Characterizations of pre-$R_0$, pre-$R_1$ spaces and $p^*$-closedness of strongly compact(countably $p$-compact) sets

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Abstract

We introduce $p^*$-closed sets and obtain new characterizations of pre-$R_0$ and pre-$R_1$ spaces. Necessary and sufficient conditions are obtained for the $p^*$-closeness of a strongly compact (countably $p$-compact) set in pre-$R_1$ (pre-sequential, pre-$R_1$), $p$-normal (pre-sequential, $p$-normal) and also in $p^*$-normal (pre-sequential, $p^*$-normal) spaces introduced in the paper.

Keywords: $p^*$-closed, pre-$R_0$, pre-$R_1$, strongly compact, countably $p$-compact, pre-accumulation, $p$-convergent, pre-sequential, $p$-normal, strongly $p$-normal, $p^*$-normal, net.

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1. Introduction

In [9] N. Levine introduced $C$-$C$ spaces as spaces in which the closed and compact sets in $X$ coincide. Further in [13] A. Wilansky introduced $KC$-spaces as spaces in which every compact set is closed, which characterizes the concept of $C$-$C$ spaces as a compact $KC$-space.

In [6] M. Ismail and P. Nyikos introduced $C$-closed spaces in which every countably compact subset is closed and as a corollary obtained the standard result that every Fréchet, $T_2$-space and more generally every sequential, $T_2$-space is $C$-closed. In [10] Á. Császár introduced $S_i$-spaces and since then $S_i$-spaces have been widely used in general topology. In particular in [12], G. L. Garg and N. Singh obtained necessary and sufficient conditions for a compact (countably compact) subset to be closed in $S_2$ (sequential, $S_2$)-spaces and in normal (sequential, normal) spaces. Further, with the introduction of preclosed sets by A. S. Mashhour [10] and preclosures by S. N. El-Deeb et al. [1], the concept of pre-$R_0$ and pre-$R_1$-spaces in terms of preclosures were introduced by M. Caldas et al. in [2] and several characterizations of these spaces were given.

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Throughout, by a space \( X \) we shall mean a topological space. In a space \( X \), \( A^c \) will denote the complement of \( A \) for any subset \( A \) of \( X \). \( R \) and \( Z^+ \) will denote the set of all real numbers and the set of all positive integers, respectively.

A subset \( A \) of a space \( X \) is preclosed [10] if closure of interior of \( A \) is contained in \( A \). The complement of a preclosed set is called a preopen set and preclosure [4] is defined in a manner analogous to closure so that preclosure is the intersection of all preclosed sets containing \( A \) and therefore it is the smallest preclosed set containing \( A \) and is denoted by \( pcl(A) \). A point \( x \in X \) is a pre-accumulation (p-convergent) point [12] of a net in \( X \) if the net is frequently (eventually) in every preopen set containing \( x \). A space \( X \) is:

1. strongly compact [11] (countably p-compact) [14] if every preopen (countable preopen) cover of \( X \) has finite subcover,
2. \( p \)-normal [12] if for each pair of disjoint closed sets of \( X \), there exist disjoint preopen sets containing them (So every normal space is \( p \)-normal),
3. pre-\( R_0 \) [2] if every preopen set contains preclosures of its singletons.
4. pre-\( R_1 \) [2] if for points \( x, y \in X \) with distinct preclosures there exist disjoint preopen sets containing \( pcl(\{x\}) \) and \( pcl(\{y\}) \),
5. pre-\( T_1 \) [8] if for each pair of distinct points \( x \) and \( y \) of \( X \), there exists a pair of preopen sets one containing \( x \) but not \( y \) and the other containing \( y \) but not \( x \).
6. pre-\( T_2 \) [8] if for each pair of distinct points \( x \) and \( y \) of \( X \), there exists a pair of disjoint preopen sets, one containing \( x \) and the other containing \( y \).

