On Contra-$\gamma$-continuous and Almost Contra-$\gamma$-continuous Multifunction

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ABSTRACT. The main goal of this paper is to introduce and study the notions of contra-$\gamma$-continuous and almost contra-$\gamma$-continuous multifunctions.

KEYWORDS. Contra-continuous; Contra-$\gamma$-continuous; Almost Contra-$\gamma$-continuous.

1. Introduction

In 1993, Noiri and Popa [1] introduced the notion of weakly precontinuous multifunctions. Recently, Ekici, Jafari and Noiri [2] introduced the notion of contra-continuous multifunctions. In this paper, we introduce and study two new concepts namely contra-$\gamma$-continuous and almost contra-$\gamma$-continuous multifunctions.

Throughout the present paper, $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) denotes a topological spaces on which no separation axioms is assumed unless explicitly stated. For a subset $K$ of a space $X$, $Cl(K)$ and $Int(K)$ represent the closure of $K$ and the interior of $A$, respectively.

Definition 1.1. A subset $U$ of a space $X$ is called:
(1) $\alpha$-open [3] if $U \subset Int(Cl(Int(U)))$.
(2) semi-open [4] if $U \subset Cl(Int(U))$.
(3) preopen [5] if $U \subset Int(Cl(U))$.
(4) $\beta$-open [6] if $U \subset Cl(Int(Cl(U)))$.
(5) regular open [7] (regular closed [7]) if $U = Int(Cl(U))$ ($U = Cl(Int(U))$).

Definition 1.2. [8] Let $(X, \tau)$ be a topological space. An operation $\gamma$ on the topology $\tau$ is a mapping from $\tau$ in to power set $P(X)$ of $X$ such that $V \subset \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of $\gamma$ at $V$.

Definition 1.3. [9] A subset $A$ of a topological spac $(X, \tau)$ is called $\gamma$-open set if for each $x \in A$ there exists an open set $U$ such that $x \in U$ and $\gamma(U) \subset A$. Then, $\tau_\gamma$ denotes the set of all $\gamma$-open set in $X$. Clearly $\tau_\gamma \subset \tau$. Complements of $\gamma$-open sets are called $\gamma$-closed.

Definition 1.4. [9] The intersection of all $\gamma$-closed sets containing $A$ is called the $\gamma$-closure of $A$ and is denoted by $\gamma(Cl(A))$.

Definition 1.5. [10] The $\tau_\gamma$-interior of $A$ is denoted by $\tau_\gamma-Int(A)$ and defined to be the union of all $\gamma$-open sets of $X$ contained in $A$.

The family of all semi-open (resp. regular closed, regular open, preopen, $\beta$-open) sets of $X$ is denoted by $SO(X)$ (resp. $RC(X)$, $RO(X)$, $PO(X)$, $BO(X)$). The union of all preopen sets of $X$ contained
in $U$ is called the preinterior of $U$ and is denoted by $pInt(U)$. The intersection of all $\alpha$-closed (resp. semi-closed, preclosed, $\beta$-closed) sets of $X$ containing $U$ is called the $\alpha$-closure (resp. semi-closure, preclosure, $\beta$-closure) of $U$ and is denoted by $\alpha Cl(U)$ (resp. $sCl(U)$, $pCl(U)$ and $\beta Cl(U)$).

A point $x \in X$ is called a $\delta$-cluster point of a subset $A$ if $Int(Cl(U)) \cap A \neq \phi$ for every open set $U$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$ and is denoted by $Cl_\delta(A)$. If $A = Cl_\delta(A)$, then $A$ is said to be $\delta$-closed [11]. The complement of a $\delta$-closed set is said to be $\delta$-open. It is shown in [11] that $Cl_\delta(S)$ is closed for each subset $S$ of $X$. A point $x \in X$ is in $\theta$-semi-closure of $A$, denoted by $\theta-sCl(A)$, if $A \cap Cl(U) \neq \phi$ for each semi-open set $U$ containing $x$. $A$ is $\theta$-semi-closed if $A = \theta-sCl(A)$. The complement of a $\theta$-semi-closed set is said to be $\theta$-semi-open [12].

