RESEARCH ARTICLE

γgb-Closed Sets in Topological Spaces

Hariwan Z. Ibrahim

Department of Mathematics, Faculty of Science, University of Zakho, Kurdistan-Region, Iraq.

(Received: 3 September 2012, Accepted: 30 October 2012)

In this paper, we introduce a new class of sets called γ-generalized b-closed sets in topological spaces (briefly γgb-closed set). Also we study some of its basic properties and investigate the relations between the associated topology.

Keywords: γ-open sets; γgb-closed sets.

AMS Subject Classification: 54A40.

1. Introduction

Vidhya and Parimelazhagan [1] introduced a new class of generalized closed sets in a topological space its called g*b-closed sets. Levine [2] introduced the notion of generalized closed (briefly g-closed) sets in topological spaces and showed that compactness, countably compactness, para compactness and normality etc are all g-closed hereditary. Dontchev [3], Maki, Devi and Balachandran [4], Mashhour, Abd-El-Monsef and El-Deeb [5], Andrijevic [6] and Nagaveni [7] introduced and investigated the concept of generalized semi-preclosed sets, generalized α-closed sets, preclosed sets, semi-preclosed sets and weakly generalized closed sets respectively. Andrijevic [8], introduced a class of generalized open sets in a topological space called b-open sets. Omari and Noorani [9] introduced and studied the concept of generalized b-closed sets (briefly gb-closed) in topological spaces.

In this paper, we introduce a new class of sets called γgb-closed set and study some of its basic properties and investigate the relations between the associated topology.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) stand for topological spaces with no separation axioms assumed unless otherwise stated. For a subset A of X, the closure of A and the interior of A will be denoted by cl(A) and int(A), respectively.

Let (X, τ) be a topological space and A a subset of X. An operation γ [10] on a topology τ is a mapping from τ into power set P(X) of X such that V ⊆ γ(V) for each V ∈ τ, where γ(V) denotes the value of γ at V. A subset A of X with an operation γ on τ is called γ-open [10] if for each x ∈ A, there exists an open set U such that x ∈ U and γ(U) ⊆ A. Then, τγ denotes the set of all γ-open set in X. Clearly τγ ⊆ τ. Complements of γ-open sets are called γ-closed. The γ-closure [10] of a subset A of X with an operation γ on τ is denoted by τγ-cl(A) and is defined to be the intersection of all γ-closed sets containing A. A topological X with an operation γ on τ is said to be γ-regular [10] if for
each $x \in X$ and for each open neighborhood $V$ of $x$, there exists an open neighborhood $U$ of $x$ such that $\gamma(U)$ contained in $V$. It is also to be noted that $\tau = \tau_\gamma$ if and only if $X$ is a $\gamma$-regular space [10].

Definition 2.1 [5] A subset $A$ of a topological space $(X, \tau)$ is called a pre-open set if $A \subseteq \text{int}(\text{cl}(A))$ and preclosed set if $\text{cl}(\text{int}(A)) \subseteq A$.

Definition 2.2 [11] A subset $A$ of a topological space $(X, \tau)$ is called a semi-open set if $A \subseteq \text{cl}(\text{int}(A))$ and semi-closed set if $\text{int}(\text{cl}(A)) \subseteq A$.

Definition 2.3 [12] A subset $A$ of a topological space $(X, \tau)$ is called an $\alpha$-open set if $A \subseteq \text{int}(\text{cl}(A)))$ and an $\alpha$-closed set if $\text{cl}(\text{int}(A))) \subseteq A$.

Definition 2.4 [6] A subset $A$ of a topological space $(X, \tau)$ is called a semi-preopen set ($\beta$-open set) if $A \subseteq \text{cl}(\text{int}(A))$ and semi-preclosed set if $\text{int}(\text{cl}(A))) \subseteq A$.

Definition 2.5 [8] A subset $A$ of a topological space $(X, \tau)$ is called a $b$-open set if $A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$ and $b$-closed set if $\text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A)) \subseteq A$.

Definition 2.6 [2] A subset $A$ of a topological space $(X, \tau)$ is called a generalized closed set (briefly $g$-closed) if $\text{cl}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 2.7 [4] A subset $A$ of a topological space $(X, \tau)$ is called a generalized $\alpha$-closed (briefly $g\alpha$-closed) if $\text{acl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$.

