Some results on pseudomonoids

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Abstract

In this paper we follow the work of López and Masa (The problem 31.10) given in \cite{1}. The purpose of this work is to study the new concepts of pseudomonoids. We also obtain some interesting results.

Keywords: Pseudogroup, pseudomonoid, diffeology space.


1. Preliminaries

In mathematics, a pseudogroup is an extension of the group. But one that grew out of the geometric approach of Sophus Lie, rather than out of abstract algebra (quasigroup, for example). A theory of pseudogroups developed by Élie Cartan in the early 1900’s.

Recall that a transformation group \( G \) on \( X \) is a subgroup in a group \( \text{Hom}(X) \), so each \( g \in G \) is a diffeomorphism of \( X \). A local homeomorphism is a homeomorphism \( f : U \rightarrow V \), where \( U \) and \( V \) are open subsets in \( X \). Certainly, the set of local homeomorphisms is not a group of transformations because the composition is not well-defined. In studying a geometrical structure, it is fruitful to study its, group of automorphisms. Usually, these automorphisms are not globally defined. They therefore do not form a group in the present sense of the word but rather a pseudogroup, or the more general case, pseudomonoid.

In general, pseudogroups were studied as a possible theory of infinite-dimensional Lie groups. The concept of a local Lie group, namely a pseudogroup of functions defined in neighborhoods of the origin of \( E \), is actually closer to Lie’s original concepts of Lie group, in the case where the transformations involved depend on a finite number of parameters, than the contemporary definition via manifolds.

Examples of infinite-dimensional pseudogroups abound, beginning with the pseudogroup of all diffeomorphisms of \( E \). The interest in mainly in sub-pseudogroup of the diffeomorphisms, and therefore with objects that have a Lie algebra analogue of vector fields. The methods proposed by Lie and Cartan for studying these objects have become more practical given the progress of computer algebra. A generalization of the

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Received: 12 January 2015 Accepted: 26 January 2015

http://dx.doi.org/10.20454/jast.2015.901

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notion of transformation group is a pseudogroup. We summarize and develop some basics of pseudomonoids in chapter two. Here, we introduce the notion of pseudogroup, following chapter 3 of W. Thurston [12] and [8, 9].

**Definition 1.1. (Pseudogroup)** Let $X$ be a topological space. Let $\Gamma$ be a local homeomorphisms set $f : U_f \to V_f$ where $U_f, V_f$ are open subsets of $X$. The set $\Gamma$ is called a pseudogroup on $X$ if satisfying the following conditions:

1. The identity belong to $\Gamma$.
2. If $f \in \Gamma$, then $f|_U \in \Gamma$ for any open $U \subseteq U_f$.
3. If $f \in \Gamma$, then $f^{-1} : V_f \to U_f$ is in $\Gamma$.
4. If $f, g \in \Gamma$ and $V_f \subseteq U_g$, then $g \circ f \in \Gamma$.
5. If $f$ is a local homeomorphism of $M$ such that each point in the domain of $f$ has a neighborhood $U$ such that $f|_U \in \Gamma$, then $f \in \Gamma$.

**Example 1.2.** [5, 6] The natural examples of pseudogroups are:

1. Let $G$ be a Lie group acting on a manifold $X$. Then we define the pseudogroup as the set of all pairs $(G|_U, U)$ where $U$ is the set of all open subsets of $X$,
2. Pseudogroup of automorphisms of a tensor field,
3. Pseudogroup of analytical (or symplectic) diffeomorphisms,
4. The pseudogroup $PL$ of piecewise linear homeomorphisms between open subsets of $\mathbb{R}^n$.

Next we are going to define the notion of a pseudomonoid is a collection of transformations which is closed under certain properties.

## 2. Pseudomonoid

We now change our focus from pseudogroup to pseudomonoid. We want to introduce a generalization of pseudogroups. In the new concept, we change the homeomorphism maps to continuous maps. In this case, the elements of $\Gamma$ aren’t necessarily invertible. So part (3) of the Definition 1.1 can’t be defined in the new concept. Thus we define the pseudomonoid following.

**Definition 2.1.** Let $X$ be a topological space and let $\Gamma$ be a set of locally continuous functions $f : U_f \to X$ where $U_f$ is an open subset of $X$. The set $\Gamma$ is called a pseudomonoid on $X$ if satisfying the following conditions:

1. The identity belong to $\Gamma$.
2. If $f \in \Gamma$, then $f|_U \in \Gamma$ for any open $U \subseteq U_f$.
3. If $f, g \in \Gamma$ and $\text{im}(f) \subseteq U_g$, then $g \circ f \in \Gamma$.

The pair $(X, \Gamma)$ is called a pseudomonoid.

**Remark 2.2.** Any pseudogroup on $X$ is a pseudomonoid on $X$ also.

**Example 2.3.** We can construct many examples:

1. For any topology space $X$ all locally continuous functions on $X$ form a pseudomonoid on its. This pseudomonoid is largest pseudomonoid on $X$. In the other words, any pseudomonoid on $X$ is a subset of its.
2. All linear maps on a Banach space $E$ and their restriction to open subsets.
2.1. Basic definitions and notations

Before we define some concepts of pseudomonoid for a topological space, we need to introduce the following definitions. See below for more details. A $C^r$-pseudomonoid, is a pseudomonoid $\Gamma$ on a $C^r$-manifold that $\Gamma$ contains $C^r$-maps (with $0 \leq r \leq \infty$).

We say $S \subset \Gamma$ generates $\Gamma$ and write $\Gamma = \langle S \rangle$, if composition and restriction to open subsets in $S$ obtain $\Gamma$ (expect identity). The orbit of a point $x \in X$ is the set

$$\Gamma(x) := \{f(x) : f \in \Gamma \text{ and } x \in U_f\}.$$  

We will denote by $\tilde{X}$ the set of all orbits of $(X, \Gamma)$. The stability or isotropy of $x$, show by $\Gamma_x$, contain all $f \in \Gamma$ leaving $x$ fixed:

$$\Gamma_x := \{f \in \Gamma : x \in U_f, f(x) = x\}.$$  

A path of $x \in X$ to $y \in X$ is a sequence $x = y_1, \cdots, y_n = y$ of points in $X$ such that $\Gamma(y_i) \cap \Gamma(y_{i+1}) \neq \emptyset$ for all $i = 1, \ldots, n-1$. We will define on $X$ the following equivalence relation: "$xRy$ if and only if there exists a path of $x$ to $y$". The equivalence classes of $R$ are said leaves of $(X, \Gamma)$. We will denote by $L_x$ the leaf contain of $x \in X$. The notation $X/\Gamma$ means the quotient topology of $R$ with quotient topology.

The union of leaves that meet $X_0 \subseteq X$ is called the saturation of $X_0$ and denoted by $\Gamma(X_0)$. We say the set $X_0$ invariant or saturated, if $X_0 = \Gamma(X_0)$. Let $\pi : X \to X/\Gamma$ be the quotient projection. Then $\Gamma(X_0) = \pi^{-1}(\pi(X_0)) = \bigcup_{x \in X_0} L_x$, where $L_x$ shows the leaf of $\Gamma$ that $x \in L_x$.

