RESEARCH ARTICLE

Coupled Fixed Point Theorems in Dislocated Quasi-metric Spaces

Duygu Akçay * and Cihangir Alaca †

* Department of Mathematics, Institute of Natural and Applied Sciences, Celal Bayar University, Muradiye Campus 45140 Manisa, Turkey.
† Department of Mathematics, Faculty of Science and Arts, Celal Bayar University, Muradiye Campus 45140 Manisa, Turkey.

(Received: 24 September 2012, Accepted: 22 October 2012)

In this paper, we prove a coupled coincidence fixed point theorem in dislocated quasi-metric spaces. Also, we give an example to validate our main theorem and some corollaries of the main result.

Keywords: Dislocated quasi-metric spaces; fixed point; coupled coincidence point.

AMS Subject Classification: 47H10, 54H25.

1. Introduction

The following definition of dislocated metric space and it’s fundamental properties was given by Hitzler and Seda [1].

Definition 1.1 Let X be a set and let \( d : X \times X \rightarrow [0, \infty) \) be a function, called a distance function. Consider the following conditions:

(M-i) For all \( x \in X \), \( d(x, x) = 0 \),
(M-ii) For all \( x, y \in X \), if \( d(x, y) = d(y, x) = 0 \), then \( x = y \),
(M-iii) For all \( x, y \in X \), \( d(x, y) = d(y, x) \),
(M-iv) For all \( x, y, z \in X \), \( d(x, y) \leq d(x, z) + d(z, y) \),
(M-iv') For all \( x, y, z \in X \), \( d(x, y) \leq \max\{d(x, z), d(z, y)\} \).

If \( d \) satisfies conditions (M-i) to (M-iv), then it is called a metric. If it satisfies conditions (M-i), (M-ii) and (M-iv), it is called a quasi-metric. If it satisfies (M-ii), (M-iii) and (M-iv), we will call it a dislocated metric (or simply d-metric). If it satisfies conditions (M-ii) and (M-iv), it is called a dislocated quasi-metric (or simply dq-metric). If a metric \( d \) satisfies the strong triangle inequality (M-iv'), then it is called an ultrametric.

The study of partial metric spaces and generalized ultrametric spaces have applications in theoretical computer science had been studied by Matthews [2]. Hitzler and Seda [1] introduced the concept of dislocated metric space as a generalization of metrics where self-distances need not be zero. They also proved a generalized version of Banach contraction mapping principle which was applied to obtain fixed point semantics for logic programs. Recently, many authors [3–14] studied different properties and fixed point results of dislocated (fuzzy or) metric spaces. Lakshmikantham and Ćirić [15] introduced coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces.

---

1 Corresponding author
Email: cihangiralaca@yahoo.com.tr
In the present paper, we define the notion of a coupled coincidence fixed point and introduce a coupled coincidence fixed point theorem in dislocated quasi-metric spaces. Also, we give an example to validate our main theorem and some corollaries of the main result.

2. Fixed Point Results

Definition 2.1 A sequence \((x_n)\) in \(d\)-metric space \((X, d)\) is called Cauchy if for all \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(m, n \geq n_0\), \(d(x_m, x_n) < \varepsilon\) or \(d(x_n, x_m) < \varepsilon\). Replacing \(d(x_m, x_n) < \varepsilon\) and \(d(x_n, x_m) < \varepsilon\) in this definition by \(\max\{d(x_m, x_n), d(x_n, x_m)\} < \varepsilon\), the sequence \((x_n)\) in \(d\)-metric space \((X, d)\) is called ‘bi’ Cauchy.

Definition 2.2 A sequence \((x_n)\) dislocated quasi-converges (for short dq-converges) to \(x\) if

\[
\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0.
\]

In this case \(x\) is called dq-limit of \((x_n)\).

Definition 2.3 A dq-metric space \((X, d)\) is called complete if every Cauchy sequences in it is dq-convergent.