Definition 2.8 [13] A subset $A$ of a topological space $(X, \tau)$ is called an $\alpha$-generalized closed (briefly $\alpha g$-closed) if $\text{acl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 2.9 [14] A subset $A$ of a topological space $(X, \tau)$ is called a generalized preclosed (briefly $gp$-closed) if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 2.10 [15] A subset $A$ of a topological space $(X, \tau)$ is called a generalized semiclosed (briefly $gs$-closed) if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 2.11 [16] A subset $A$ of a topological space $(X, \tau)$ is called a generalized semiclosed (briefly $sg$-closed) if $\text{scp}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 2.12 [9] A subset $A$ of a topological space $(X, \tau)$ is called a generalized $b$-closed set (briefly $gb$-closed) if $\text{bcl}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 2.13 [7] A subset $A$ of a topological space $(X, \tau)$ is called a weakly generalized closed set (briefly $wg$-closed) if $\text{cl}(\text{int}(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 2.14 [3] A subset $A$ of a topological space $(X, \tau)$ is called a generalized semi-preclosed set (briefly $gsp$-closed) if $\text{scp}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 2.15 [17] A subset $A$ of a topological space $(X, \tau)$ is called a generalized $^*$closed set (briefly $g^*$-closed) if $\text{clf}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $g$-open in $X$.

Definition 2.16 [1] A subset $A$ of a topological space $(X, \tau)$ is called a $g^*b$-closed set if $\text{bcl}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $g$-open in $X$.

3. $\gamma lb$-Closed Sets

In this section we introduce the concept of $\gamma lb$-closed sets in topological space and we investigate the group of structure of the set of all $\gamma lb$-closed sets.

Definition 3.1 A subset $A$ of a topological space $(X, \tau)$ is called a $\gamma lb$-closed set if $\text{bcl}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\gamma$-open in $X$.

Theorem 3.2 Every $\gamma$-closed set is $\gamma lb$-closed set.
$\gamma$-gb-Closed Sets in Topological Spaces

Let $A$ be a $\gamma$-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since $A$ is $\gamma$-closed, $\gamma^{-}\text{cl}(A) = A$. Since $\text{bcl}(A) \subseteq \text{cl}(A) \subseteq \gamma^{-}\text{cl}(A) = A$. Therefore $\text{bcl}(A) \subseteq U$. Hence $A$ is a $\gamma$-gb-closed set in $X$.

The converse of the above theorem need not be true as seen from the following example.

Example 3.3 Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Define an operation $\gamma$ on $\tau$ by

$\gamma(A) = \begin{cases} A & \text{if } A = \{b\} \text{ or } \{a, c\}, \\ X & \text{otherwise}. \end{cases}$

Let $A = \{a\}$, since the only $\gamma$-open supersets of $A$ are $\{a, c\}$ and $X$, then $A$ is a $\gamma$gb-closed set but not a $\gamma$-closed set of $(X, \tau)$.

Theorem 3.4 Every closed set is $\gamma$gb-closed set.

Proof Let $A$ be a closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since $A$ is closed, $\text{cl}(A) = A$. Therefore $\text{bcl}(A) \subseteq U$. Hence $A$ is a $\gamma$gb-closed set in $X$.

The converse of the above theorem need not be true as seen from the following example.

Example 3.5 Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{b\}, \{b, c\}\}$ and $\gamma(A) = A$ for all $A \in \tau$. Let $A = \{a, b\}$. Here $A$ is a $\gamma$gb-closed set but not a closed set of $(X, \tau)$.

Theorem 3.6 Every b-closed set is $\gamma$gb-closed set.

Proof Let $A$ be a b-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since $A$ is b-closed, $\text{bcl}(A) = A$. Therefore $\text{bcl}(A) \subseteq U$. Hence $A$ is a $\gamma$gb-closed set in $X$.

The converse of the above theorem need not be true as seen from the following example.

Example 3.7 Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{b\}, \{a, b\}\}$ and $\gamma(A) = A$ for all $A \in \tau$. Let $A = \{b, c\}$. Here $A$ is a $\gamma$gb-closed set but not a b-closed set of $(X, \tau)$.

Theorem 3.8 Every gb-closed set is $\gamma$gb-closed set.

Proof Let $A$ be a gb-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since every $\gamma$-open set is open and $A$ is gb-closed, $\text{bcl}(A) \subseteq U$. Hence $A$ is $\gamma$gb-closed.

The converse of the above theorem need not be true as seen from the following example.

Example 3.9 From Example 3.3, let $A = \{a\}$. Here $A$ is a $\gamma$gb-closed set but not a gb-closed set of $(X, \tau)$.