Throughout the paper $F : X \rightarrow Y$ presents a multifunction. For a multifunction $F : X \rightarrow Y$, we shall denote the upper and lower inverse of a subset $U$ of $Y$ by $F^+(U)$ and $F^-(U)$, respectively for which $F^+(U) = \{x \in X : F(x) \subset U\}$ and $F^-(U) = \{x \in X : F(x) \cap U \neq \phi\}$ [13]. The graph multifunction $G_F : X \rightarrow X \times Y$ of a multifunction $F : X \rightarrow Y$ is defined as follows $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

**Lemma 1.6.** [1] Let $X$ and $Y$ be topological spaces and let $A \subset X$ and $B \subset Y$. The following properties hold for a multifunction $F : X \rightarrow Y$:

1. $G^+_F(A \times B) = A \cap F^+(B)$.
2. $G^-_F(A \times B) = A \cap F^-(B)$.

**Definition 1.7.** A multifunction $F : X \rightarrow Y$ is called lower (upper) contra-continuous [2] (resp. lower (upper) contra-precontinuous [14]) if for each $x \in X$ and each closed set $K$ such that $x \in F^-(K)$ ($x \in F^+(K)$), there exists an open (resp. preopen) set $U$ containing $x$ such that $U \subset F^-(K)$ ($U \subset F^+(K)$).

**Definition 1.8.** [1] A multifunction $F : X \rightarrow Y$ is called lower (upper) weakly precontinuous if for each $x \in X$ and each open set $U$ of $Y$ such that $x \in F^-(U)$ ($x \in F^+(U)$), there exists a preopen set $V$ of $X$ containing $x$ such that $V \subset F^-(Cl(U))$ ($V \subset F^+(Cl(U))$).

2. Contra-$\gamma$-continuity and almost contra-$\gamma$-continuity

**Definition 2.1.** A multifunction $F : X \rightarrow Y$ is called:

1. lower contra-$\gamma$-continuous at $x \in X$ if for each closed set $V$ with $x \in F^-(V)$, there exists a $\gamma$-open set $U$ containing $x$ such that $U \subset F^-(V)$.
2. upper contra-$\gamma$-continuous at $x \in X$ if for each closed set $V$ with $x \in F^+(V)$, there exists a $\gamma$-open set $U$ containing $x$ such that $U \subset F^+(V)$.
3. lower (upper) contra-$\gamma$-continuous if $F$ has this property at each point of $X$.

**Definition 2.2.** A multifunction $F : X \rightarrow Y$ is called:

1. lower almost contra-$\gamma$-continuous at $x \in X$ if for each regular closed set $V$ with $x \in F^-(V)$, there exists a $\gamma$-open set $U$ containing $x$ such that $U \subset F^-(V)$.
2. upper almost contra-$\gamma$-continuous at $x \in X$ if for each regular closed set $V$ with $x \in F^+(V)$, there exists a $\gamma$-open set $U$ containing $x$ such that $U \subset F^+(V)$.
3. lower (upper) almost contra-$\gamma$-continuous if $F$ has this property at each point of $X$.

**Theorem 2.3.** If $F : X \rightarrow Y$ is an upper (lower) almost contra-$\gamma$-continuous multifunction, then $F$ is upper (lower) weakly precontinuous.

**Proof.** Let $x \in X$ and $V$ be an open subset of $Y$ with $F(x) \subset V$. This implies that $Cl(V)$ is a regular closed set with $F(x) \subset Cl(V)$. Since $F$ is upper almost contra-$\gamma$-continuous, then there exists a $\gamma$-open set $U$ containing $x$ such that $U \subset F^+(Cl(V))$. Hence, $F$ is upper weakly precontinuous.
The converse of this implication is not reversible as shown in the following example.