$\Gamma$ is called transitive if for any $x, y \in X$, there exists $f \in \Gamma$ that $f(x) = y$. A sub-pseudomonoid of $\Gamma$ is a subset $\Sigma \subseteq \Gamma$ which is a pseudomonoid on $X$ also.

**Definition 2.4.** Let $\Gamma_1$ and $\Gamma_2$ are two pseudomonoids on $X_1$ and $X_2$ (resp). The structure $(X_2, \Gamma_2)$ is called a strong sub-pseudomonoid of $(X_1, \Gamma_1)$ if $X_2$ is a subspace of $X_1$, and $\{i \circ g : g \in \Gamma_2\} \subseteq \{f \circ i : f \in \Gamma_1\}$ where $i : X_2 \to X_1$ is inclusion map.

**Example 2.5.** (1) Let $\Gamma$ be the set of all locally continuous maps of $X$. Then any pseudomonoid on $X$ is a sub-pseudomonoid of $\Gamma$.

(2) Let $\Gamma$ be contain of identity and all identity restrictions to open subsets of $X$. Then $\Gamma$ is a sub-pseudomonoid of any pseudomonoid on $X$.

(3) Let $M$ be a smooth manifold and let $C^r(M)$ be all locally $C^r$-maps on $M$. Then $C^{r_1}(M)$ is a sub-pseudomonoid of $C^{r_2}(M)$ where $0 \leq r_2 \leq r_1 \leq \infty$.

(4) $(\mathbb{R}^m, C^{\infty}(\mathbb{R}^m))$ is a strong sub-pseudomonoid of $(\mathbb{R}^n, C^{\infty}(\mathbb{R}^n))$ for all $m \leq n$.

**Lemma 2.6.** Let $\Gamma$ be a pseudomonoid on $X$. If $\Gamma_x$ is isotropy of $x$, then the pseudomonoid is generated by $\Gamma_x$ is the set $\{g|_U : U$ is an open subset of $U_g, x \in U_g, g(x) = x\}$.

**Proof.** $\Lambda$ stands for the set. We have divided the proof into two parts: First, we show that $\Lambda$ is a pseudomonoid, next we will prove that $\Lambda = \langle \Gamma_x \rangle$. Clearly, identity belongs to $\Lambda$ and (2) of pseudomonoid definition is valid by the definition of $\Lambda$. Let $f_1, f_2$ belong to $\Lambda$ and $f_2 \circ f_1$ is definition. Then there are $g_1, g_2 \in \Gamma$ such that $f_i = g_i|_{U_i}, U_i \subset U_g, x \in U_g$ and $g_i(x) = x, i = 1, 2$. Define $g = g_2 \circ g_1$, there is a $U \subset U_g$ that $f_2 \circ f_1 = g|_U$. Thus (3) of pseudomonoid definition is valid. Finally, we conclude that $\Lambda$ is a pseudomonoid. Obviously, $\Gamma_x$ is a subset of $\Lambda$. Since $\Lambda$ is a pseudomonoid and $\Gamma_x$ is a subset of $\Lambda$, we have $\langle \Gamma_x \rangle \subseteq \Lambda$. By definition, $\Lambda \subseteq \langle \Gamma_x \rangle$, So $\Lambda = \langle \Gamma_x \rangle$ and this finishes the proof.

**Lemma 2.7.** Let $S$ be a subset of $\Gamma$ and $S$ generates $\Gamma$. Then

$$\Gamma(x) = \{f_1 \circ \cdots \circ f_n(x) \mid f_i \in S, \ n \in \mathbb{N}, \text{ if the composition is defined on } U_x \} \cup \{x\}$$

for any $x \in X$. 
Proposition 2.8. If $X$ be $T_1$-space and if $\pi$ is a closed map, then any leaf of $\Gamma$ is a closed subset of $X$.

Proof. The singleton $\{x\}$ is a closed set, because $X$ is $T_1$-space. It is evident that $\pi$ is a closed map. Thus $\pi(x)$ is a closed set. From continuously of $\pi$, we conclude $\pi^{-1}(\pi(x)) = L_x$ is a closed set in $X$ which completes the proof.

Proposition 2.9. If $X_0 \subseteq X$ is an invariant subset under $\Gamma$, then $X_0^c = X - X_0$ is invariant under $\Gamma$ too.

Proof. Since $X_0$ is invariant. Consequently, $X_0$ is the union of some leaves of $\Gamma$. Thus $X_0^c$ is union some leaves in $\Gamma$ that do not intersect with $X_0$, therefore $X_0^c$ is invariant.

Example 2.10. In following, we give several examples of pseudomonoids.

1) Consider a pseudomonoid $(X, \Gamma)$ where $X = \mathbb{R}^n$ with $\Gamma = \langle f \rangle$, which $f$ is defined by

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$f(x_1, x_2, \ldots, x_n) \mapsto (\sum_{i=1}^{n} x_i, 0, \ldots, 0).$$

Then the isotropy $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is the set $\Gamma_a = \{id|_U : a \in U \subseteq \mathbb{R}^n\}$ when $a_j \neq 0$ for some $j \geq 2$ and $\Gamma_a = \{g \in \Gamma : a \in U_g\}$ when $a_j = 0$ for all $j \geq 2$. The orbit of $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is $\Gamma(a) = \{a, (\sum_{i=1}^{n} a_i, 0, \ldots, 0)\}$ and the leaf contains $a \in \mathbb{R}^n$ is $L_a = \{(x_1, \ldots, x_n) | \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i\}$. So $X/\Gamma \cong \mathbb{R}$.

2) Let $X = S^n$ and $\Gamma$ is generated by

$$f : S^n \rightarrow S^n$$

$$x \mapsto -x.$$ 

Then the isotropy of $a \in S^n$ is $\Gamma_a = \{id|_U : a \in U \subseteq S^n\}$, and we have $X/\Gamma \cong \mathbb{R}P^n$, because $L_a = \Gamma(a) = \{a, -a\}$.

3) Suppose that $X = \mathbb{R}^n \setminus \{0\}$ and consider the pseudomonoid $\langle f \rangle$ on $X$, where

$$f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$$

$$x \mapsto (\|x\|, 0, \ldots, 0).$$

For any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \{0\}$ the isotropy of its $\Gamma_a = \{g \in \Gamma : a \in U_g\}$ if $a_1 > 0$ and $a_j = 0$ for all $j \geq 2$, other wise $\Gamma_a = \{id|_U : a \in U\}$. For any $a \in \mathbb{R}^n \setminus \{0\}$, we have the orbit of its $\Gamma(a) = \{a, (\|a\|, 0, \ldots, 0)\}$. The leaf contain of $a$ is

$L_a = \{x \in \mathbb{R}^n \setminus \{0\} : \|x\| = \|a\|\}.$

Therefore, we conclude $X/\Gamma \cong \mathbb{R}$.

4) Assume $(X, \Gamma)$ is a pseudomonoid where, $X = \mathbb{R}^n \setminus \{0\}$. The set $\Gamma$ is generated by

$$f : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$$

$$x \mapsto x/\|x\|.$$ 

Then the isotropy set $\Gamma_a = \{g \in \Gamma : a \in U_g\}$ if $\|a\| = 1$, and other wise $\Gamma_a = \{id|_U : a \in U\}$. We have $\Gamma(a) = \{a, \lambda a/\|a\|\}$, $L_a = \{\lambda a : \lambda \in \mathbb{R}^+\}$. Finally, we earn that $X/\Gamma \cong S^{n-1}$.