Theorem 3.10 Every $\alpha$-closed set is $\gamma$gb-closed set.

Proof Let $A$ be a $\alpha$-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since $A$ is $\alpha$-closed, $\text{bcl}(A) \subseteq \alpha\text{cl}(A) = A \subseteq U$. Therefore $\text{bcl}(A) \subseteq U$. Hence $A$ is $\gamma$gb-closed.

The converse of the above theorem need not be true as seen from the following example.

Example 3.11 From Example 3.7, let $A = \{b, c\}$. Here $A$ is a $\gamma$gb-closed set but not an $\alpha$-closed set of $(X, \tau)$.

Theorem 3.12 Every semi-closed set is $\gamma$gb-closed set.

Proof Let $A$ be a semi-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since $A$ is semi-closed, $\text{bcl}(A) \subseteq \text{sc}(A) = A \subseteq U$. Therefore $\text{bcl}(A) \subseteq U$. Hence $A$ is a $\gamma$gb-closed set in $X$.

The converse of the above theorem need not be true as seen from the following example.

Example 3.13 Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{b, c\}\}$ and $\gamma(A) = A$ for all $A \in \tau$. Let $A = \{b\}$. Here $A$ is a $\gamma$gb-closed set but not a semi-closed set of $(X, \tau)$. 

Theorem 3.14 Every preclosed set is $\gamma$gb-closed set.

Proof Let $A$ be a preclosed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since $A$ is preclosed,\[ bcl(A) \subseteq pcl(A) = A \subseteq U. \] Therefore $bcl(A) \subseteq U$. Hence $A$ is $\gamma$gb-closed set in $X$. \hfill \blacksquare \\

The converse of the above theorem need not be true as seen from the following example.

Example 3.15 From Example 3.5, let $A = \{a, b\}$. Here $A$ is a $\gamma$gb-closed set but not a pre-closed set of $(X, \tau)$.

Theorem 3.16 Every $g^*$-closed set is $\gamma$gb-closed set.

Proof Let $A$ be a $g^*$-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since every open set is $g$-open and $A$ is $g^*$-closed, $bcl(A) \subseteq cl(A) \subseteq U$. Therefore $bcl(A) \subseteq U$. Hence $A$ is a $\gamma$gb-closed set in $X$. \hfill \blacksquare \\

The converse of the above theorem need not be true as seen from the following example.

Example 3.17 Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{b\}\}$ and $\gamma(A) = A$ for all $A \in \tau$. Let $A = \{a\}$. Here $A$ is a $\gamma$gb-closed set but not a $g^*$-closed set of $(X, \tau)$.

Theorem 3.18 Every $\alpha$-closed set is $\gamma$gb-closed set.

Proof Let $A$ be a $\alpha$-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since every open set is $\alpha$-open and $A$ is $\alpha$-closed, $bcl(A) \subseteq \alpha cl(A) \subseteq U$. Therefore $bcl(A) \subseteq U$. Hence $A$ is a $\gamma$gb-closed set in $X$. \hfill \blacksquare \\

The converse of the above theorem need not be true as seen from the following example.

Example 3.19 From Example 3.5, let $A = \{a, b\}$. Here $A$ is a $\gamma$gb-closed set but not a $\alpha$-closed set of $(X, \tau)$.

Theorem 3.20 Every $g$-closed set is $\gamma$gb-closed set.

Proof Let $A$ be a $g$-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since $A$ is $g$-closed, $bcl(A) \subseteq cl(A) \subseteq U$. Hence $A$ is $\gamma$gb-closed. \hfill \blacksquare \\

The converse of the above theorem need not be true as seen from the following example.

Example 3.21 From Example 3.5, let $A = \{c\}$. Here $A$ is a $\gamma$gb-closed set but not a $g$-closed set of $(X, \tau)$.

Theorem 3.22 Every $g^*$b-closed set is $\gamma$gb-closed set.

Proof Let $A$ be a $g^*$b-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since every open set is $g$-open and $A$ is $g^*$b-closed, $bcl(A) \subseteq U$. Hence $A$ is $\gamma$gb-closed. \hfill \blacksquare \\

The converse of the above theorem need not be true as seen from the following example.

Example 3.23 Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\}\}$ and $\gamma(A) = X$ for all $A \in \tau$. Let $A = \{a, b\}$. Here $A$ is a $\gamma$gb-closed set but not a $g^*$b-closed set of $(X, \tau)$.