**Example 2.4.** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\varnothing, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\} \). Define a multifunction \( F : (X, \tau) \rightarrow (X, \tau) \) by \( F(a) = \{b\}, F(b) = \{b\}, F(c) = \{a\}, F(d) = \{d\} \) and \( \gamma(A) = A \) for all \( A \in \tau \). Then \( F \) is upper weakly precontinuous but it is not upper almost contra-\( \gamma \)-continuous.

**Remark 2.5.** The following diagram hold for a multifunction \( F : X \rightarrow Y \):

\[
\begin{array}{c}
\text{upper/lower weakly precontinuous} \\
\uparrow \\
\text{upper/lower almost contra-\( \gamma \)-continuous} \\
\uparrow \\
\text{upper/lower contra-\( \gamma \)-continuous} \\
\downarrow \\
\text{upper/lower contra-continuous} \\
\downarrow \\
\text{upper/lower contra-precontinuous}
\end{array}
\]

The converses of these implications are not true in general as shown in the following examples and Example 2.4.

**Example 2.6.** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\varnothing, X, \{a\}, \{a, b\}, \{a, c\}, \{a, c, d\}\} \). Define a multifunction \( F : (X, \tau) \rightarrow (X, \tau) \) by \( F(a) = \{d\}, F(b) = \{c\}, F(c) = \{a\}, F(d) = \{b\} \) and \( \gamma(A) = A \) for all \( A \in \tau \). Then \( F \) is upper contra-precontinuous but it is not upper contra-\( \gamma \)-continuous.

**Theorem 2.7.** Let \( F : X \rightarrow Y \) be a multifunction. The following are equivalent:

1. \( F \) is upper contra-\( \gamma \)-continuous.
2. \( F^+(V) \) is a \( \gamma \)-open set for any closed subset \( V \) of \( Y \).
3. \( F^-(V) \) is a \( \gamma \)-closed set for any open subset \( V \) of \( Y \).
4. For each \( x \in X \) and each closed set \( K \) containing \( F(x) \), there exists a \( \gamma \)-open set \( U \) containing \( x \) such that if \( y \in U \), then \( F(y) \subset K \).

**Proof.** (1) \( \Leftrightarrow \) (2): Let \( V \) be a closed subset in \( Y \) and \( x \in F^+(V) \). Since \( F \) is upper contra-\( \gamma \)-continuous, then there exists a \( \gamma \)-open set \( U \) containing \( x \) such that \( U \subset F^+(V) \). Hence, \( F^+(V) \) is \( \gamma \)-open. The converse is similar.

(2) \( \Leftrightarrow \) (3): It follows from the fact that \( F^+(Y \setminus V) = X \setminus F^-(V) \) for every subset \( V \) of \( Y \).

(1) \( \Leftrightarrow \) (4): Obvious.

**Theorem 2.8.** Let \( F : X \rightarrow Y \) be a multifunction. The following are equivalent:

1. \( F \) is lower contra-\( \gamma \)-continuous.
2. \( F^-(K) \) is a \( \gamma \)-open set for any closed subset \( K \) of \( Y \).
3. \( F^+(V) \) is a \( \gamma \)-closed set for any open subset \( V \) of \( Y \).
4. For each \( x \in X \) and each closed set \( K \) such that \( F(x) \cap K \neq \varnothing \), there exists a \( \gamma \)-open set \( U \) containing \( x \) such that if \( y \in U \), then \( F(y) \cap K \neq \varnothing \).

**Proof.** The proof is similar to that of Theorem 2.7.

**Theorem 2.9.** Let \( F : X \rightarrow Y \) be a multifunction. The following are equivalent:

1. \( F \) is upper almost contra-\( \gamma \)-continuous.
2. \( F^+(A) \) is \( \gamma \)-open for any regular closed subset \( A \) of \( Y \).
3. \( F^-(U) \) is \( \gamma \)-closed for any regular open subset \( U \) of \( Y \).