In this paper, we introduce a generalization of preclosed sets, namely \( p^* \)-closed sets and study the \( p^* \)-closedness of strongly compact sets. For any set to be \( p^* \)-closed in any space \( X \), it is trivially necessary that the set be a union of \( p^* \)-closed sets, of the form \( G \cup F \), or of the form \( G \cap F \) where \( G \) and \( F \) are arbitrary \( p^* \)-open and \( p^* \)-closed sets in \( X \), respectively. In Section 2, we show that some of these conditions are also sufficient for a strongly compact (countably \( p \)-compact) set to be \( p^* \)-closed in pre-\( R_1 \) (pre-sequential, pre-\( R_1 \)) and in \( p^* \)-normal (pre-sequential, \( p^* \)-normal) spaces. Among other results we obtain new characterizations of pre-\( R_0 \)-spaces and pre-\( R_1 \)-spaces in terms of \( p^* \)-closed sets and we use these to give characterizations of \( p^* \)-closeness of a strongly compact set in a pre-\( R_1 \) (pre-sequential, pre-\( R_1 \)) spaces and in \( p^* \)-normal (pre-sequential, \( p^* \)-normal) spaces. Further, sufficient conditions for the \( p^* \)-closeness of a strongly compact (countably \( p \)-compact) in \( p \)-normal (pre-sequential, \( p \)-normal) and strongly \( p \)-normal (pre-sequential, strongly \( p \)-normal) spaces and for the equality of the union of \( p^* \)-closures and the \( p^* \)-closure of the union of arbitrary families of sets in pre-\( R_1 \) (pre-sequential, pre-\( R_1 \)) spaces and in \( p^* \)-normal (pre-sequential, \( p^* \)-normal) spaces are also obtained. Strong compactness (Countably \( p \)-compactness) of the \( p^* \)-closure of a strongly compact (countably \( p \)-compact) set in pre-\( R_1 \) (pre-sequential, pre-\( R_1 \)) spaces is also obtained.

The following results will be used in the next section.

**Lemma 1.1.** [4] Let \( X \) be a space and \( A \) is subset of \( X \). Then the following hold:

(i) For a point \( x \) of \( X \), \( x \in pcl(A) \) if and only if \( A \) intersects with every preopen set containing \( x \).
(ii) \( A \) is preclosed in \( X \) if and only if \( A = pcl(A) \).
(iii) \( pcl(pcl(A)) = pcl(A) \) i.e., \( pcl(A) \) is a preclosed set.

**Lemma 1.2.** [2] For a space \( X \), the following conditions are equivalent:

(i) \( X \) is pre-\( R_0 \),
(ii) For a pair of points \( x \) and \( y \) in \( X \), \( x \) is in preclosure of \( y \) if and only if \( y \) is in preclosure of \( x \),
(iii) \( X \) is pre-\( T_1 \) i.e., every singleton is preclosed.

**Remark 1.3.** From definition of pre-\( R_0 \) space it is obvious that a space \( X \) is pre-\( R_0 \) if and only if every preopen set is union of preclosures of its singletons and therefore by Lemma [14] it follows that a space \( X \) is pre-\( R_0 \) if and only if every preopen set is a union of preclosed sets.
Lemma 1.4. \cite{2} For a space $X$, the following conditions are equivalent:

(i) $X$ is pre-$R_1$,
(ii) $X$ is pre-$T_2$.

Remark 1.5. (a) \cite{2, Proposition 4.1} A space $X$ is pre-$R_1$ then it is pre-$R_0$.

(b) By part (a) and Lemma 1.4 it follows that a space $X$ is pre-$R_1$ if and only if for every pair of points $x, y$ whenever there is a preopen set containing $x$ but not $y$, then they have disjoint preopen sets.

The following Lemma is proved in \cite{7} for nets with well ordered directed domains. However, it is easy to see that it holds for all nets.

Lemma 1.6. \cite{7, Theorem 3.2} A space $X$ is strongly compact if and only if every net in $X$ pre-accumulates to some point of $X$.

Lemma 1.7. \cite{1, Theorem 2.9 (2)} Let $X$ be a space and $A$ is subset of $X$. If $A$ is preclosed in $X$, then no net in $A$ $p$-converges to a point of $A^c$.

2. Results

We first introduce the following sets.

Definition 2.1. (i) A subset $A$ of space $X$ is said to be $p^*$-closed if no net in $A$ $p$-converges to a point of $A^c$. The complement of a $p^*$-closed set is said to be $p^*$-open.

(ii) The intersection of all $p^*$-closed sets of $X$ containing $A$ is said to be the $p^*$-closure of $A$ and will be denoted by $p^*cl(A)$.

Remark 2.2. (a) Arbitrary intersection $p^*$-closed sets is $p^*$-closed.

(b) Every preclosed set is $p^*$-closed (Lemma 1.7).

(c) Let $X$ be a space and $A$ be subset of $X$. Then the following hold:

(i) $A$ is $p^*$-closed in $X$ if and only if $A = p^*cl(A)$.

