When countably compact preserving (countably compact) maps are sequentially subcontinuous (inversely sequentially subcontinuous)

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Abstract

We give a necessary and sufficient condition for a map to be countably compact preserving (countably compact) and use it to give a new characterization of sequential subcontinuity (inverse sequential subcontinuity) for a closed (continuous) map. It is also proved as a consequence that under appropriate restrictions on the domain and the co-domain of a map, preservation of countably compact sets is a necessary and sufficient condition for an inversely sequentially subcontinuous map to be sequentially subcontinuous.

Keywords: Countably compact, Fréchet, KC, sequentially subcontinuous, inversely sequentially subcontinuous.

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1. Introduction

Since the introduction of subcontinuity by Fuller [1], various authors have obtained results pertaining to these concepts. For instance, compactness and perfectness of maps have been discussed by Garg and Goel in [2]. This author has also introduced weak subcontinuity [3] and obtained a necessary and sufficient condition for a map to be relatively compact preserving under appropriate restrictions on the domain or range of the map.

In this paper, we first obtain a necessary and sufficient condition for a map to be countably compact preserving (countably compact) by restricting the set in which the cluster point of the image sequence (the sequence) lies for a convergent sequence (image sequence), where it is assumed that the domain (co-domain) is Fréchet and KC space. We use this result to give a new characterization of sequential subcontinuity (inverse sequential subcontinuity) for closed (continuous) maps. As a result, we also obtain a necessary and sufficient condition for an inversely sequentially subcontinuous map to be sequentially subcontinuous under appropriate restrictions.

Throughout by a space we shall mean a topological space on which no separation axioms are assumed unless mentioned explicitly. A space is called:

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Lemma 2.1.

2. Results

that we have taken.

Remark 1.2 if it is inversely sequentially subcontinuous.

We shall also refer to the following:

Theorem 2.2. The following result characterizes maps preserving (inversely preserving) countably compact subsets of a

\[ \text{Theorem 2.2. Let } f : X \rightarrow Y, \text{ where } X \text{ is arbitrary and } Y \text{ is Fréchet } T_2. \text{ Then } f \text{ is closed if it is inversely sequentially subcontinuous.} \]

Remark 1.2. The definition of sequential subcontinuity and inverse sequential subcontinuity is a little different from that given in [2], however in the context of this paper it is more appropriate to take the definition that we have taken.

2. Results

The proof of the following Lemma is straightforward and is omitted.

Lemma 2.1. In a Fréchet, \( T_1 \)-space, if a sequence \( \{ x_n \} \) has a cluster point \( x \), then \( \{ x_n \} \) has a subsequence converging to \( x \).

The following result characterizes maps preserving (inversely preserving) countably compact subsets of a Fréchet and KC space.

Theorem 2.2. Let \( f : X \rightarrow Y \) be any map, where \( X \) (\( Y \)) is Fréchet and KC space. Then \( f \) is countably compact preserving (countably compact) if and only if for any sequence \( \{ x_n \} \) \((\{ f(x_n) \})\) in \( X \) (\( Y \)), \( x_n \rightarrow x \) in \( X \) \((f(x_n) \rightarrow y \) in \( Y \)) implies that the sequence \( \{ f(x_n) \} \) \((\{ x_n \})\) has a cluster point in the subset \( S = \bigcup_n \{ f(x_n), f(x) \} \) \((R = \bigcup_n \{ f^{-1}(x_n), f^{-1}(y) \})\) of \( Y \) (\( X \)).

Proof. For arbitrary spaces \( X \) and \( Y \), if \( f \) is countably compact preserving (countably compact) and \( x_n \rightarrow x \) in \( X \) \((f(x_n) \rightarrow y \) in \( Y \)), then \( \bigcup_n \{ x_n, x \} \) \((\bigcup_n \{ f(x_n), y \})\) being compact in \( X \) (\( Y \)) implies that the set \( S \) \((R)\) is a countably compact subset of \( Y \) (\( X \)). Therefore, the sequence \( \{ f(x_n) \} \) \((\{ x_n \})\) has a cluster point in \( S \) \((R)\).