5) Choose $X = \mathbb{R}$ and $\Gamma = \langle f \rangle$, which $f(x) = x + 1$. It is a simple matter to check that $\Gamma_a = \{id|_U : a \in U\}$, $\Gamma(a) = \{a + n|n \in \mathbb{N} \cup \{0\}\}$ and $L_a = \{a + n|n \in \mathbb{Z}\}$. Therefore $X/\Gamma \cong S^1$. 


(6) Let \( X = \mathbb{R}^n \) and \( \Gamma = \langle f_i, g_i : i = 1, \ldots, n \rangle \), where

\[
  f_i, g_i : \mathbb{R}^n \to \mathbb{R}^n \\
  f_i(x_1, \ldots, x_n) = (x_1, \ldots, x_i + 1, \ldots, x_n), \\
  g_i(x_1, \ldots, x_n) = (x_1, \ldots, x_i + \alpha_i, \ldots, x_n).
\]

Then by using the Lemma 2.7 we have:

\[
  \Gamma_x = \{ \text{id}|_U : x \in U \} \\
  \Gamma(x) = \{(x_1 + m_1 + k_1 \alpha_1, \ldots, x_n + m_n + k_n \alpha_n) : m_i, k_i \in \mathbb{N} \cup \{0\} \} \\
  L_x = \{(x_1 + m_1 + k_1 \alpha_1, \ldots, x_n + m_n + k_n \alpha_n) : m_i, k_i \in \mathbb{Z}\}.
\]

Now, if \( \alpha_i \in \mathbb{Q} \) for all \( i = 1, \ldots, n \), then \( X/\Gamma \cong \mathbb{T}^n(\text{n-torus}) \). If \( \alpha_i \notin \mathbb{Q} \) for some \( i = 1, \ldots, n \), then

\[
  X/\Gamma \cong \mathbb{T}^n_{\alpha_1} \times \cdots \times \mathbb{T}^n_{\alpha_n}.
\]

**Example 2.11.** (a) The \( n \)-th symmetric product of topological space \( Y \) \[13\]. Let \((Y, \ast)\) be a pointed topological space. Suppose that \( X = \Pi_{i=1}^n Y \) and \( \Gamma \) is a pseudomonoid such that it is generated by the maps

\[
  f_{i,j} : X \to X \\
  (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \mapsto (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)
\]

for any \( 1 \leq i, j \leq n \) and \( i \neq j \). Then

\[
  \Gamma((x_1, \ldots, x_n)) = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : \sigma \in S_n \},
\]

where \( S_n \) is the symmetric group. Consequently, we have \( X/\Gamma = SP^n Y \).

(b) The infinite symmetric product of topological space \( Y \) \[13\]. Let \( Y \) be a topological space. Consider the pseudomonoid \((X, \Gamma)\) such that \( X = \Pi_{m \in \mathbb{N}} Y \). Here the set \( \Gamma \) is a pseudomonoid such that it is generated by the following maps on \( X \).

\[
  f_{i,j} : X \to X \\
  (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \mapsto (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)
\]

for all \( i, j \in \mathbb{N} \) and \( i \neq j \). Therefore, we have \( X/\Gamma = SP Y = \text{colim} (SP^n Y) \)

**Lemma 2.12.** Let \( \{\Gamma_i\}_{i \in I} \) is a collection of pseudomonoids on \( X \). Then \( \bigcap_{i \in I} \Gamma_i \) is a pseudomonoid on \( X \) too.

**Proof.** Because any \( \Gamma_i \) contain the identity, therefore the \( \bigcap_{i \in I} \Gamma_i \) contain the identity. Let \( f, g \in \bigcap_{i \in I} \Gamma_i \) that \( im(f) \subseteq U_g \). Then \( f, g \in \Gamma_i \) for any \( i \in I \). Thus \( g \circ f \in \Gamma_i \) for all \( i \in I \). Hence \( g \circ f \in \bigcap_{i \in I} \Gamma_i \). Let \( f \in \bigcap_{i \in I} \Gamma_i \) and \( U \subseteq U_f \). Then \( f \in \Gamma_i \) for any \( i \in I \), thus \( f|_{\overline{U}} \in \Gamma_i \) for any \( i \in I \). Thus \( f|_{\overline{U}} \in \bigcap_{i \in I} \Gamma_i \) and this finishes the proof.

**Proposition 2.13.** Let \( S \) is a family of locally continuous maps on \( X \). Then the intersection all pseudomonoids on \( X \) that contain \( S \) is the pseudomonoid such that it is generated by \( S \).

**Proof.** Suppose \( \Gamma_1 \) is the pseudomonoid is generated by \( S \) and \( \Gamma_2 \) is the intersection all pseudomonoids on \( X \) and contain \( S \). By the Lemma 2.12 the set \( \Gamma_2 \) is a pseudomonoid. Because \( \Gamma_1 \) contain \( S \), thus by to define \( \Gamma_2 \subseteq \Gamma_1 \). From any pseudomonoid contain \( S \) is a superset of \( \Gamma_1 \). We have \( \Gamma_1 \subseteq \Gamma_2 \). Thus, we showed that \( \Gamma_1 = \Gamma_2 \).
Lemma 2.16. If \( \Gamma \) is a pseudomonoid on \( X \) and 
\[
\Gamma|_U = \{ f \in \Gamma : U_f \subseteq U, \ im(f) \subseteq U \}
\]
where \( U \) an open subset of \( X \), then \( \Gamma|_U \) is a pseudomonoid on \( U \).

Proof. It is a simple matter to \( id_U \in \Gamma|_U \). Let \( f \in \Gamma|_U \) and \( U' \subseteq U \subseteq U \), because \( f|_{U'} \) is in \( \Gamma \) and 
\[im(f) \subseteq U\] we have \( f|_{U'} \in \Gamma|_U \). Let \( f, g \in \Gamma|_U \) that \( im(f) \subseteq U_g \). From \( g \circ f \in \Gamma \), \( U_{gof} \subseteq U_f \subseteq U \) and 
\[im(g \circ f) \subseteq im(g) \subseteq U, \] we have \( g \circ f \in \Gamma|_U \).

The restriction of \( \Gamma \) to a subspace \( X_0 \subseteq X, \) \( \Gamma|_{X_0} \), contains continuous maps between open subsets of \( X_0 \) that can be locally extended to maps in \( \Gamma \). If \( X_0 \) is open in \( X \), then \( \Gamma|_{X_0} \) consists of maps in \( \Gamma \) whose domains and images are subsets of \( X_0 \) (the Proposition 2.14).

Lemma 2.15. If \( X_0 \subseteq X \) is a subspace of \( X \), then 
\[
\Gamma|_{X_0}(x) \subseteq \Gamma(x) \cap X_0
\]
for any \( x \in X_0 \).