Theorem 3.24 Every $\alpha g$-closed set is $\gamma$gb-closed set.

Proof Let $A$ be an $\alpha g$-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since every $\gamma$-open set is open and $A$ is $\alpha g$-closed, $bcl(A) \subseteq \alpha cl(A) \subseteq U$. Hence $A$ is $\gamma$gb-closed. \hfill \blacksquare \\

The converse of the above theorem need not be true as seen from the following example.

Example 3.25 Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{c\}\}$ and $\gamma(A) = X$ for all $A \in \tau$. Let $A = \{c\}$. Here $A$ is a $\gamma$gb-closed set but not an $\alpha g$-closed set of $(X, \tau)$.

Theorem 3.26 Every gp-closed set is $\gamma$gb-closed set.

Proof Let $A$ be a gp-closed set in $X$ such that $A \subseteq U$, where $U$ is $\gamma$-open. Since every $\gamma$-open set is open and $A$ is gp-closed, $bcl(A) \subseteq pcl(A) \subseteq U$. Hence $A$ is $\gamma$gb-closed. \hfill \blacksquare
The converse of the above theorem need not be true as seen from the following example.

**Example 3.27** From Example 3.25, let \( A = \{ c \} \). Here \( A \) is a \( \gamma \)-gb-closed set but not a gp-closed set of \((X, \tau)\).

**Theorem 3.28** Every gs-closed set is \( \gamma \)-gb-closed set.

**Proof** Let \( A \) be a gs-closed set in \( X \) such that \( A \subseteq U \), where \( U \) is \( \gamma \)-open. Since every \( \gamma \)-open set is open and \( A \) is gs-closed, \( bcl(A) \subseteq scl(A) \subseteq U \). Hence \( A \) is \( \gamma \)-gb-closed.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.29** From Example 3.25, let \( A = \{ c \} \). Here \( A \) is a \( \gamma \)-gb-closed set but not a gs-closed set of \((X, \tau)\).

**Theorem 3.30** Every sg-closed set is \( \gamma \)-gb-closed set.

**Proof** Let \( A \) be a sg-closed set in \( X \) such that \( A \subseteq U \), where \( U \) is \( \gamma \)-open. Since every open set is semi-open and \( A \) is sg-closed, \( bcl(A) \subseteq scl(A) \subseteq U \). Hence \( A \) is \( \gamma \)-gb-closed.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.31** From Example 3.25, let \( A = \{ c \} \). Here \( A \) is a \( \gamma \)-gb-closed set but not a sg-closed set of \((X, \tau)\).

**Remark 3.32** We have the following implications but none of this implications are reversible.

\[
\begin{align*}
&\text{closed} \quad \gamma\text{-closed} \quad \text{sg-closed} \quad \text{gs-closed} \quad \text{ga-closed} \\
&\quad \text{b-closed} \quad \gamma\text{gb-closed} \quad \text{gp-closed} \\
&\quad \text{gb-closed} \quad \alpha\text{-closed} \quad \text{semi-closed} \quad \text{ag-closed} \quad \text{g-closed} \\
&\quad \text{preclosed} \quad g^*\text{-closed} \\
&\quad \quad \quad \quad \quad \quad \text{g^*b-closed}
\end{align*}
\]

**Theorem 3.33** If \( X \) is a \( \gamma \)-regular space then every \( \gamma \)-gb-closed is gsp-closed.

**Proof** Let \( A \) be a \( \gamma \)-gb-closed set in \( X \) such that \( A \subseteq U \), where \( U \) is open. Since every open set is \( \gamma \)-open and \( A \) is \( \gamma \)-gb-closed, \( spcl(A) \subseteq bcl(A) \subseteq U \). Hence \( A \) is gsp-closed.

**Theorem 3.34** If \( A \) is open and wg-closed then \( A \) is a \( \gamma \)-gb-closed.

**Proof** Let \( A \) be an open set and a wg-closed set in \( X \) such that \( A \subseteq U \), where \( U \) is \( \gamma \)-open. Since every \( \gamma \)-open set is open and \( A \) is wg-closed, \( bcl(A) \subseteq cl(A) = cl(int(A)) \subseteq U \). Hence \( A \) is \( \gamma \)-gb-closed.

**Theorem 3.35** If \( A \) is \( \gamma \)-open and \( \gamma \)-gb-closed then \( A \) is b-closed.

**Proof** Let \( A \) be \( \gamma \)-open and \( \gamma \)-gb-closed. As \( A \subseteq A \), we have \( bcl(A) \subseteq A \), also \( A \subseteq bcl(A) \), therefore \( bcl(A) = A \). Hence \( A \) is b-closed.