\( \square \)
(4) $F^-(Int(Cl(A)))$ is $\gamma$-closed for every open subset $A$ of $Y$.
(5) $F^+(Cl(Int(A)))$ is $\gamma$-open for every closed subset $A$ of $Y$.

(6) for each $x \in X$ and for each $V \in SO(Y)$ with $F(x) \subset V$, there exists a $\gamma$-open subset $U$ of $X$ containing $x$ such that $F(U) \subset Cl(V)$.

(7) $F^+(V) \subset \tau_{\gamma}Int(F^+(Cl(V)))$ for every $V \in SO(Y)$.

Proof. (1) $\Rightarrow$ (2): Let $A \in RC(Y)$ and $x \in F^+(A)$. Since $F$ is upper almost contra-$\gamma$-continuous, then there exists a $\gamma$-open subset $U$ containing $x$ such that $U \subset F^+(A)$. Thus, $F^+(A)$ is $\gamma$-open.

(2) $\Leftrightarrow$ (1): Obvious.

(2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5): It follows from the fact that $F^+(Y \setminus A) = X \setminus F^-(A)$ for every subset $A$ of $Y$.

(3) $\Leftrightarrow$ (4): Let $A$ be an open subset of $Y$. Since $Int(Cl(A))$ is regular open, then $F^-(Int(Cl(A)))$ is $\gamma$-closed. The converse is obvious.

(5) $\Leftrightarrow$ (2): It is similar to that of (3) $\Leftrightarrow$ (4).

(6) $\Rightarrow$ (7): Let $V \in SO(Y)$ and $x \in F^+(V)$. Then $F(x) \subset V$. By (6), there exists a $\gamma$-open set $U$ in $X$ containing $x$ such that $F(U) \subset Cl(V)$. This implies that $x \in U \subset F^+(Cl(V))$. Hence, $x \in \tau_{\gamma}Int(F^+(Cl(V)))$ and $F^+(V) \subset \tau_{\gamma}Int(F^+(Cl(V)))$.

(7) $\Rightarrow$ (2): Let $A \in RC(Y)$. Since $A \in SO(Y)$, then $F^+(A) \subset \tau_{\gamma}Int(F^+(A))$. Hence, $F^+(A)$ is $\gamma$-open in $X$.

(2) $\Rightarrow$ (6): Let $x \in X$ and $V \in SO(Y)$ with $F(x) \in V$. Since $Cl(V) \in RC(Y)$, then there exists a $\gamma$-open set $A$ in $X$ containing $x$ such that $x \in A \subset F^+(Cl(V))$. Hence, $F(A) \subset Cl(V)$.

Theorem 2.10. Let $F : X \to Y$ be a multifunction. The following are equivalent:

(1) $F$ is lower almost contra-$\gamma$-continuous.
(2) $F^-(A)$ is $\gamma$-open for any regular closed subset $A$ of $Y$.
(3) $F^+(U)$ is $\gamma$-closed for any regular open subset $U$ of $Y$.
(4) $F^+(Int(Cl(A)))$ is $\gamma$-closed for every open subset $A$ of $Y$.
(5) $F^-(Cl(Int(A)))$ is $\gamma$-open for every closed subset $A$ of $Y$.
(6) for each $x \in X$ and for each $V \in SO(Y)$ with $F(x) \cap V \neq \emptyset$, there exists a $\gamma$-open subset $U$ of $X$ containing $x$ such that $F(u) \cap Cl(V) \neq \emptyset$ for each $u \in U$.
(7) $F^-(V) \subset \tau_{\gamma}Int(F^-(Cl(V)))$ for every $V \in SO(Y)$.

Proof. It is similar to that of Theorem 2.9.

Recall that a topological space $(X, \tau)$ is said to be semi-regular if for each open set $U$ of $X$ and for each point $x \in U$, there exists a regular open set $V$ such that $x \in V \subset U$.