Proof. Let \( y \in \Gamma|_{X_0}(x) \). Thus there is a \( f \in \Gamma|_{X_0}(x) \) that \( y = f(x) \). It follows that \( f \) can be locally extended to \( f' \in \Gamma \), therefore \( y = f(x) = f'(x) \in \Gamma(x) \). Finally, we get \( \Gamma|_{X_0}(x) \subseteq \Gamma(x) \cap X_0 \).

Lemma 2.16. If \( U \) is an open subset of \( X \), then 
\[
\Gamma|_U(x) = \Gamma(x) \cap U
\]
for any \( x \in U \).

Proof. We can prove \( \Gamma|_{U}(x) \subseteq \Gamma(x) \cap U \) as the Lemma 2.15. Now, assume \( y \in \Gamma(x) \cap U \), there is a continuous map \( f \in \Gamma \) that \( y = f(x) \). Because \( y \) and \( x \) are in \( U \). We can restrict \( f \) to an open subset \( U' \subseteq U \) and contain \( x \), where \( f(U') \subseteq U \). So \( f' = f|_{U'} \) is a member of \( \Gamma|_{U} \). Thus, we have \( y = f'(x) \in \Gamma|_{U}(x) \).

Note 2.17. In the Lemma 2.15 may not be equal (see the following example).

Example 2.18. Assume \( X = \mathbb{R}^n \) and \( \Gamma = \langle f > \) where, 
\[
f : \mathbb{R}^n \rightarrow \mathbb{R}^n
\]
\[(x_1, \ldots, x_n) \mapsto (1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + \ldots + x_n)\]
and choose \( X_0 = \mathbb{R} \times \{0\}^{n-1} \). Simplify, we get 
\[
\Gamma|_{X_0} = \langle id_{X_0} \rangle \Rightarrow \Gamma|_{X_0}(0, \ldots, 0) = \{(0, \ldots, 0)\}.
\]
But \( (1, 0, \ldots, 0) \in \Gamma((0, \ldots, 0)) \). Therefore \( \Gamma((0, \ldots, 0)) \cap X_0 = \{(0, \ldots, 0), (1, 0, \ldots, 0)\} \), which is not equal \( \Gamma|_{X_0}((0, \ldots, 0)) \).

Let \( \Gamma_1, \ldots, \Gamma_n \) are pseudomonoid on \( X_1, \ldots, X_n \), resp. Then the pseudomonoid is generated by \( S = \{f_i \times \cdots \times f_n : f_i \in \Gamma_i \} \) with \( 1 \leq i \leq n \} \) and it is called product of \( \Gamma_1 \times \cdots \times \Gamma_n \) on \( X_1 \times \cdots \times X_n \).

Proposition 2.19. If \( \Gamma = \Pi_{i=1}^n \Gamma_i \) is product pseudomonoid on \( \Pi_{i=1}^n X_i \), then 
\[
\Gamma(x_1, \ldots, x_n) = \Pi_{i=1}^n \Gamma_i(x_i)
\]
for any \((x_1, \ldots, x_n) \in \Pi_{i=1}^n X_i \).
Proof. From the Lemma 2.7, the proposition is proved.

Given \( X = \bigcup_{i \in I} X_i \) that any \( \Gamma_i \) is a pseudomonoid on \( X_i \), for \( i \in I \). The pseudomonoid is generated by \( \bigcup_{i \in I} \Gamma_i \) is called composition of pseudomonoids \( \Gamma_i \) and show by \( \bigcup_{i \in I} \Gamma_i \).

Lemma 2.20. Under the above assumptions, for \( x \in X_j \) where, \( j \in I \) we have
\[
(\bigcup_{i \in I} \Gamma_i)(x) = \Gamma_j(x).
\]

Proof. We can prove from the Lemma 2.7 and the composition definition.

Proposition 2.21. If \( (X, \Gamma) \) is a pseudomonoid and if \( \Gamma' \) is a sub-pseudomonoid of \( \Gamma \) on \( X \), then the following properties are being held:

1. Isotropy: For any \( x \in X \), \( \Gamma_x \cap \Gamma' = \Gamma'_x \).
2. Orbit: \( \Gamma'(x) \subseteq \Gamma(x) \), for every \( x \in X \).
3. Leaves: Suppose that \( L'_x \) and \( L_y \) are the leaves of \( \Gamma' \) contain of \( x \) and \( y \) in \( X \), resp. Then, the following statements are being held:
   - if \( L'_x = L'_y \), then \( L_x = L_y \).
   - \( L'_x \subseteq L_x \).
4. Subspace: If \( X_0 \) is a subspace of \( X \), then \( \Gamma'|_{X_0} \subseteq \Gamma|_{X_0} \). Thus, for any open subset \( U \) of \( X \), \( \Gamma'|_U \subseteq \Gamma|_U \).
5. Product: Let \( \Gamma'_i \) is a sub-pseudomonoid of \( \Gamma_i \) on topological space \( X_i \), for any \( i = 1, \ldots, n \). Then
   \[ \Gamma'_1 \times \cdots \times \Gamma'_n \subseteq \Gamma_1 \times \cdots \times \Gamma_n. \]
6. Composition: If for any \( i \in I \), \( \Gamma'_i \) to be sub-pseudomonoid of \( \Gamma_i \), then \( \bigcup_{i \in I} \Gamma'_i \subseteq \bigcup_{i \in I} \Gamma_i \).

Proof. Simplify, all statements is proved expect (3). For the first part of (3) proof, suppose that \( L'_x = L'_y \), then there is a sequence \( x = x_1, \ldots, x_n = y \) of \( X \) that \( \Gamma'(x_j) \cap \Gamma'(x_{j+1}) \neq \emptyset \) with \( (j = 1, \ldots, n - 1) \). From the (2) we have \( \Gamma(x_j) \cap \Gamma(x_{j+1}) \neq \emptyset \) with \( (j = 1, \ldots, n - 1) \). We thus get \( L_x = L_y \). For proved the second part of (3) we assume \( y \) is a member of \( L'_x \). We have \( L'_x = L'_y \). By the first part, we have \( y \in L_x \). Therefore, here \( L'_x \subseteq L_x \).

Definition 2.22. Let \( \Gamma \) is a pseudomonoid on \( X \). Then, a subset \( \mu \subseteq X \) is called minimal, if it satisfies the following properties:

1. \( \mu \) is nonempty, closed and invariant,
2. if \( \mu' \subseteq \mu \) is a closed, invariant subset, then either \( \mu' = \emptyset \) or \( \mu' = \mu \).

Proposition 2.23. If \( \Gamma \) is a pseudomonoid on Hausdorff compact space \( X \), then there is a minimal subset of \( X \).

Proof. Show \( Z \) the family of nonempty, compact and invariant subsets of \( X \). Clearly \( Z \) is nonempty. Given a sequence \( \mu_1 \supset \mu_2 \supset \cdots \) of members of \( Z \), thus \( \mu = \bigcap_{i=1}^\infty \mu_i \) is nonempty, invariant, compact and \( \mu \subseteq \mu_i \) for every \( i \in \mathbb{N} \). By Zorn’s lemma, \( Z \) have a minimal member, what is a minimal subset of \( X \).

We want to define étalé morphisms by involving arbitrary local maps, precisely, we introduce the following concept.