**Theorem 3.36** The intersection of a \( \gamma \)-gb-closed set and a \( \gamma \)-closed set is always \( \gamma \)-gb-closed.

**Proof** Let \( A \) be \( \gamma \)-gb-closed and \( F \) be \( \gamma \)-closed. Assume that \( U \) is \( \gamma \)-open set such that \( A \cap F \subseteq U \), set \( G = X \setminus F \). Then \( A \subseteq U \cup G \), since \( G \) is \( \gamma \)-open, then \( U \cup G \) is \( \gamma \)-open and since \( A \) is \( \gamma \)-gb-closed, then
\( bcl(A) \subseteq U \cup G. \) Now, \( bcl(A \cap F) \subseteq bcl(A) \cap bcl(F) = bcl(A) \cap F \subseteq (U \cup G) \cap F = (U \cap F) \cup (G \cap F) = (U \cap F) \cup \phi \subseteq U. \) Hence \( A \cap F \) is \( \gamma_{gb} \)-closed.

The union of two \( \gamma_{gb} \)-closed sets need not be \( \gamma_{gb} \)-closed in general. It is shown by the following example.

Example 3.37 Let \( X = \{a, b, c\} \) with the topology \( \tau = \{X, \phi, \{a, b\}\} \) and \( \gamma(A) = A \) for all \( A \in \tau. \) Let \( A = \{a\} \) and \( B = \{b\}. \) Here \( A \) and \( B \) are \( \gamma_{gb} \)-closed but \( A \cup B = \{a, b\} \) is not \( \gamma_{gb} \)-closed, since \( \{a, b\} \) is \( \gamma \)-open and \( bcl(\{a, b\}) = X. \)

Definition 3.38 \[9\] Let \( A \) be a subset of a space \( X. \) A point \( x \in X \) is said to be a \( b \)-limit point of \( A \) if for each \( b \)-open set \( \gamma \) containing \( x, \) we have \( U \cap (A \setminus \{x\}) \neq \phi. \) The set of all \( b \)-limit points of \( A \) is called the \( b \)-derived set of \( A \) and is denoted by \( D_b(A). \)

Corollary 3.1 \[9\] If \( D(A) \subseteq D_b(A) \) for every subset \( A \) of \( X. \) Then for any subsets \( F \) and \( B \) of \( X, \) we have \( bCl(F \cup B) = bCl(F) \cup bCl(B). \)

Theorem 3.39 If \( D(A) \subseteq D_b(A) \) for every subset \( A \) of \( X. \) Then the finite union of \( \gamma_{gb} \)-closed sets is always a \( \gamma_{gb} \)-closed set.

Proof Let \( A \) and \( B \) be two \( \gamma_{gb} \)-closed sets, and let \( A \cup B \subseteq U, \) where \( U \) is \( \gamma \)-open. Since \( A \) and \( B \) are \( \gamma_{gb} \)-closed sets, therefore \( bcl(A) \subseteq U \) and \( bcl(B) \subseteq U \) implies \( bcl(A) \cup bcl(B) \subseteq U. \) So, by Corollary 3.1 we have \( bcl(A) \cup bcl(B) = bcl(A \cup B). \) Therefore \( bcl(A \cup B) \subseteq U. \) Hence \( A \cup B \) is a \( \gamma_{gb} \)-closed set.

The intersection of two \( \gamma_{gb} \)-closed sets need not be \( \gamma_{gb} \)-closed in general. It is shown by the following example.

Example 3.40 Let \( X = \{a, b, c\} \) with the topology \( \tau = \{X, \phi, \{a, b\}\} \) and \( \gamma(A) = A \) for all \( A \in \tau. \) Let \( A = \{a, c\} \) and \( B = \{b, c\}. \) Clearly, \( A \) and \( B \) are \( \gamma_{gb} \)-closed sets, since \( X \) is their only \( \gamma \)-open superset. But \( C = \{c\} = A \cap B \) is not \( \gamma_{gb} \)-closed, since \( C \subseteq \{c\} \in \tau \) and \( bcl(C) = X \not\subseteq \{c\}. \)

Theorem 3.41 If a subset \( A \) of \( X \) is \( \gamma_{gb} \)-closed and \( A \subseteq B \subseteq bcl(A), \) then \( B \) is a \( \gamma_{gb} \)-closed set in \( X. \)