Definition 2.4 Let \((X, d_1)\) and \((Y, d_2)\) be a dq-metric spaces and let \(f : X \to Y\) be a function. Then \(f\) is continuous if for each sequence \((x_n)\) which is \(d_1\)-convergent to \(x_0\) in \(X\), the sequence \((f(x_n))\) is \(d_2\)-convergent to \(f(x_0)\) in \(Y\).

Definition 2.5 Let \((X, d)\) be a dq-metric spaces. A map \(f : X \to X\) is called contraction if there exists \(0 \leq \lambda < 1\) such that

\[
d(f(x), f(y)) \leq \lambda d(x, y)
\]

for all \(x, y \in X\).

Lemma 2.6 Every subsequence of dq-convergent sequence to a point \(x_0\) is dq-convergent to \(x_0\).

Definition 2.7 An element \((x, y) \in X \times X\) is called a coupled fixed point of a mapping \(F : X \times X \to X\) if \(F(x, y) = x\) and \(F(y, x) = y\).

Definition 2.8 An element \((x, y) \in X \times X\) is called a coupled coincidence point of a mappings \(F : X \times X \to X\) and \(g : X \to X\) if \(F(x, y) = gx\) and \(F(y, x) = gy\).

Definition 2.9 Let \(X\) be a nonempty set. Then we say that the mappings \(F : X \times X \to X\) and \(g : X \to X\) are commutative if \(gF(x, y) = F(gx, gy)\).

Lemma 2.10 Let \((X, d)\) be a dq-metric spaces. Let \(F : X \times X \to X\) and \(g : X \to X\) be two mappings such that

\[
d(F(x, y), F(u, v)) \leq \lambda [d(gx, gu) + d(gy, gv)]
\]

for all \(x, y, u, v \in X\). Assume that \((x, y)\) is a coupled coincidence point of the mappings \(F\) and \(g\). If \(\lambda \in \left[0, \frac{1}{2}\right]\), then

\[
F(x, y) = gx = gy = F(y, x).
\]

Proof Since \((x, y)\) is a coupled coincidence point of the mappings \(F\) and \(g\), we have \(F(x, y) = gx\) and \(F(y, x) = gy\). Assume that \(gx \neq gy\). Then by (1), we get

\[
d(gx, gy) = d(F(x, y), F(y, x)) \leq \lambda [d(gx, gy) + d(gy, gx)].
\]
Also by (1), we have

\[ d(gy, gx) = d(F(y, x), F(x, y)) \leq \lambda [d(gy, gx) + d(gx, gy)]. \]

Therefore

\[ d(gx, gy) + d(gy, gx) \leq 2\lambda [d(gx, gy) + d(gy, gx)]. \]

Since \( 2\lambda < 1 \), we get

\[ d(gx, gy) + d(gy, gx) < d(gx, gy) + d(gy, gx) \]

which is a contradiction. So \( gx = gy \) and hence

\[ F(x, y) = gx = gy = F(y, x). \]

\[ \square \]

**Theorem 2.11** Let \((X, d)\) be a \(dq\)-metric spaces. Let \( F : X \times X \to X \) and \( g : X \to X \) be two continuous mappings such that

\[ d(F(x, y), F(u, v)) \leq \lambda [d(gx, gu) + d(gy, gv)] \]

for all \( x, y, u, v \in X \). Assume that \( F \) and \( g \) satisfy the following conditions:

(i) \( F(X \times X) \subseteq g(X) \).

(ii) \( g(X) \) is complete \( dq\)-metric,

(iii) \( g \) commutes with \( F \).

If \( \lambda \in (0, 1) \), then there is a unique \( x \) in \( X \) such that \( gx = F(x, x) = x \).