Definition 2.24. Let \( \Gamma_1 \) and \( \Gamma_2 \) are pseudomonoid on \( X_1 \) and \( X_2 \), respectively. A morphism \( \Phi : (X_1, \Gamma_1) \to (X_2, \Gamma_2) \) is a maximal collection of continuous maps of open subsets of \( X_1 \) to \( X_2 \) satisfying the following properties:

1. If \( \phi \in \Phi \), \( h \in \Gamma_1 \) and \( h_2 \in \Gamma_2 \), then \( h_2 \circ \phi \circ h_1 \in \Phi \).
Proposition 2.25. If \( \Phi : (X_1, \Gamma_1) \to (X_2, \Gamma_2) \) is a morphism, then
\[
\Gamma_2(\psi(x)) = \bigcup_{\phi \in \Phi} \phi(\Gamma_1(x) \cap \text{dom} \phi)
\]
for all \( x \in X \) and for every \( \psi \in \Phi \) that \( x \in \text{dom} \psi \).

**Proof.** Fix \( \psi \in \Phi \). Let \( y \in \Gamma_2(\psi(x)) \) is an arbitrary element. Then for some \( h_2 \in \Gamma_2 \) we have \( y = h_2(\psi(x)) \). Since \( h_2 \circ \psi \) is a member of \( \Phi \), \( y \in \bigcup_{\phi \in \Phi} \phi(\Gamma_1(x) \cap \text{dom} \phi) \). Therefore \( \Gamma_2(\psi(x)) \subseteq \bigcup_{\phi \in \Phi} \phi(\Gamma_1(x) \cap \text{dom} \phi) \).

If \( y \in \bigcup_{\phi \in \Phi} \phi(\Gamma_1(x) \cap \text{dom} \phi) \), then we have \( y \in \phi(\Gamma_1(x) \cap \text{dom} \phi) \) for some \( \phi \in \Phi \). There is a \( h_1 \in \Gamma_1 \) that \( y = \phi(h_1(x)) \). By the property (3) of morphism definition, there is \( h_2 \in \Gamma_2 \) that \( h_2 \circ \psi = \phi \circ h_1 \) on some neighborhood of \( x \), therefore \( y \in \Gamma_2(\psi(x)) \). Thus, \( \bigcup_{\phi \in \Phi} \phi(\Gamma_1(x) \cap \text{dom} \phi) \subseteq \Gamma_2(\psi(x)) \). The proof is now complete.

**Corollary 2.26.** The morphism \( \Phi \) induces a map \( \Phi_{orb} : \tilde{X}_1 \to \tilde{X}_2 \) is defined by
\[
\Phi_{orb}(\Gamma_1(x)) = \Gamma_2(\phi(x)).
\]
for any \( \phi \in \Phi \) that \( x \in \text{dom} \phi \).

**Proof.** Define \( \Phi_{orb}(\Gamma_1(x)) = \bigcup_{\phi \in \Phi} \phi(\Gamma_1(x) \cap \text{dom} \phi) \).

**Proposition 2.27.** Let \( \Phi : (X_1, \Gamma_1) \to (X_2, \Gamma_2) \) is a morphism. Then \( \Phi \) induces a continuous map from \( X_1/\Gamma_1 \) into \( X_2/\Gamma_2 \) that be defined by:
\[
\overline{\Phi} : X_1/\Gamma_1 \to X_2/\Gamma_2
\]
\[
L_x \mapsto L'_{\phi(x)}
\]
such that \( \phi \in \Phi \), \( x \in \text{dom} \phi \), and \( L \) and \( L' \) are leaves in \( \Gamma \) and \( \Gamma' \), resp.

**Proof.** We first show that, if \( \Gamma_1(x) \cap \Gamma_2(y) \neq \emptyset \), then \( \Gamma_2(\psi(x)) \cap \Gamma_2(\psi'(y)) \neq \emptyset \) for any \( \psi, \psi' \in \Phi \) where, \( x \in \text{dom} \psi \) and \( y \in \text{dom} \psi' \). Let \( t \in \Gamma_1(x) \cap \Gamma_2(y) \), then there are \( h_1 \) and \( h_1' \) in \( \Gamma_1 \) that \( t = h_1(x) = h_1'(y) \). By the property (2) morphism definition, there is \( \phi \) that \( t \in \text{dom} \phi \). From the property (3) of morphism, there are two maps \( h_2 \) and \( h_2' \) in \( \Gamma_2 \) where, \( \psi(x) \in \text{dom} h_2 \), \( \psi'(y) \in \text{dom} h_2' \) and \( h_2 \circ \psi = \phi \circ h_1, h_2' \circ \psi' = \phi \circ h_1' \). Hence \( \phi(t) \in \Gamma_2(\psi(x)) \cap \Gamma_2(\psi'(y)) \).

The set of morphisms \( (X_1, \Gamma_1) \to (X_2, \Gamma_2) \) will be denoted by \( C(\Gamma_1, \Gamma_2) \). A morphism \( \Phi : (X_1, \Gamma_1) \to (X_2, \Gamma_2) \) is said to be of class \( C^\infty \) if \( \Gamma_1 \) and \( \Gamma_2 \) are \( C^\infty \) pseudomonoids and \( \Phi \) consists of \( C^\infty \) maps; the set of \( C^\infty \) morphisms \( (X_1, \Gamma_1) \to (X_2, \Gamma_2) \) will be denoted by \( C^\infty(\Gamma_1, \Gamma_2) \).

**Proposition 2.28.** Let \( \Phi \) be a continuous map collection of open subsets of \( X_1 \) to \( X_2 \) satisfying the properties (1) - (3) of Definition [2.24]. Then \( \Phi \) is a morphism \( (X_1, \Gamma_1) \to (X_2, \Gamma_2) \) if and only if \( \Phi \) is closed under combinations of maps.
PROOF. Suppose that $\Phi : (X_1, \Gamma_1) \to (X_2, \Gamma_2)$ is a morphism. Let $\Psi$ contain all possible combinations of maps in $\Phi$. Then $\Psi$ satisfies (1)-(3) of Definition 2.24 and $\Psi$ contains $\Phi$, by the maximality of $\Phi$, $\Phi = \Psi$. Now, Suppose that $\Phi$ is closed under map combinations of and let us show the maximality of Definition 2.24. Assume $\Phi$ is contained in another collection $\Psi$ of continuous maps of open subsets of $X_1$ to $X_2$ satisfying the (1)-(3) of Definition 2.24. Take any $\psi \in \Psi$. By property (2) for $\Phi$, for every $x \in \text{dom } \psi$ there is some $\phi \in \Phi$ with $x \in \text{dom } \phi$. From property (3) of $\Psi$, there is some $h_2 \in \Gamma_2$ with $\phi(x) \in \text{dom } h_2$ that $h_2 \circ \phi = \psi$ on some neighborhood of $x$. But $h_2 \circ \phi \in \Phi$ because $\Phi$ satisfies property (1). Therefore, every germ of $\Psi$ is some member germ of $\Phi$, yielding that $\psi$ is a combination of members of $\Phi$. Thus $\psi \in \Psi$ as desired because $\Phi$ is closed under combinations of maps.