(ii) $p^*cl(A)$ is a $p^*$-closed set.

(iii) $p^*cl(A) \subset pcl(A)$.

Definition 2.3. A space $X$ is said to be pre-sequential if for every non-preclosed subset $A$ of $X$ there is a sequence $\{x_n\}$ in $A$ which $p$-converges to a point of $A^c$.

We begin with the following new characterizations of pre-$R_0$ spaces.

Theorem 2.4. For a space $X$, the following conditions are equivalent:

(i) for every pair of points $x, y$ of $X$ whenever there is a $p^*$-open set containing $x$ but not $y$, then there is also a preopen set containing $y$ but not $x$,

(ii) for every pair of points $x, y$ of $X$ whenever $x$ is in every preopen set containing $y$, then $y$ is in every $p^*$-open set containing $x$,

(iii) every $p^*$-open set is union of preclosures of its singletons,

(iv) every $p^*$-open set is a union of preclosed sets,

(v) $X$ is pre-$R_0$.

Proof. (i)$\Rightarrow$ (ii): Let $x$ is in every preopen set containing $y$. To prove, $y$ is in every $p^*$-open set containing $x$. Let if possible there exists a $p^*$-open set $U$ containing $x$ but not containing $y$. By (i) there exists a preopen $V$ containing $y$ but not $x$. So there exists a $p^*$-open set $V$ containing $y$ but not $x$ which is a contradiction and hence the result.
(ii)⇒ (iii): Let \( G \) be a \( p^* \)-open set, let \( y \in pcl\{x\} \) for some \( x \in G \). Then by Lemma 1.1(i), \( x \) is in every \( p^* \)-open set of \( y \). Then by (ii) we have \( y \) is in every \( p^* \)-open set containing \( x \) so \( y \) is in \( G \). As \( x \) was arbitrarily chosen, \( pcl\{x\} \subset G \) for all \( x \in G \) and hence (iii) holds.

(iii)⇒ (iv): The proof is obvious by Lemma 1.1(iii).

(iv) ⇒ (i): Let \( A \) be a \( p^* \)-open set containing \( x \) but not \( y \). By (iv), \( A = \bigcup_{\alpha} F_{\alpha} \), where each \( F_{\alpha} \) is a preclosed set. Then \( x \in F_{\alpha} \) and \( y \in F_{\alpha}^c \) for some \( \alpha \) and so \( F_{\alpha}^c \) is a preopen set containing \( y \) but not \( x \).

(iv) ⇒ (v): It follows from the fact that every preopen set is \( p^* \)-open and Remark 1.3.

(v) ⇒ (iv): This follows from the fact that in a pre-\( R_0 \) space every singleton is preclosed (Lemma 1.2 above).

**Theorem 2.5.** A space \( X \) is pre-\( R_0 \) if and only if any subset \( A \) of \( X \) is a union of preclosed sets, whenever \( A^c \) is a union of \( p^* \)-closed sets.

**Proof.** The proof follows from the fact that a space \( X \) is pre-\( R_0 \) if and only if every \( p^* \)-open set is a union of preclosed sets (Theorem 2.4).

We now obtain the following new characterization of pre-\( R_1 \) spaces.

**Theorem 2.6.** For a space \( X \), the following conditions are equivalent:

(i) \( X \) is pre-\( R_1 \),

(ii) for every pair of points \( x, y \) in \( X \), whenever there is a \( p^* \)-open set containing \( x \) but not \( y \), then they have disjoint preopen set.

**Proof.** (i) ⇒ (ii): It follows from Lemma 1.4.

(ii) ⇒ (iii): It follows by fact that every preopen is \( p^* \)-open and Remark 1.3(b).

We now obtain the following characterization for the \( p^* \)-closedness of a strongly compact (countably \( p \)-compact) subset in a pre-\( R_1 \) (pre-sequential, pre-\( R_1 \)) space.

**Theorem 2.7.** For a strongly compact (countably \( p \)-compact) subset \( K \) of a pre-\( R_1 \) (pre-sequential, pre-\( R_1 \)) space \( X \), the following conditions are equivalent:

(i) \( K \) is \( p^* \)-closed,

(ii) either \( K \) or \( K^c \) is a union of \( p^* \)-closed sets,

(iii) both \( K \) and \( K^c \) are unions of \( p^* \)-closed sets.

**Proof.** For any set \( K \), (i) implies (ii) is obvious and (ii) implies (iii) follows from Theorem 2.5 above since every preclosed set is \( p^* \)-closed.

