv,y_{2n+2}) \} \right. 
\]

We have the following four cases.

(k) If $M(v,x_{2n+1}) = d(Sv,y_{2n+1})$, then from (6), we have

\[
d(Sv,v) \leq d(Sv,y_{2n+1}) - \varphi(d(Sv,y_{2n+1})) + d(y_{2n+2},v) \\
\leq d(Sv,v) + d(v,y_{2n+1}) - \varphi(d(Sv,y_{2n+1})) + d(y_{2n+2},v) \\
\varphi(d(Sv,y_{2n+1})) \leq d(v,y_{2n+1}) + d(y_{2n+2},v) \\
\ll \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{for each } n \geq n_0 = n_0(c) \\
= c.
\]

Therefore for each $c \in intP$, there exists $n_0 = n_0(c) \in Z^+$ such that $\varphi(d(Sv,y_{2n+1})) \ll c$ for all $n \geq n_0$. i.e., $\lim_{n \to \infty} \varphi(d(Sv,y_{2n+1})) = 0$. Since $\varphi$ is continuous, we get $\lim_{n \to \infty} d(Sv,y_{2n+1}) = 0$. Hence $Sv = \lim_{n \to \infty} y_{2n+1}$. Therefore $Sv = v$.

(l) If $M(v,x_{2n+1}) = 0$, then from (6), we have $d(Sv,v) \leq d(y_{2n+2},v) + 0 - \varphi(0) = d(y_{2n+2},v) \ll c$ for all $n \geq n_0$. Hence $0 \leq d(Sv,v) \ll c$, for each $c \in intP$. Therefore $d(Sv,v) = 0$.

(m) If $M(v,x_{2n+1}) = d(y_{2n+1}, y_{2n+2})$, then from (6), we have

\[
d(Sv,v) \leq d(y_{2n+1},y_{2n+2}) - \varphi(d(y_{2n+1},y_{2n+2})) + d(y_{2n+2},v) \\
\leq d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},v) \\
\ll \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{for each } n \geq n_0 = n_0(c) \\
= c.
\]

Hence $0 \leq d(Sv,v) \ll c$, for each $c \in intP$. Therefore $d(Sv,v) = 0$.

(n) If $M(v,x_{2n+1}) = d(Sv,y_{2n+2})$, then as in case (k), we have $Sv = v$. Thus

\[
v = fv = Sv.
(7)
\]

Since $S(X) \subseteq g(X)$, then there exists $u \in X$ such that $v = Sv = gu$.

\[
d(v,Tu) = d(Sv,Tu) \leq M(v,u) - \varphi(M(v,u))
\]
\( M(v,u) \in \left\{ d(fv,gu), d(fv,Sv), d(gu,Tu), d(fv,Tu), d(gu,Sv) \right\} \)
\[ = \{ 0, 0, d(v,Tu), d(v,Tu), 0 \} \]
\[ = \{ 0, d(v,Tu) \}. \]

If \( M(v,u) = 0 \), then \( d(v,Tu) = 0 \) so that \( Tu = v \). If \( M(v,u) = d(v,Tu) \), then \( d(v,Tu) \leq d(v,Tu) - \varphi(d(v,Tu)) \). It follows that \( \varphi(d(v,Tu)) = 0 \). Hence \( d(v,Tu) = 0 \). Thus \( Tu = v = gu \).

Since \((T,g)\) is weakly compatible, we have \( gv = Tv \).

\[ d(v,Tv) = d(Sv,Tv) \leq M(v,v) - \varphi(M(v,v)) \]
\[ M(v,v) \in \{ d(v,Tv), 0, 0, d(v,Tv), d(Tv,v) \} \]
\[ = \{ d(v,Tv), 0 \}. \]

Therefore \( d(v,Tv) \leq d(v,Tv) - \varphi(d(v,Tv)) \). It follows that \( \varphi(d(v,Tv)) = 0 \). Hence \( d(v,Tv) = 0 \). Thus

\[ gv = Tv = v. \] (8)

From (7) and (8), \( v \) is a common fixed point of \( f, g, S \) and \( T \).