Theorem 2.11. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, where $(Y, \sigma)$ is semi-regular, the following are equivalent:

(1) $F$ is upper contra-$\gamma$-continuous (u.c.$\gamma$-c.).
(2) $F^+(Cl\delta(B))$ is $\gamma$-open for every subset $B$ of $Y$.
(3) $F^+(K)$ is $\gamma$-open for every $\delta$-closed set $K$ of $Y$.
(4) $F^-(V)$ is $\gamma$-closed for every $\delta$-open set $V$ of $Y$.

Proof. (1) $\Rightarrow$ (2): Let $B$ be any subset of $Y$. Then $Cl\delta(B)$ is closed and by Theorem 2.7, $F^+(Cl\delta(B))$ is $\gamma$-open.

(2) $\Rightarrow$ (3): Let $K$ be a $\delta$-closed set of $Y$. Then $Cl\delta(K) = K$. By (2), $F^+(K)$ is $\gamma$-open.
Theorem 2.15. For a multifunction $\tau: Y \to \mathcal{P}$, if $\tau$ is semi-regular, then the following are equivalent:

(1) $F$ is lower contra-$\gamma$-continuous (l.c.$\gamma$.c.).
(2) $F^-(\text{Cl}(B))$ is $\gamma$-open for every subset $B$ of $Y$.
(3) $F^-(K)$ is $\gamma$-open for every $\delta$-closed set $K$ of $Y$.
(4) $F^+(V)$ is $\gamma$-closed for every $\delta$-open set $V$ of $Y$.

Proof. The proof is similar to the proof of Theorem 2.11.

\begin{proof}

\end{proof}

Definition 2.13. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be contra-$\gamma$-continuous if $f^{-1}(V)$ is $\gamma$-closed in $X$ for each open set $V$ of $Y$.

Corollary 2.14. For a function $f: (X, \tau) \to (Y, \sigma)$, where $(Y, \sigma)$ is semi-regular, the following properties are equivalent:

(1) $f$ is contra-$\gamma$-continuous.
(2) $f^{-1}(\text{Cl}(B))$ is $\gamma$-open for every subset $B$ of $Y$.
(3) $f^{-1}(K)$ is $\gamma$-open for every $\delta$-closed set $K$ of $Y$.
(4) $f^{-1}(V)$ is $\gamma$-closed for every $\delta$-open set $V$ of $Y$.

Theorem 2.15. For a multifunction $F: (X, \tau) \to (Y, \sigma)$, the following are equivalent:

(1) $F$ is lower almost contra-$\gamma$-continuous (l.a.$\gamma$.c.).
(2) $F^-(V)$ is $\gamma$-open for every $\theta$-semi-open set $V$ of $Y$.
(3) $F^+(K)$ is $\gamma$-closed for every $\theta$-semi-closed set $K$ of $Y$.
(4) $\tau^{-}\text{Cl}(F^+(\text{Int}(\text{Cl}(B)))) \subset F^+(\text{sCl}(B))$ for every subset $B$ of $Y$.
(5) $\tau^{-}\text{Cl}(F^+(B)) \subset F^+(\theta-\text{sCl}(B))$ for every subset $B$ of $Y$.
(6) $F(\tau^{-}\text{Cl}(A)) \subset \theta-\text{sCl}(F(A))$ for every subset $A$ of $X$.

Proof. (1) $\Rightarrow$ (2): Let $G$ be any $\theta$-semi-open set of $Y$. There exists a family of regular closed sets $\{K_\alpha: \alpha \in \Lambda\}$ such that $G = \bigcup\{K_\alpha: \alpha \in \Lambda\}$. It follows from Theorem 2.10 (2), that $F^-(G) = \bigcup\{F^-(K_\alpha): \alpha \in \Lambda\}$ is $\gamma$-open.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (4): Let $B$ be any subset of $Y$. Then $\text{Int}(\text{Cl}(B))$ is regular open and it is $\theta$-semi-closed in $Y$. Therefore we have that $F^+(\text{Int}(\text{Cl}(B)))$ is $\gamma$-closed and $\tau^{-}\text{Cl}(F^+(\text{Int}(\text{Cl}(B)))) = F^+(\text{Int}(\text{Cl}(B))) \subset F^+(\text{sCl}(B))$.