To prove (iii) implies (i), it is sufficient to assume that \( K = \bigcup_{\alpha} F_{\alpha} \), where each \( F_{\alpha} \) is a \( p^* \)-closed set in \( X \). If \( K \) is not \( p^* \)-closed (hence not preclosed), then there exists a net \( \{x_\lambda\} \) (a sequence \( \{x_n\} \) in \( K \) such that \( x_\lambda \) \( p \)-converges to \( a \) \( (x_n \ p \)-converges to \( a \) \) and \( a \in K^c \). Then as \( K \) is strongly compact (countably \( p \)-compact) implies that the net \( \{x_\lambda\} \) (sequence \( \{x_n\} \) has a pre-accumulation point \( b \) in \( K \) (Lemma 1.6). Therefore, there exists an \( \alpha \) such that \( b \in F_{\alpha} \) and \( a \notin F_{\alpha} \). Then \( F_{\alpha}^c \) is a \( p^* \)-open set containing \( a \) not containing \( b \), and since \( X \) is pre-\( R_1 \) it follows that they have disjoint preopen sets (Theorem 2.6), contradicting to the fact that \( \{x_\lambda\} \) \( p \)-converges to \( a \) \( (\{x_n\} \ p \)-converges to \( a \) \) and \( b \) is a pre-accumulation point of \( \{x_\lambda\} \) (\( \{x_n\} \)). Hence \( K \) must be \( p^* \)-closed and (i) follows.

**Corollary 2.8.** Let \( X \) be a pre-\( R_1 \) (pre-sequential, pre-\( R_1 \)) space and \( G \) and \( F \) be arbitrary \( p^* \)-open and \( p^* \)-closed sets in \( X \), respectively. Then:

(a) for a strongly compact (countably \( p \)-compact) subset \( K \) of \( X \), the following conditions are equivalent:
We now obtain the following characterization for the compact (countably $p$-compact) subset in a $A$-space the preclosure of any point $x$ is contained in every preopen set of the point and $p^*$-closed, if it is strongly compact (countably $p$-compact).  

Definition 2.11. A space $X$ is said to be $p^*$-normal if for each pair of disjoint $p^*$-closed sets of $X$, there exist disjoint preopen sets containing them.

Definition 2.12. A space $X$ is said to be strongly $p$-normal if for each pair of disjoint preclosed sets of $X$, there exist disjoint preopen sets containing them.

Definition 2.13. A $p^*$-normal space is strongly $p$-normal and a strongly $p$-normal space is $p$-normal.

Proof. The proof follows from the fact that in a pre-$R_0$ and therefore, in a pre-$R_1$ space, every $p^*$-open set is a union of preclosed sets as every pre-$R_1$ space is a pre-$R_0$ space.

Corollary 2.9. Let $K$ be a strongly compact (countably $p$-compact) subset of a pre-$R_1$ (pre-sequential, pre-$R_1$) space $X$. Then:

(a) $p^*\text{cl}(K) = \bigcup\{p^*\text{cl}\{x\} : x \in K\}$ and $p^*\text{cl}(K)$ is strongly compact (countably $p$-compact).

(b) if $S$ is any set such that $K \subseteq S \subseteq p^*\text{cl}(K)$, then $S$ is strongly compact (countably $p$-compact).

Proof. First, we prove that the set $E = \bigcup\{p^*\text{cl}\{x\} : x \in K\} \subseteq p^*\text{cl}(K)$ is strongly compact (countably $p$-compact). Let $G$ be an arbitrary (countable) family of preopen sets covering $E$. Then $G$ is a preopen (countable preopen) cover of $K$ and as $K$ is strongly compact (countably $p$-compact) so there exist finitely many sets $G_1$, $G_2$, . . . , $G_n$ in $G$ such that $K \subseteq \bigcup_{i=1}^{n} G_i$. Since in a pre-$R_0$, and therefore, in a pre-$R_1$ space the preclosure of any point $x$ is contained in every preopen set of the point and $p^*$-closure of a set is contained in preclosure of the set it follows that $E \subseteq \bigcup_{i=1}^{n} G_i$, implying thereby that $E$ is strongly compact (countably $p$-compact). Since $E$ is also a union of $p^*$-closed sets, it follows from Theorem 2.7 above that $E$ is $p^*$-closed. Hence $p^*\text{cl}(K) = E$ and $p^*\text{cl}(K)$ is strongly compact (countably $p$-compact). This proves (a). Also, (b) follows from (a).