Let \( z \) be another common fixed point of \( f, g, S \) and \( T \).

\[ d(v,z) = d(Sv,Tz) \leq M(v,z) - \varphi(M(v,z)). \]
\[ M(v,z) \in \{ d(v,z), 0, 0, d(v,z), d(z,v) \} \]
\[ = \{ d(v,z), 0 \}. \]

As in above, we can show that \( v = z \). Thus \( v \) is the unique common fixed point of \( f, g, S \) and \( T \).

\[ \blacksquare \]

Now we give the following example which illustrates Theorem 2.1.

Example 2.2 Let \( E = C_R^1[0,1] \) with \( \|x\| = \|x\|_\infty + \|x'\|_\infty \) and \( P = \{ x \in E : x \geq 0 \} \). Let \( X = [0,1] \) and \( d(x,y) = |x-y| \psi, \psi(t) = e^t, t \geq 0 \). Let \( f, g, S, T : X \to X \) be defined by \( Sx = \frac{x+3}{4}, Tx = \frac{x^2+3}{4}, fx = \frac{x+1}{2} \) and \( gx = \frac{x^2+1}{2} \) and \( \varphi : P \to P \) by \( \varphi(x) = \frac{x}{2} \). Then

\[ d(Sx,Ty) = \left| \frac{x}{4} - \frac{y^2}{4} \right| \psi = \frac{1}{2} d(fx,gy) = d(fx,gy) - \varphi(d(fx,gy)). \]
Let $x_0 = \frac{1}{2}$. Then

$$y_1 = Sx_0 = \frac{7}{8} = g(\frac{\sqrt{3}}{2}),$$
$$y_2 = T(\frac{\sqrt{3}}{2}) = \frac{15}{16} = f(\frac{7}{8}),$$
$$y_3 = S(\frac{7}{8}) = \frac{31}{32} = g(\frac{\sqrt{15}}{4}),$$
$$y_4 = T(\frac{\sqrt{15}}{4}) = \frac{63}{64} = f(\frac{31}{32}),$$
$$y_5 = S(\frac{31}{32}) = \frac{127}{128} = g(\frac{\sqrt{63}}{8}), \ldots, \text{ etc.}$$

Clearly,

$$y_n = \frac{2^n + 2 - 1}{2^n + 2}, \quad n = 1, 2, 3, \ldots$$

and

$$d(y_n, y_{n+1}) = \left| \frac{1}{2^{n+3}} - \frac{1}{2^{n+4}} \right| \psi = \frac{1}{2^{n+4}} \psi \to 0 \text{ as } n \to \infty.$$

Clearly 1 is the unique common fixed point of $S$, $T$, $f$ and $g$.

Corollary 2.1 Let $(X, d)$ be a complete cone metric space with solid cone $P$ and let $S, T : X \to X$ be mappings such that

1. $d(Sx, Ty) \leq M(x, y) - \varphi(M(x, y))$, for all $x, y \in X$, where $\varphi : P \to P$ is continuous, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t \in P - \{0\}$ and

$$M(x, y) \in \{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\},$$

2. the pair $(S, T)$ is asymptotically regular at some point $x_0 \in X$.

Then $S$ and $T$ have a unique common fixed point in $X$.

Corollary 2.2 Let $(X, d)$ be a cone metric space with solid cone $P$ and let $S, f : X \to X$ be mappings such that

1. $d(Sx, Sy) \leq M(x, y) - \varphi(M(x, y))$, for all $x, y \in X$, where $\varphi : P \to P$ is continuous, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t \in P - \{0\}$ and

$$M(x, y) \in \{d(fx, fy), d(fx, Sx), d(fy, Sy), d(fx, Sy), d(fy, Sx)\},$$

2. $S(X) \subseteq f(X),$
3. $S$ is asymptotically regular at some point $x_0 \in X$ with respect to $f$,
4. $f(X)$ is a complete subspace of $X$ and,
5. the pair $(S, f)$ is weakly compatible.

Then $S$ and $f$ have a unique common fixed point in $X$. 
References