(4) $\Rightarrow$ (5): Let $B$ be any subset of $Y$. For any regular open set $V$ with $B \subset V$, we have $\tau^{-}\text{Cl}(F^+(B)) \subset \text{Cl}(F^+(V)) = \tau^{-}\text{Cl}(\text{Cl}(F^+(V))) \subset F^+(\text{sCl}(V)) = F^+(V)$. Therefore, $\tau^{-}\text{Cl}(F^+(B)) \subset F^+(\text{Int}(\text{Cl}(V)))$. Hence, $F^-(V) \subset F^-(\text{Cl}(V)) \subset F^-(\text{Cl}(V)) \subset F^-(\text{Cl}(V))$. By Theorem 2.10 (7), $F$ is lower almost contra-$\gamma$-continuous.

(5) $\Rightarrow$ (6): Let $A$ be a subset of $X$ and $B = F(A)$. Then $A \subset F^+(B)$ and $\tau^{-}\text{Cl}(A) \subset \tau^{-}\text{Cl}(F^+(B)) \subset F^+(\theta-\text{sCl}(B))$. Therefore, we have $F(\tau^{-}\text{Cl}(A)) \subset F(F^+(\theta-\text{sCl}(B))) \subset \theta-\text{sCl}(B) = \theta-\text{sCl}(F(A))$. 

□
(6) ⇒ (5): Let \( B \) be any subset of \( Y \). Then we have \( F(\tau_\gamma Cl(F^+(B))) \subseteq \theta Cl(F(F^+(B))) \subseteq \theta Cl(B) \) and hence \( \tau_\gamma Cl(F^+(B)) \subseteq F^+(\theta - s Cl(B)) \).

**Theorem 2.16.** The following are equivalent for a multifunction \( F : (X, \tau) \to (Y, \sigma) \):

1. \( F \) is upper almost contra-\( \gamma \)-continuous (u.a.c.\( \gamma \)-c).
2. \( \tau_\gamma Cl(F^+(Int(K))) \subseteq F^+(K) \) for every semi-closed set \( K \) of \( Y \).
3. \( \tau_\gamma Cl(F^+(Int(s Cl(B)))) \subseteq F^+(s Cl(B)) \) for every subset \( B \) of \( Y \).
4. \( F^+(s Cl(B)) \subseteq \tau_\gamma - Int(F(Cl(s Cl(B)))) \) for every subset \( B \) of \( Y \).

**Proof.** (1) ⇒ (2): Let \( K \) be a semi-closed set of \( Y \). Then \( Y \setminus K \) is semi-open. By Theorem 2.9 (7), it follows that \( F^+(Y \setminus K) \subseteq \tau_\gamma - Int(F^+(Y \setminus Int(K))) \). Hence \( X \setminus F^-(K) \subseteq \tau_\gamma - Int(F^+(Y \setminus Int(K))) = \tau_\gamma - Int(X \setminus F^-(Int(K))) = X \setminus \tau_\gamma Cl(F^-(Int(K))) \). Hence, \( \tau_\gamma Cl(F^-(Int(K))) \subseteq F^-(K) \).

(2) ⇒ (3): Let \( B \) be any subset of \( Y \). Then \( s Cl(B) \) is semi-closed in \( Y \) and hence \( \tau_\gamma Cl(F^-(Int(s Cl(B)))) \subseteq F^-(s Cl(B)) \).