Corollary 2.10. In a pre-$R_1$ (pre-sequential, pre-$R_1$) space $X$,

(a) a union of $p^*$-closed sets or an intersection of $p^*$-open sets is $p^*$-closed, if it is strongly compact (countably $p$-compact),

(b) if $\varepsilon$ is a family of subsets of $X$ such that $\bigcup\{p^*\text{cl}(E) : E \in \varepsilon\}$, in particular $\bigcup\{E : E \in \varepsilon\}$, is strongly compact (countably $p$-compact), then $p^*\text{cl}(\bigcup\{E : E \in \varepsilon\}) = \bigcup\{p^*\text{cl}(E) : E \in \varepsilon\}$.

Proof. Proof of (a) is obvious from Theorem 2.7. For the proof of part (b), we note that in view of Corollary 2.9(b), the condition “$\bigcup\{p^*\text{cl}(E) : E \in \varepsilon\}$ is strongly compact (countably $p$-compact)” is weaker than the condition “$\bigcup\{E : E \in \varepsilon\}$ is strongly compact (countably $p$-compact)”.

So far we have discussed $p^*$-closedness of a strongly compact (countably $p$-compact) set in pre-$R_1$ (pre-sequential, pre-$R_1$) spaces. Since in [5] closeness of compact sets has been studied in normal (sequential, normal) spaces it is relevant to study $p^*$-closedness of strongly compact (countably $p$-compact) set in p-normal (pre-sequential, p-normal) spaces introduced in [2] and in the following spaces introduced below.

Definition 2.11. A space $X$ is said to be $p^*$-normal if for each pair of disjoint $p^*$-closed sets of $X$, there exist disjoint preopen sets containing them.

Definition 2.12. A space $X$ is said to be strongly $p$-normal if for each pair of disjoint preclosed sets of $X$, there exist disjoint preopen sets containing them.

Definition 2.13. A $p^*$-normal space is strongly $p$-normal and a strongly $p$-normal space is $p$-normal.

In Section 3, we will give examples to show that converse of the above implications do not hold. We now obtain the following characterization for the $p^*$-closedness of a strongly compact (countably $p$-compact) subset in a $p^*$-normal (pre-sequential, $p^*$-normal) space.
Theorem 2.14. In a $p^*$-normal (pre-sequential, $p^*$-normal) space $X$, strongly compact (countably $p$-compact) set $K$ is $p^*$-closed if and only if $K$ is a union of $p^*$-closed sets and $K^c$ is of the form $G \cup F$, where $G$ and $F$ are arbitrary preopen and $p^*$-closed sets, respectively.

Proof. Since necessity is obvious for any set $K$, we need only prove the sufficiency part. Let $K = \bigcup \alpha F_\alpha$, where each $F_\alpha$ is a $p^*$-closed set in $X$ and $K^c = G \cup F$, where $G$ is preopen and $F$ is $p^*$-closed in $X$. If $K$ is not $p^*$-closed (hence not preclosed), then there exists a net $\{x_\lambda\}$ (a sequence $\{x_n\}$) in $K$ such that $x_\lambda$ p-converges to $a$ ($x_n$ $p$-converges to $a$) and $a \in K^c$. Since $a \in \text{pcl}(K)$, $a$ cannot belong to $G$. Therefore, $a \in F$. Then as $K$ is strongly compact (countably $p$-compact) implies that the net $\{x_\lambda\}$ (sequence $\{x_n\}$) has a pre-accumulation point $b$ in $K$ (Lemma 1.6). Therefore, there exists an such that $b \in F\alpha$ and $a \notin F_\alpha$. Thus $a$ and $b$ belong to the disjoint $p^*$-closed sets $F$ and $F_\alpha$, respectively and can be separated by disjoint preopen sets, since $X$ is $p^*$-normal. This contradicts to the fact that $x_\lambda$ $p$-converges to $a$ ($x_n$ $p$-converges to $a$) and $b$ is a pre-accumulation point of $\{x_\lambda\}$ ($\{x_n\}$). Hence $K$ must be $p^*$-closed.