**Proof** Let \( x_0, y_0 \in X \). By condition (i), Since \( F(X \times X) \subseteq g(X) \), we can choose \( x_1, y_1 \in X \) such that \( gx_1 = F(x_0, y_0) \) and \( gy_1 = F(y_0, x_0) \). Again since \( F(X \times X) \subseteq g(X) \), we can choose \( x_2, y_2 \in X \) such that \( gx_2 = F(x_1, y_1) \) and \( gy_2 = F(y_1, x_1) \). Continuing this process we can construct two sequences \((x_n)\) and \((y_n)\) in \( X \) such that \( gx_{n+1} = F(x_n, y_n) \) and \( gy_{n+1} = F(y_n, x_n) \). For \( n \in N \), by (2), we have

\[ d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \leq \lambda [d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)]. \]

From

\[ d(gx_{n-1}, gx_n) = d(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1})) \leq \lambda [d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})] \]

and

\[ d(gy_{n-1}, gy_n) = d(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1})) \leq \lambda [d(gy_{n-2}, gy_{n-1}) + d(gx_{n-2}, gx_{n-1})] \]

we have

\[ d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n) \leq 2\lambda [d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})] \]
holds for all $n \in N$. Thus we get that
\[
d(gx_n, gx_{n+1}) \leq \lambda [d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)]
\leq 2\lambda^2 [d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})]
\leq \cdots
\leq \frac{1}{2}(2\lambda)^n [d(gx_0, gx_1) + d(gy_0, gy_1)].
\]

Thus for each $n \in N$ we have
\[
d(gx_n, gx_{n+1}) \leq \frac{1}{2}(2\lambda)^n [d(gx_0, gx_1) + d(gy_0, gy_1)].
\tag{3}
\]

Let $m, n \in N$ with $m > n$. By axiom (M-iv) of the definition of dislocated quasi metric spaces, we have;
\[
d(gx_n, gx_m) \leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m)
\]
since $2\lambda < 1$, by (3), we get that
\[
d(gx_n, gx_m) \leq \frac{1}{2} \left( \sum_{i=n}^{m-1} (2\lambda)^i \right) [d(gx_0, gx_1) + d(gy_0, gy_1)]
\leq \frac{(2\lambda)^n}{2(1-2\lambda)} [d(gx_0, gx_1) + d(gy_0, gy_1)].
\]

Letting $m, n \to \infty$, we have
\[
\lim_{m,n \to \infty} d(x_n, gx_m) = 0.
\]

Thus $(gx_n)$ is a Cauchy sequence in $g(X)$. Similarly, we may show that $(gy_n)$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete dq-metric, we get that $(gx_n)$ and $(gy_n)$ are dq-convergent to some $x \in X$ and $y \in X$, respectively. Since $F$ and $g$ are continuous, and also, $F$ and $g$ are commute, we have
\[
 ggx_{n+1} = g(F(x_n, y_n)) = F(gx_n, gy_n),
\]
and
\[
 ggy_{n+1} = g(F(y_n, x_n)) = F(gy_n, gx_n).
\]

Thus we get that
\[
gx = g(\lim_{n \to \infty} gx_n) = \lim_{n \to \infty} g(F(x_{n-1}, y_{n-1})) = \lim_{n \to \infty} F(gx_{n-1}, gy_{n-1})
= F(\lim_{n \to \infty} gx_{n-1}, \lim_{n \to \infty} gy_{n-1}) = F(x, y).
\]