Conversely, let \( X \) (\( Y \)) be Fréchet and KC space and assume that the given condition holds. Let \( K \) be a countably compact subset of \( X \) (\( Y \)). Let \( \{ f(x_n) \} \) \((\{ x_n \})\) be any sequence in \( f(K) \) \((f^{-1}(K))\). Then \( \{ x_n \} \) can be chosen to be a sequence in \( K \) \((\{ f(x_n) \})\) is a sequence in \( K \) and so has a cluster point \( x \) \((y)\) in \( K \). Since \( X \) (\( Y \)) is Fréchet and KC space, by Lemma 2.1 there exists a subsequence \( \{ x_{n_k} \} \) \((\{ f(x_{n_k}) \})\) of \( \{ x_n \} \) \((\{ f(x_n) \})\) such that \( x_{n_k} \rightarrow x \) in \( X \) \((f(x_{n_k}) \rightarrow y \) in \( Y \)) and so by hypothesis the sequence \( \{ f(x_{n_k}) \} \) and therefore, the sequence \( \{ f(x_n) \} \) has a cluster point in \( S \subset f(K) \) \((\text{the sequence } \{ x_{n_k} \} \text{ and therefore, the sequence } \{ x_n \} \text{ has a cluster point in } R \subset f^{-1}(K))\). Therefore, \( f(K) \) \((f^{-1}(K))\) is countably compact. Hence \( f \) is countably compact preserving (countably compact).

Since the concept of countably compact and compact coincide for second countable, more generally for hereditarily Lindelöf as well as for metric spaces we have the following.

Corollary 2.3. Let \( f : X \rightarrow Y \) be any map, where \( X \) (\( Y \)) is Fréchet and KC space and \( Y \) (\( X \)) is either hereditarily Lindelöf or a metric space. Then \( f \) is compact preserving (compact) if and only if for any sequence \( \{ x_n \} \) \((\{ f(x_n) \})\) in \( X \) (\( Y \)), \( x_n \rightarrow x \) in \( X \) \((f(x_n) \rightarrow y \) in \( Y \)) implies that the sequence \( \{ f(x_n) \} \) \((\{ x_n \})\) has a cluster point in the set \( \bigcup_n \{ f(x_n), f(x) \} \) \((\bigcup_n \{ f^{-1}(x_n), f^{-1}(y) \})\).
Theorem 2.4. Let \( f : X \rightarrow Y \) be a closed (continuous) map, where \( X \) (\( Y \)) is Fréchet and KC space. Then \( f \) is countably compact preserving (countably compact) if and only if \( f \) is sequentially subcontinuous (inversely sequentially subcontinuous).

Proof. If any map \( f \) is countably compact preserving (countably compact) and a sequence \( \{x_n\} \) (\( \{f(x_n)\} \)) has a cluster point \( x \) (\( y \)) in \( X \) (\( Y \)), by Lemma 2.1, the sequence \( \{x_n\} \) (\( \{f(x_n)\} \)) has a subsequence \( \{x^*_n\} \) (\( \{f(x^*_n)\} \)) such that \( x^*_n \rightarrow x \) in \( X \) (\( f(x^*_n) \rightarrow y \) in \( Y \)). Then by Theorem 2.2, the subsequence \( \{f(x^*_n)\} \) (\( \{x^*_n\} \)) and so the sequence \( \{f(x_n)\} \) (\( \{x_n\} \)) has a cluster point in \( Y \) (\( X \)). Hence \( f \) is sequentially subcontinuous (inversely sequentially subcontinuous).

Conversely, if a closed (continuous) map is sequentially subcontinuous (inversely sequentially subcontinuous) and \( x_n \rightarrow x \) in \( X \) (\( f(x_n) \rightarrow y \) in \( Y \)), then the sequence \( \{f(x_n)\} \) (\( \{x_n\} \)) has a cluster point \( y^* \) (\( x^* \)). Since, \( f \) is closed (continuous) and \( X \) (\( Y \)) is KC space, the cluster point \( y^* \) (\( x^* \)) must belong to the closed set \( \bigcup_n \{f(x_n), f(x)\} \) (\( \bigcup_n \{f^{-1}(x_n), f^{-1}(y)\} \)). Hence by Theorem 2.2, \( f \) is countably compact preserving (countably compact).

Combining the above theorem with Theorem 1.1 above we obtain the following necessary and sufficient condition for an inversely sequentially subcontinuous map to be sequentially subcontinuous.

Theorem 2.5. Let \( f : X \rightarrow Y \), where \( X \) and \( Y \) are both Fréchet, \( T_2 \) and \( f \) is inversely sequentially subcontinuous. Then \( f \) is sequentially subcontinuous if and only if \( f \) is countably compact preserving.

3. Concluding remarks

In summary we have proved for closed (continuous) maps, the equivalence of sequential subcontinuity (inverse sequential subcontinuity) and countably compact preserving (countably compact) maps. It is left to the reader to obtain similar relationship with compactness and subcontinuity.

References