**Proposition 2.29.** Let $\Phi$ be a morphism from $(X_1, \Gamma_1)$ into $(X_2, \Gamma_2)$. If $\Gamma_1$ is sub-pseudomonoid of $\Gamma_1$ on $X_1$, then $\Phi : (X_1, \Gamma_1') \to (X_2, \Gamma_2)$ is a morphism too.

**Proof.** Because $\Phi$ is a morphism, satisfies (1)-(3) of Definition 2.24 and it is closed under combination, by the Proposition 2.28 $\Phi : (X_1, \Gamma_1') \to (X_2, \Gamma_2)$ is closed under combination and satisfying (1)-(3) of Definition 2.24 Therefore $\Phi : (X_1, \Gamma_1') \to (X_2, \Gamma_2)$ is a morphism.

**Proposition 2.30.** If $\Phi : (X_1, \Gamma_1) \to (X_2, \Gamma_2)$ is a morphism, then $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Phi \cup \{id_X\}$ is a pseudomonoid on the topological sum $X = X_1 \sqcup X_2$. So $\Gamma|_{X_1} = \Gamma_1$ and $\Gamma|_{X_2} = \Gamma_2$.

**Lemma 2.31.** Let $\Phi_0$ be a collection of continuous maps of open subsets of $X_1$ to $X_2$ satisfying the following properties:

(2)' $\Gamma_1$-saturation of the domains of maps in $\Phi_0$ cover $X$.

(3)' There is a subset $S$ of generators of $\Gamma_1$ that satisfies the following property: If $\phi, \psi \in \Phi_0, h \in S$ and $x \in \text{dom } \phi \cap h^{-1}(\text{dom } \psi)$ then there is some $h' \in \Gamma_2$ with $\phi(x) \in \text{dom } h'$ and so $h' \circ \phi = \psi \circ h$ on some neighborhood of $x$.

Then there is a unique morphism $\Phi : (X_1, \Gamma_1) \to (X_2, \Gamma_2)$ containing $\Phi_0$. In this case, we say that $\Phi_0$ generates $\Phi$.

**Proof.** Let $\Phi$ be the family of continuous maps from open subset $X$ to $X'$ that satisfying the following conditions:

- All composites $h' \circ \phi \circ h$ with $\phi \in \Phi_0, h \in \Gamma_1$ and $h' \in \Gamma_2$, wherever defined.
- All possible combinations of composites of the above type are in $\Phi$.

Clearly, $\Phi$ satisfies properties (1)-(3) of Definition 2.24. By the Lemma 2.28 $\Phi$ is closed under combinations of maps, it is a morphism. Finally, the uniqueness of $\Phi$ follows because, if $\Psi$ is another morphism $(X_1, \Gamma_1) \to (X_2, \Gamma_2)$ containing $\Phi_0$, then it also contains $\Phi$ by property (i) for $\Psi$ and its maxinoimality, and thus $\Phi = \Psi$ by the maximality of $\Phi$.

The composition of two morphisms,

$$(X_1, \Gamma_1) \xrightarrow{\Phi} (X_2, \Gamma_2) \xrightarrow{\Psi} (X_3, \Gamma_3)$$

is the morphism $\Psi \circ \Phi : (X_1, \Gamma_1) \to (X_3, \Gamma_3)$ be generated by all compositions of maps in $\Phi$ with maps in $\Psi$. With this operation, the pseudomonoids morphisms form a category PsMo. The identity morphism $\text{id}_{(X, \Gamma)}$ of PsMo at $X$ is the morphism be generated by $\text{id}_X$, note that $\Gamma \subseteq \text{id}_{(X, \Gamma)}$.

The restriction of a morphism $\Phi : (X, \Gamma) \to (X', \Gamma')$ to a subspace $X_0 \subset X$ is the morphism $\Phi|_{X_0} : (X_0, \Gamma|_{X_0}) \to (X', \Gamma')$ consisting of all maps of open subsets of $X_0$ to $X$ that can be locally extended to maps in $\Phi$. 
If \( X_0 \) is open in \( X \), then \( \Gamma|_{X_0} \) consists of all maps in \( \Phi \) whose domain is contained in \( X_0 \). The inclusion map \( X_0 \to X \) generates a morphism \((X_0, \Gamma|_{X_0}) \to (X, \Gamma)\), whose composition with \( \Phi \) is \( \Phi|_{X_0} \).

Given \( X = \bigcup_i X_i \) and suppose that \( \Phi_i : (X_i, \Gamma|_{X_i}) \to (X', \Gamma') \) for each \( i \). Let \( \Phi \) be the family of continuous maps \( \phi : U \to X' \), where \( U \) is an open subset of \( X \), such that \( \phi|_{U \cap X_i} \in \Phi_i \) for all \( i \). If \( \Phi \) is a morphism and the maps in each \( \Phi_i \) can be locally extended to maps in \( \Phi \), then \( \Phi|_{X_i} = \Phi_i \) for all \( i \), and \( \Phi \) is called the combination of the morphisms \( \Phi_i \).

When every \( X_i \) is open in \( X \), then the combination of morphisms \( \Phi_i \) is defined just when \( \Phi_i|_{X_i \cap X_j} = \Phi_j|_{X_i \cap X_j} \) for all \( i \) and \( j \). We define the *image* of a morphism \( \Phi : (X, \Gamma) \to (X', \Gamma') \) be the set \( \text{im} \Phi = \bigcup_{\phi \in \Phi} \text{im} \phi \). If \( \Gamma \) is a pseudomonoid on \( X \) and \( X_0 \subset X \), define direct image

\[
\Phi(X_0) = \text{im} \Phi|_{X_0} = \bigcup_{\phi \in \Phi} \phi(X_0 \cap \text{dom} \phi)
\]

We say \( \Phi \) is constant if \( \text{im} \Phi \) is one orbit (\( \Phi_{\text{orb}} \) is constant). If \( \Gamma' \) is a pseudomonoid on \( X' \) and \( X'_0 \subset X' \), define the inverse image

\[
\Phi^{-1}(X'_0) = \bigcup_{\phi \in \Phi} \phi^{-1}(X'_0 \cap \text{im} \phi)
\]

**Proposition 2.32.** Let \( X'_0 \) is a subspace of \( X' \) and \( \text{im} \Phi \subset X'_0 \). Then the restrictions \( \phi : \text{dom} \phi \to X'_0 \), for any \( \phi \in \Phi \), form a morphism that we denote by \( \Phi : (X, \Gamma) \to (X'_0, \Gamma'|_{X'_0}) \). The morphism is called the restriction of \( \Phi \) too.

The product of two morphism, \( \Phi_i : (X_i, \Gamma_i) \to (X'_i, \Gamma'_i), i = 1, 2, \) is the morphism

\[
\Phi_1 \times \Phi_2 : (X_1 \times X_2, \Gamma_1 \times \Gamma_2) \to (X'_1 \times X'_2, \Gamma'_1 \times \Gamma'_2)
\]

be generated by product maps in \( \Phi_1 \) and \( \Phi_2 \). The pair of two morphism \( \Phi_i : (X, \Gamma) \to (X'_i, \Gamma'_i), i = 1, 2, \) is the morphism \( (\Phi_1, \Phi_2) : (X, \Gamma) \to (X'_1 \times X'_2, \Gamma'_1 \times \Gamma'_2) \) be generated by the pairs \( (\phi_1, \phi_2) \), where \( \phi_1 \in \Phi_1 \) and \( \phi_2 \in \Phi_2 \) have the same domain.