Hence $gx = F(x, y)$. Similarly, we may show that $gy_n = F(y, x)$. By Lemma 2.10, $(x, y)$ is a coupled fixed point of the mappings $F$ and $g$. So
\[
 gx = F(x, y) = F(y, x) = gy.
\]
Since \((gx_{n+1})\) is subsequence of \((gx_n)\) we have that \((gx_{n+1})\) is \(dq\)-convergent to \(x\). Thus
\[
d(gx_{n+1}, gx) = d(gx_{n+1}, F(x, y)) = d(F(x_n, y_n), F(x, y)) \\
\leq \lambda [d(gx_n, gx) + d(gy_n, gy)].
\]
Letting \(n \to \infty\) and using the fact that \(d\) is continuous on its variables, we get that
\[
d(x, gx) \leq \lambda [d(x, gx) + d(y, gy)].
\]
Similarly, we may show that
\[
d(y, gy) \leq \lambda [d(x, gx) + d(y, gy)].
\]
Thus
\[
d(x, gx) + d(y, gy) \leq 2\lambda [d(x, gx) + d(y, gy)].
\]
Since \(2\lambda < 1\), the last inequality happens only if \(d(x, gx) = 0\) and \(d(y, gy) = 0\). Similarly,
\[
d(gx, gx_{n}) = d(F(x, y), gx_{n+1}) = d(F(x, y), F(x_n, y_n)) \\
\leq \lambda [d(gx, gx_n) + d(gy, gy_n)].
\]
Letting \(n \to \infty\) and using the fact that \(d\) is continuous on its variables, we get that:
\[
d(gx, x) \leq \lambda [d(gx, x) + d(gy, y)].
\]
Similarly, we may show that
\[
d(gy, y) \leq \lambda [d(gx, x) + d(gy, y)].
\]
Thus
\[
d(gx, x) + d(gy, y) \leq 2\lambda [d(gx, x) + d(gy, y)].
\]
Since \(2\lambda < 1\), the last inequality happens only if \(d(gx, x) = 0\) and \(d(gy, y) = 0\). From Definition 2.1 (M-ii), we have \(x = gx\) and \(y = gy\). Thus we get
\[
gx = F(x, x) = x.
\]
To prove the uniqueness, let \(z \in X\) with \(z \neq x\) such that
\[
z = gz = F(z, z).
\]
Then
\[
d(x, z) = d(F(x, x), F(z, z)) \leq \lambda [d(gx, gz) + d(gx, gz)] \\
= 2\lambda d(gx, gz).
\]
Since \(2\lambda < 1\), we get \(d(x, z) < d(x, z)\), which is a contradiction. Thus \(F\) and \(g\) have a unique common fixed point. \(\blacksquare\)
Corollary 2.1 Let \((X, d)\) be a complete dq-metric spaces. Let \(F : X \times X \rightarrow X\) be a continuous mapping such that
\[
d(F(x, y), F(u, v)) \leq \lambda [d(x, u) + d(y, v)]
\]
for all \(x, y, u, v \in X\). If \(\lambda \in [0, \frac{1}{2})\), then there is a unique \(x\) in \(X\) such that \(F(x, x) = x\).

Proof Define \(g : X \rightarrow X\) by \(g(x) = x\). Then \(F\) and \(g\) satisfy all the hypothesis of theorem. Hence the result follows.

Example 2.12 Let \(X = [0, 1]\). Define \(d : X \times X \rightarrow [0, \infty)\) by \(d(x, y) = |x - y| + |x|\) for all \(x, y \in X\). Then \((X, d)\) is a complete dq-metric spaces. Define a map \(F : X \times X \rightarrow X\) by \(F(x, y) = \frac{1}{6}xy\) for all \(x, y \in X\). Also, define \(g : X \rightarrow X\) by \(g(x) = \frac{1}{2}x\) for all \(x \in X\). Since 
\[
|xy - uv| \leq |x - u| + |y - v| \quad \text{and} \quad |xy| \leq |x| + |y|
\]
holds for all \(x, y, u, v \in X\). We have
\[
\begin{align*}
d(F(x, y), F(u, v)) &= \frac{1}{6} |xy - uv| + \frac{1}{6} |xy| \\
&\leq \frac{1}{6} [ |x - u| + |y - v| ] + \frac{1}{6} [ |x| + |y| ] \\
&\leq \frac{1}{3} [d(gx, gu) + d(gy, gv)]
\end{align*}
\]
holds for all \(x, y, u, v \in X\). It is an easy matter to see that \(F\) and \(g\) satisfy all the hypothesis of Theorem. Thus \(F\) and \(g\) have a unique common fixed point. Here \(F(0, 0) = g(0) = 0\).

3. Acknowledgements

The authors would like to thank the Editor-In-Chief Professor A. M. Zahran and the Executive Editor Dr. A. Ghareeb for good cooperation and the referee(s) for giving useful suggestions in the improvement of this paper.

References