(3) ⇒ (4): Let \( B \) be any subset of \( Y \). Then we have \( X \setminus F^+(s Cl(B)) = F^-(s Cl(Y \setminus B)) \subseteq \tau_\gamma Cl(F^-(Int(s Cl(Y \setminus B)))) = \tau_\gamma Cl(F^-(Int(Y \setminus s Cl(B)))) = \tau_\gamma Cl(F^-(Y \setminus Cl(s Cl(B)))) = \tau_\gamma Cl(X \setminus F^+(Cl(s Cl(B)))) = X \setminus \tau_\gamma - Int(F^+(Cl(s Cl(B)))) \). Hence, \( F^+(s Cl(B)) \subseteq \tau_\gamma - Int(F^+(Cl(s Cl(B)))) \).

(4) ⇒ (1): Let \( V \) be any semi-open set of \( Y \). Then \( V = s Cl(V) \) and hence \( F^+(V) \subseteq \tau_\gamma - Int(F^+(Cl(V))) \). By Theorem 2.9 (7), \( F \) is upper almost contra-\( \gamma \)-continuous.

**Theorem 2.17.** The following are equivalent for a multifunction \( F : (X, \tau) \to (Y, \sigma) \):

1. \( F \) is lower almost contra-\( \gamma \)-continuous.
2. \( \tau_\gamma Cl(F^+(Int(K))) \subseteq F^+(K) \) for every semi-closed set \( K \) of \( Y \).
3. \( \tau_\gamma Cl(F^+(Int(s Cl(B)))) \subseteq F^+(s Cl(B)) \) for every subset \( B \) of \( Y \).
4. \( F^-(s Cl(B)) \subseteq \tau_\gamma - Int(F^-(Cl(s Cl(B)))) \) for every subset \( B \) of \( Y \).

**Proof.** The proof is similar to the proof of Theorem 2.16.

**Definition 2.18.** A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be almost contra-\( \gamma \)-continuous if \( f^{-1}(V) \) is \( \gamma \)-closed for each regular open set \( V \) in \( Y \).

**Remark 2.19.** By Theorems 2.15-2.17 we obtain new characterizations for almost contra-\( \gamma \)-continuous functions. For example:

**Corollary 2.20.** The following are equivalent for a function \( f : (X, \tau) \to (Y, \sigma) \):

1. \( f \) is almost contra-\( \gamma \)-continuous.
2. \( f^{-1}(V) \) is \( \gamma \)-open for every \( \theta \)-semi-open set \( V \) of \( Y \).
3. \( f^{-1}(K) \) is \( \gamma \)-closed for every \( \theta \)-semi-closed set \( K \) of \( Y \).
4. \( \tau_\gamma Cl(f^{-1}(Int(Cl(B)))) \subseteq f^{-1}(s Cl(B)) \) for every subset \( B \) of \( Y \).
5. \( \tau_\gamma Cl(f^{-1}(B)) \subseteq f^{-1}(\theta - s Cl(B)) \) for every subset \( B \) of \( Y \).
6. \( f(\tau_\gamma Cl(A)) \subseteq \theta Cl(f(A)) \) for every subset \( A \) of \( X \).
7. \( \tau_\gamma Cl(f^{-1}(Int(K))) \subseteq f^{-1}(K) \) for every semi-closed set \( K \) of \( Y \).
8. \( \tau_\gamma Cl(f^{-1}(Int(s Cl(B)))) \subseteq f^{-1}(s Cl(B)) \) for every subset \( B \) of \( Y \).
9. \( f^{-1}(s Cl(B)) \subseteq \tau_\gamma - Int(f^{-1}(Cl(s Cl(B)))) \) for every subset \( B \) of \( Y \).

**Theorem 2.21.** Let \( F : X \to Y \) be a multifunction. The following are equivalent:

1. \( F \) is upper (lower) almost contra-\( \gamma \)-continuous.
2. \( F^+(Cl(A))(F^-{f Cl(A)}) \) is \( \gamma \)-open in \( X \) for every \( A \in \beta O(Y) \).
3. \( F^+(Cl(A))(F^-{f Cl(A)}) \) is \( \gamma \)-open in \( X \) for every \( A \in SO(Y) \).