3. **Top, PsGr and PsMo Categories**

Here we focus on Top, PsGr and PsMo Categories and want to prove the following results.

Let Top denote the category of topological spaces and continuous maps between them. There is a canonical injective covariant functor \( \text{Top} \to \text{PsMo} \) which the pseudomonoid be generated by \( \text{id}_X \) to each space \( X \), and assigns the morphism be generated by \( f \) to each map.

Conclude, we can consider Top as a subcategory of PsMo. If PsGr be the pseudogroup category [1], then

\[
\text{Top} \subset \text{PsGr} \subset \text{PsMo}
\]

Let \( Y \) be a topological space and \( \Gamma \) is a pseudomonoid on \( X \). Then any continuous map \( Y \to X \) generates a morphism \((Y, < \text{id}_Y>) \to (X, \Gamma)\).

**Lemma 3.1.** A continuous map \( f : X \to Y \) generates a morphism \((X, \Gamma) \to (Y, < \text{id}_Y>)\) if and only if \( f \) is constant on the \( \Gamma \)-orbits.
A pseudomonoid \( \Gamma \) is said to be path connected if, for any pair of leaf \( L \) and \( L' \) of \( \Gamma \), there is a morphism \( \Phi : I \to \Gamma \) with \( \Phi(0) = L \) and \( \Phi(1) = L' \), where \( I = [0, 1] \) is a pseudomonoid in the above sense. Clearly, if \( \Gamma \) is path connected, then \( X/\Gamma \) is path connected too.

A homotopy between two morphism \( \Phi_0, \Phi_1 : (X, \Gamma) \to (X', \Gamma') \) is a morphism \( \Psi : (X, \Gamma) \times (I, \text{id}_I) \to (X', \Gamma') \) such that \( \Phi_0 \) and \( \Phi_1 \) can be identified to the restrictions of \( \Psi \) to \( X \times \{0\} \) and \( X \times \{1\} \). Since the restriction of \( X \times I \) to each slice \( X \times \{t\} \) can be identified with \( X \), a homotopy \( \Psi \) can be considered as a family of its restrictions \( \Psi_t = \Psi|_{X \times \{t\}} : (X, \Gamma) \to (X', \Gamma') \).

A morphism \( \Psi : (X, \Gamma) \to (X', \Gamma') \) is said a homotopy equivalence if there is a morphism \( \Phi' : (X', \Gamma') \to (X, \Gamma) \) that \( \Phi' \circ \Phi \) and \( \Phi \circ \Phi' \) are homotopic to the identity morphisms \( \text{id}_{(X, \Gamma)} \) and \( \text{id}_{(X', \Gamma')} \), respectively. If \( \text{id}_{(X, \Gamma)} \) is homotopic to a constant morphism, then \( (X, \Gamma) \) is called to be contractible.

### 4. \((X, \Gamma)\)-structure

Let \( M \) be a non-empty set and \( X \) is a topological space. Then a parametrization of \( X \) into \( M \) is a map \( \phi : U \to M \) where, \( U \subseteq X \) is open subset. A cover for \( M \) is a collection \( \mathcal{A} = \{(\phi_a, U_a)\}_{a \in I} \) of parametrizations of \( M \) where, \( M = \cup_{a \in I} \phi_a(U_a) \).

**Definition 4.1.** Let \( \Gamma \) be a pseudomonoid on \( X \). A \((X, \Gamma)\)-atlas on \( M \), is a cover for \( M \), \( \mathcal{A} = \{(U_a, \phi_a)\}_{a \in I} \), that for any \( \phi_a \) and for any \( f \in \Gamma \),

\[
\phi_a \circ f \in \mathcal{A}
\]

where \( \phi_a \circ f \) is defined. A set endowed with a \((X, \Gamma)\)-atlas is called a \((X, \Gamma)\)-structure and denote by \((M, \mathcal{A})\) where, \( \mathcal{A} \) is the \((X, \Gamma)\)-atlas on \( M \).

**Proposition 4.2.** Any pseudomonoid \((X, \Gamma)\) is a \((X, \Gamma)\)-structure too.

**Remark 4.3.** Let \((X, \Gamma)\) be a pseudomonoid. Suppose that \( \mathcal{C} = \{\psi : U \subseteq X \to M\} \) is a cover for \( M \). Then \( \{\psi \circ f : \psi \in \mathcal{C}, f \in \Gamma\} \) is called the \((X, \Gamma)\)-atlas generated by \( \mathcal{C} \) and denote by \( <\mathcal{C}> \).

**Example 4.4.** \((n\text{-manifolds})\) Let \( \Gamma \) be all local diffeomorphisms on \( \mathbb{R}^n \). If \( M \) is a \( n \)-manifold with maximal atlas \( \mathcal{A} \). Then \( \mathcal{A} \) is a \((\Gamma, \mathbb{R}^n)\)-atlas for \( M \), from here \( M \) has a \((\mathbb{R}^n, \Gamma)\)-structure.

**Remark 4.5.** According to the above example, the \((\mathbb{R}^n, \Gamma)\)-structures category is greater than the \( n \)-manifolds category.

**Lemma 4.6.** Let \( \Gamma \) is the pseudomonoid of previous example on \( \mathbb{R}^n \) and suppose that \( \mathcal{A} \) is a \((\mathbb{R}^n, \Gamma)\)-atlas for \( M \). Then \((M, \mathcal{A})\) is a smooth manifold if and only if \( \mathcal{A} \) satisfies the following properties:

1. All elements of \( \mathcal{A} \) are one-to-one and onto.
2. For any \( \phi, \psi \in \mathcal{A} \) and \( p \in \text{im} \psi \cap \text{im} \phi \) there is \( f \in \Gamma \), where \( \phi = \psi \circ f \) on an open subset contain \( \phi^{-1}(p) \).

**Example 4.7.** \((\text{Diffeological spaces} [7])\) A diffeology on a non-empty set \( M \) is a parametrizations collection \( \mathcal{D} \) of \( \mathbb{R}^n \)'s \((n \in \mathbb{N})\) to \( M \) such that:

1. Any constant parametrization is an element of \( \mathcal{D} \) (Covering).
2. If \( P : \cup_{i \in I} U_i \to M \) is a parametrization and if \( P|_{U_i} \in \mathcal{D} \) for all \( i \in I \). Then \( P \in \mathcal{D} \) (Locality).
3. For every parametrization \( P : U \to M \) and for any open subset \( V \) of \( \mathbb{R}^m \), if \( F : V \to U \) is a smooth map, \( P \circ F \in \mathcal{D} \) (Smooth compatibility).
A set $M$ equipped with a diffeology is called a diffeological space. Now, consider $X = \bigsqcup_{n=1}^{\infty} \mathbb{R}^n$ and $\Gamma$ is all smooth maps $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ that $U$ and $V$ are open subsets. If $M$ is a diffeological space be equipped by the diffeology $D$. Then $D$ is a $(X, \Gamma)$-atlas for $M$. Consequently $(M, D)$ is a $(\bigsqcup_{n=1}^{\infty} \mathbb{R}^n, \Gamma)$-structure.