Corollary 2.15. Let $X$ be a $p^*$-normal (pre-sequential, $p^*$-normal) space and $G$ and $F$ are arbitrary $p^*$-open and preclosed sets respectively. Then:

(a) a union of $p^*$-closed sets in $X$ is $p^*$-closed, if it is strongly compact (countably $p$-compact) and is of the form $G \cap F$,

(b) if $\varepsilon$ is a family of subsets of $X$ such that $\bigcup \{p^*\text{cl}(E) : E \in \varepsilon\}$ is strongly compact (countably $p$-compact) and is of the form $G \cap F$, then $p^*\text{cl}(\bigcup \{E : E \in \varepsilon\}) = \bigcup \{p^*\text{cl}(E) : E \in \varepsilon\}$.

The proofs of the following Theorems 2.16 and 2.17 are similar to the proof of Theorem 2.14.

Theorem 2.16. In a $p$-normal (pre-sequential, $p$-normal) space $X$, strongly compact (countably $p$-compact) set $K$ is $p^*$-closed if $K$ is a union of closed sets and $K^c$ is of the form $C \cup D$, where $C$ and $D$ are arbitrary preopen and closed sets, respectively.

Theorem 2.17. In a strongly $p$-normal (pre-sequential, strongly $p$-normal) space $X$, strongly compact (countably $p$-compact) set $K$ is $p^*$-closed if $K$ is a union of preclosed sets and $K^c$ is of the form $C \cup D$, where $C$ and $D$ are arbitrary preopen and preclosed sets, respectively.

3. Examples

Example 3.1. [16] 5.1 Problem 114] Let $X = Z^+$, together with the topology, $T = \{G \subset X|G = \emptyset$ or $G^c \cap 2Z^+$ is finite, where $2Z^+$ denotes the set of all even positive integers$\}$. Here $K = 2Z^+$ is a $p^*$-closed set which is not preclosed. Then $(X, T)$ is a strongly $p$-normal but not $p^*$-normal as $2Z^+$ and $2Z^+ - 1$ are disjoint $p^*$-closed sets which do not have disjoint preopen sets containing them and it is not normal as singletons $\{1\}$ and $\{2\}$ are disjoint closed sets which have no disjoint open sets containing them, respectively.

Example 3.2. [13] Example 57] Let $X = \{x \in Z^+|x \geq 2\}$, together with the topology generated by the sets of the form $U_n = \{x \in Z^+|x$ divides $n\}$, for $n \geq 2$. Then $X$ is a normal space and hence $p$-normal. But it is not strongly $p$-normal as singletons $\{4\}$ and $\{6\}$ are disjoint preclosed sets which have no disjoint preopen sets containing them, respectively.

Example 3.3 shows that $p^*$-closeness of $K$ cannot be replaced by preclosedness of $K$ in Theorem 2.7.

Example 3.3. Let $X = \{a, b, c\}$ with topology $T = \{\emptyset, \{a, b\}, X\}$. Then $(X, T)$ is a pre-$R_1$ space. Further, $K = \{a, b\}$ is a strongly compact set such that both $K$ and $K^c$ are union of preclosed sets but $K$ is not preclosed.
Example 3.4 below shows that converse of Theorem 2.16 does not hold.

Example 3.4. (this is a special case of [16, 8.1 Problem 1]) Let \( p \notin R \) and \( X = R \cup p \), with the topology \( T = \{ G \subset X | G \text{ is open in } R \text{ with the usual topology or } G = X \} \). Then \( (X, T) \) is \( p \)-normal space and the singleton set \( K = \{ x \} \) for some \( x \in R \) is a \( p^* \)-closed set and \( K^c \) is of the form \( C \cup D \) where \( C \) is preopen and \( D \) is closed, but \( K \) is not union of closed sets.

Example 3.5 shows that \( p^* \)-closeness of \( K \) cannot be replaced by preclosedness of \( K \) in Theorem 2.17.

Example 3.5. Let \( X = \{ a, b, c, d \} \) with topology \( T = \{ \emptyset, \{ c \}, \{ a, b \}, \{ a, b, c \}, X \} \). Then \( K = \{ a, b \} \) is a \( p^* \)-closed set which is not preclosed. Also, \( (X, T) \) is a strongly \( p \)-normal space which is not \( p^* \)-normal as \( \{ a, b \} \) and \( \{ a, c, d \} \) are disjoint \( p^* \)-closed sets which have no disjoint preopen sets containing them. Further, \( K = \{ a, b \} \) is a strongly compact set which is a union of \( p^* \)-closed sets and \( K^c \) is of the form \( C \cup D \), where \( C \) and \( D \) are arbitrary preopen and preclosed sets respectively but \( K \) is not preclosed.

References

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