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(4) \( F^{-}(\text{Int} (\text{Cl} (A))) (F^{+}(\text{Int} (\text{Cl} (A)))) \) is \( \gamma \)-closed in \( X \) for every \( A \in \text{PO}(Y) \).

\[ \text{Proof. } (1) \Rightarrow (2): \text{Let } A \in \beta O(Y). \text{ By Theorem 2.22 of [15], } \text{Cl} (A) \text{ is regular closed. Thus, } F^{+} (\text{Cl} (A)) \text{ is } \gamma \text{-open.} \]

(2) \Rightarrow (3): It follows from the fact that \( SO(Y) \subset \beta O(Y) \).

(3) \Rightarrow (4): Let \( A \in \text{PO}(Y) \). This implies that \( Y \setminus \text{Int} (\text{Cl} (A)) \) is regular closed and semi-open. Hence, \( X \setminus F^{-} (\text{Int} (\text{Cl} (A))) = F^{+} (Y \setminus \text{Int} (\text{Cl} (A))) = F^{+} (\text{Cl} (Y \setminus \text{Int} (\text{Cl} (A)))) \) is \( \gamma \)-open. Thus, \( F^{-} (\text{Int} (\text{Cl} (A))) \) is \( \gamma \)-closed.

(4) \Rightarrow (1): Let \( A \in \text{RO}(Y) \). This implies that \( A \in \text{PO}(Y) \) and \( F^{-} (A) = F^{-} (\text{Int} (\text{Cl} (A))) \) is \( \gamma \)-closed. Hence, \( F \) is upper almost contra-\( \gamma \)-continuous. \[ \square \]

**Theorem 2.22.** Let \( F : X \rightarrow Y \) be a multifunction. If the graph multifunction of \( F \) is upper contra-\( \gamma \)-continuous, then \( F \) is upper contra-\( \gamma \)-continuous.

\[ \text{Proof. } \text{Let } G_{F} : X \rightarrow X \times Y \text{ be upper contra-\( \gamma \)-continuous and } x \in X. \text{ Let } A \text{ be any closed set of } Y \text{ containing } F(x). \text{ Since } X \times A \text{ is closed in } X \times Y \text{ and } G_{F}(x) \subset X \times A, \text{ there exists a } \gamma \text{-open set } U \text{ containing } x \text{ such that } G_{F}(U) \subset X \times A. \text{ By Lemma 1.6, } U \subset G_{F}^{+}(X \times A) = F^{+} (A) \text{ and } F(U) \subset A. \text{ Thus, } F \text{ is upper contra-\( \gamma \)-continuous.} \]

**Theorem 2.23.** Let \( F : X \rightarrow Y \) be a multifunction. If \( G_{F} : X \rightarrow X \times Y \) is lower contra-\( \gamma \)-continuous, then \( F \) is lower contra-\( \gamma \)-continuous.

\[ \text{Proof. } \text{Let } G_{F} \text{ be lower contra-\( \gamma \)-continuous and } x \in X. \text{ Let } A \text{ be any closed set in } Y \text{ such that } x \in F^{-} (A). \text{ This implies that } X \times A \text{ is closed in } X \times Y \text{ and } \]

\[ G_{F}(x) \cap (X \times A) = (\{x\} \times F(x)) \cap (X \times A) = \{x\} \times (F(x) \cap A) \neq \phi. \]

Since \( G_{F} \) is lower contra-\( \gamma \)-continuous, there exists an \( \gamma \)-open set \( U \) containing \( x \) such that \( U \subset G_{F}^{-}(X \times A) \). By Lemma 1.6, \( U \subset F^{-} (A) \). Thus, \( F \) is lower contra-\( \gamma \)-continuous. \[ \square \]

**References**