**Lemma 4.8.** Suppose that $X = \bigsqcup_{n=1}^{\infty} \mathbb{R}^n$ and $\Gamma$ is all smooth maps $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ that $U$ and $V$ are open subsets. Then a $(X, \Gamma)$-structure, on $(M, \mathcal{A})$, is a diffeology if and only if $\mathcal{A}$ satisfies the following conditions:

1. $\mathcal{A}$ contain all constant parametrizations,
2. $\mathcal{A}$ is closed under combination maps.

**Definition 4.9.** Let $\mathcal{A}$ be a $(X, \Gamma)$-atlas on $M$. Then the weakest topology on $M$ such that all parametrizations in $\mathcal{A}$ are continuous is called $(X, \Gamma)$-topology on $M$ induced by $\mathcal{A}$.

**Lemma 4.10.** Suppose that $(M, \mathcal{A})$ is a $(X, \Gamma)$-structure. If $V \subseteq M$ is open subset, then

$$\mathcal{A}|_V = \{(\phi^{-1}(V), \phi|_{\phi^{-1}(V)}) : \phi \in \mathcal{A}\}$$

is a $(X, \Gamma)$-atlas on $V$. Therefore, $(V, \mathcal{A}|_V)$ is a $(X, \Gamma)$-structure.

**Definition 4.11.** Let $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$ are $(X, \Gamma)$-structure. Then $\alpha : M_1 \to M_2$ is called $(X, \Gamma)$-map if for any $\phi \in \mathcal{A}_1$, $\alpha \circ \phi \in \mathcal{A}_2$.

**Definition 4.12.** Let $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$ are $(X, \Gamma)$-structure. Then $\alpha : M_1 \to M_2$ is called $(X, \Gamma)$-equivalent if $\alpha$ satisfies the following properties:

1. $\alpha$ is one-to-one and onto,
2. $\alpha$ and $\alpha^{-1}$ are $(X, \Gamma)$-maps.

**Remark 4.13.** We say $(M_1, \mathcal{A}_1)$ is local $(X, \Gamma)$-equivalent to $(M_2, \mathcal{A}_2)$ if for $p \in M_1$ there are open subsets $U \subseteq M_1, V \subseteq M_2$ such that $p \in U$ and there is a $(X, \Gamma)$-equivalent $\beta : U \to V$.

**Lemma 4.14.** Assume that $\Gamma$ is all local diffeomorphism maps on $\mathbb{R}^n$. Suppose that $\mathcal{A}$ is a $(\mathbb{R}^n, \Gamma)$-atlas on $M$. Then, there is $\mathcal{A}' \subseteq \mathcal{A}$ that $(M, \mathcal{A}')$ is a $n$-manifold if and only if $(M, \mathcal{A})$ is local $(\mathbb{R}^n, \Gamma)$-equivalent to $(\mathbb{R}^n, \Gamma)$.

**Example 4.15.** (Frölicher spaces[4]) A Frölicher structure on a set $S$ is a couple $(C, F)$ where $C \subseteq S^\mathbb{R}$ and $F \subseteq \mathbb{R}^S$ such that $C = D_*F$ and $F = D^*C$ holds, with

$$D_*F = \{c : \mathbb{R} \to S : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in F\}$$

$$D^*C = \{f : S \to \mathbb{R} : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in C\}.$$

A Frölicher space is a triple $(S, C, F)$ where $S$ is a set and the couple$(C, F)$ is a Frölicher structure on it. Let $X = \mathbb{R}$ and $\Gamma$ be pseudomonoid generated by $C^\infty(\mathbb{R}, \mathbb{R})$. Therefore $C$ is a cover for $S$. So by Lemma 4.3 the $(S, < C >)$ is a $(X, \Gamma)$-structure. Hence, any Frölicher spaces equipped a $(X, \Gamma)$-atlas.

**Lemma 4.16.** Let $(S, \mathcal{A})$ is a $(\mathbb{R}, < C^\infty(\mathbb{R}, \mathbb{R}) >)$-structure. Then $S$ admits a Frölicher structure $(C, F)$ such that $C = D_*F$, $F = D^*C_0$ and $C_0 = \{\phi \in \mathcal{A} : \text{domain}(\phi) = \mathbb{R}\}$.

**Example 4.17.** (Sikorski spaces[11]) Suppose that $S$ is a nonempty set. A Sikorski structure on $S$ is a non-empty collection $\mathscr{F} = \{f : S \to \mathbb{R}\}$, with weakest topology on $S$ that all elements of $\mathscr{F}$ are continuous, satisfying the following conditions:

1. For any $m \in \mathbb{N}$, if $f_1, \cdots, f_m \in \mathscr{F}$ and $G \in C^\infty(\mathbb{R}^m)$, then $G(f_1, \cdots, f_m) \in \mathscr{F}$. 

Let $g: S \to \mathbb{R}$ be a function on $S$. Suppose that for any $p \in S$ there are $f_p \in F$ and an open subset $U_p \subset S$ such that $g|_{U_p} = f_p|_{U_p}$. Then $g \in F$.

A Sikorski space is a couple $(S, F)$ where $S$ is a non-empty set and $F$ is a Sikorski structure on $S$. Now, get $C = D_* F$ and $F = D^* C$. Thus, $(C, F)$ is a Frölicher structure on $S$ such that $F \subseteq F$. By the Example 4.15, $(S, < C >)$ is a $(\mathbb{R}, < C^\infty(\mathbb{R}, \mathbb{R}) >)$-structure.

**Lemma 4.18.** Any Sikorski structure on a non-empty set $S$ generates a Frölicher structure $S$. Therefore $S$ admits a $(\mathbb{R}, < C^\infty(\mathbb{R}, \mathbb{R}) >)$-atlas.

**Lemma 4.19.** Consider $M$ is a $n$-manifold and $C^\infty(M)$ contain all smooth functions on $M$. Then $C^\infty(M)$ is a Sikorski structure on $M$. Therefore, any manifold admits a Sikroski structure.

**Remark 4.20.** Now we continue considering the above lemma and in particular, we can conclude that, $M$ admits other Sikroski structure. For example, suppose $F \subseteq C^\infty(M)$ contain all functions such that $df = 0$.

**Lemma 4.21.** Let $(X', \Gamma')$ and $(X, \Gamma)$ are two pseudomonoids where $X'$ is an open subset of $X$ and $\Gamma' \subseteq \Gamma$. If $(M, \mathcal{A}')$ is a $(X', \Gamma')$-structure. Then there exists a $(X, \Gamma)$-atlas, denoted by $\mathcal{A}$, on $M$ such that $\mathcal{A}' \subseteq \mathcal{A}$.

**Proposition 4.22.** Any Frölicher space admits a diffeology.

**Corollary 4.23.** We conclude from this paper that, any $n$-manifold admits a Sikroski structure. Hence, any Sikroski space admits a Frölicher structure. Finally, any Frölicher space admits a diffeology.

**References**