Proof Let \( A \) be a \( \gamma_{gb} \)-closed set such that \( A \subseteq B \subseteq bcl(A). \) Let \( U \) be a \( \gamma \)-open set of \( X \) such that \( B \subseteq U. \) Since \( A \) is \( \gamma_{gb} \)-closed, we have \( bcl(A) \subseteq U. \) Now \( bcl(A) \subseteq bcl(B) \subseteq bcl(bcl(A)) = bcl(A) \subseteq U. \) That is \( bcl(B) \subseteq U. \) where \( U \) is \( \gamma \)-open. Therefore \( B \) is a \( \gamma_{gb} \)-closed set in \( X. \)

The converse of the above theorem need not be true in general as seen from the following example.

Example 3.42 Let \( X = \{a, b, c\} \) with the topology \( \tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}\} \) and \( \gamma(A) = A \) for all \( A \in \tau. \) Let \( A = \{b\} \) and \( B = \{b, c\}. \) Then \( A \) and \( B \) are \( \gamma_{gb} \)-closed sets in \( (X, \tau). \) But \( A \subseteq B \not\subseteq bcl(A). \)

Theorem 3.43 For each \( x \in X, \) \( \{x\} \) is \( \gamma \)-closed or \( X \setminus \{x\} \) is \( \gamma_{gb} \)-closed in \( X. \)

Proof Suppose that \( \{x\} \) is not \( \gamma \)-closed, then \( X \setminus \{x\} \) is not \( \gamma \)-open. Let \( U \) be any \( \gamma \)-open set such that \( X \setminus \{x\} \subseteq U, \) implies \( U = X. \) Therefore \( bcl(X \setminus \{x\}) \subseteq U. \) Hence \( X \setminus \{x\} \) is \( \gamma_{gb} \)-closed.

Theorem 3.44 A subset \( A \) of \( X \) is \( \gamma_{gb} \)-closed if \( bcl(\{x\}) \cap A \neq \phi, \) holds for every \( x \in bcl(A). \)

Proof Let \( U \) be a \( \gamma \)-open set such that \( A \subseteq U \) and let \( x \in bcl(A). \) By assumption, there exists a point \( z \in bcl(\{x\}) \) and \( z \in A \subseteq U. \) It follows that \( U \cap \{x\} \neq \phi, \) hence \( x \in U, \) this implies \( bcl(A) \subseteq U. \) Therefore \( A \) is \( \gamma_{gb} \)-closed.

Theorem 3.45 If a subset \( A \) of a space \( X \) is \( \gamma_{gb} \)-closed then \( bcl(A) \setminus A \) does not contain any non-empty \( \gamma \)-closed set.

Proof Suppose that \( A \) is a \( \gamma_{gb} \)-closed set in \( X. \) We prove the result by contradiction. Let \( F \) be a \( \gamma \)-closed set such that \( F \subseteq bcl(A) \setminus A \) and \( F \neq \phi. \) Then \( F \subseteq X \setminus A \) which implies \( A \subseteq X \setminus F. \) Since \( A \) is \( \gamma_{gb} \)-closed and \( X \setminus F \) is \( \gamma \)-open, therefore \( bcl(A) \subseteq X \setminus F, \) that is \( F \subseteq X \setminus bcl(A). \) Hence
REFERENCES

79

$F \subseteq bcl(A) \cap (X \setminus bcl(A)) = \phi$. This shows that, $F = \phi$ which is a contradiction. Hence $bcl(A) \setminus A$ does not contain any non-empty $\gamma$-closed set in $X$. ■

Theorem 3.46 A $\gamma$gb-closed set $A$ is b-closed if $bcl(A) \setminus A$ is $\gamma$-closed.

**Proof** Let $bcl(A) \setminus A$ be a $\gamma$-closed set and $A$ be $\gamma$gb-closed. Then by Theorm 3.45, $bcl(A) \setminus A$ does not contain any non-empty $\gamma$-closed subset. Since $bcl(A) \setminus A$ is $\gamma$-closed and $bcl(A) \setminus A = \phi$, this shows that $A$ is b-closed. ■

Theorem 3.47 If $A$ is $\gamma$gb-closed and $\gamma$-closed then $bcl(A) \setminus A$ is $\gamma$-closed.

**Proof** If $A$ is a $\gamma$gb-closed set which is also $\gamma$-closed, then by Theorm 3.45, $bcl(A) \setminus A = \phi$, which is $\gamma$-closed. ■

References