RESEARCH ARTICLE

Fixed point Theorems on Dislocated and Dislocated Quasi-Metric spaces

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(Received: 17 July 2012, Accepted: 17 October 2012)

In this paper, we prove some fixed point theorems and a common fixed point theorem in dislocated and dislocated quasi metric spaces.

Keywords: Dislocated metric space; fixed point.

AMS Subject Classification: 54H25, 47H10.

1. Introduction

In 1922 Banach proved a celebrated fixed point theorem for contractive mappings in complete metric space. It is well-known as a Banach fixed point theorem. As a generalization of metrics where the self distance for any point need not to be equal to zero, Hitzler and Seda introduced the notion of dislocated metric spaces [1]. These metrics play a very important role in Topology. Zeyada et al. [2] initiated the concept of dislocated quasi-metric space and generalized the results of Hitzler and Seda in dislocated quasi-metric space. Recently, the study in such spaces followed by Isufati [3] and Aage and Salunke [4]. In [3], the author proved some fixed point results in dislocated quasi-metric spaces involving continuous contraction mappings exist in the literature of usual complete metric space. The following definitions will be needed in the sequel.

Definition 1.1 Let $X$ be nonempty set and let $d : X \times X \rightarrow [0, \infty)$ be a function satisfying following conditions:

(i) $d(x, y) = d(y, x) = 0$ implies $x = y$,
(ii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a dislocated quasi-metric on $X$. If $d$ satisfies $d(x, x) = 0$ for all $x \in X$, then the dislocated quasi-metric is called a quasi-metric on $X$. If $d$ satisfies $d(x, y) = d(y, x)$ for all $x, y \in X$, then the dislocated quasi-metric is called a dislocated metric on $X$.

Definition 1.2 A sequence $\{x_n\}$ in dq-metric space (dislocated quasi metric space) $(X, d)$ is called a Cauchy sequence if for given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ implies $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$.

Definition 1.3 A sequence $\{x_n\}$ is said to be dislocated quasi-converges to $x$ if $\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{n \rightarrow \infty} d(x, x_n) = 0$. In this case $x$ is called a dq-limit of $\{x_n\}$ and we write $\{x_n\} \rightarrow x$.

Lemma 1.4 dq-limit in a dq-metric space is unique.

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Definition 1.5 A dq-metric space \((X, d)\) is called complete if every Cauchy sequence in it is dq-convergent.

Definition 1.6 Let \((X, d_1)\) and \((Y, d_2)\) be dq-metric spaces and let \(f : X \to Y\) be a function. Then \(f\) is said to be continuous at \(x_0 \in X\), if the sequence \(\{fx_n\}\) is \(d_2q\)-convergent to \(fx_0 \in Y\) whenever the sequence \(x_n\) in \(X\) is \(d_1q\)-convergent to \(x_0\).

Definition 1.7 Let \((X, d)\) be a dq-metric space. A map \(T : X \to X\) is called contraction if there exists \(0 \leq \lambda < 1\) such that \(d(Tx, Ty) \leq \lambda d(x, y)\) for all \(x, y \in X\).

Definition 1.8 Let \((X, d)\) be a dq-metric space and let \(T : X \times X\) be continuous contraction mapping. Then \(T\) has unique fixed point.

Our first result is the following theorem.

2. Main Results

Theorem 2.1 Let \((X, d)\) be a complete dislocated quasi-metric space. Let \(T : X \to X\) be a continuous mapping satisfies the condition:

\[
d(Tx, Ty) \leq a_1d(x, y) + a_2(d(x, Tx) + d(y, Ty)) + a_3(d(x, Ty) + d(y, Tx)) + a_4\frac{d(x, Tx)d(y, Ty)}{d(x, y)} + a_5\frac{d(x, Tx)(1 + d(x, Tx))}{1 + d(x, y)},
\]

for all \(x, y \in X\) and \(a_i \geq 0\), \(i = 1, 2, \ldots, 5\) with \(a_1 + 2a_2 + 2a_3 + a_4 + a_5 < 1\). Then \(T\) has a unique fixed point.

Proof Let \(\{x_n\}\) be a sequence in \(X\), defined as sequence as follows. Let \(x_0 \in X\), \(x_1 = T(x_0)\), \(x_2 = T(x_1)\), \ldots, \(x_n = T(x_{n-1})\), \(x_{n+1} = T(x_n)\), \ldots

Using (1), we obtain

\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq a_1d(x_{n-1}, x_n) + a_2(d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)) + a_3(d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)) + a_4\frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)} + a_5\frac{d(x_{n-1}, Tx_{n-1})(1 + d(x_{n-1}, Tx_{n-1}))}{1 + d(x_{n-1}, x_n)}
\]

\[
= a_1d(x_{n-1}, x_n) + a_2d(x_{n-1}, x_{n+1}) + a_3d(x_{n-1}, x_{n+1}) + a_4\frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + a_5\frac{d(x_{n-1}, x_n)(1 + d(x_{n-1}, x_n))}{1 + d(x_{n-1}, x_n)}
\]

\[
\leq a_1d(x_{n-1}, x_n) + a_2d(x_{n-1}, x_{n+1}) + a_3d(x_{n-1}, x_{n+1}) + a_4d(x_{n-1}, x_{n+1}) + a_5d(x_{n-1}, x_{n+1})
\]

\[
\leq (a_1 + a_2 + a_3 + a_5)d(x_{n-1}, x_n) + (a_2 + a_3 + a_4)d(x_n, x_{n+1}).
\]

Therefore

\[
d(x_n, x_{n+1}) \leq \frac{a_1 + a_2 + a_3 + a_5}{1 - (a_2 + a_3 + a_4)}d(x_{n-1}, x_n) = kd(x_{n-1}, x_n).
\]
Where \( k = \left( \frac{a_1+a_2+a_3+a_4}{1-(a_2+a_3+a_4)} \right) \), with \( 0 \leq k < 1 \). Similarly we show that \( d(x_{n-1}, x_n) \leq kd(x_{n-2}, x_{n-1}) \).
Continuing in this way we have
\[
d(x_n, x_{n+1}) \leq k^n d(x_0, x_1),
\]
since \( 0 \leq k < 1 \), \( k^n \to 0 \) as \( n \to \infty \). Hence \( \{x_n\} \) is a Cauchy sequence in complete dislocated quasi-metric space \( X \). There exists a point \( u \in X \) such that \( x_n \to u \). By continuity of \( T \),
\[
T(u) = T(\lim x_n) = \lim T(x_n) = \lim x_{n+1} = u.
\]
Thus \( u \) is a fixed point \( T \).

**Uniqueness:** Let \( u \) be fixed point of \( T \).
\[
d(u, u) = d(Tu, Tu) \leq a_1 d(u, u) + a_2 d(u, u) + d(v, v) + a_3 d(u, u) + d(u, u)
\]
\[
+ a_4 \left( \frac{d(u, u)d(u, u)}{d(u, u)} \right) + a_5 \left( \frac{d(u, u)(1 + d(u, u))}{1 + d(u, u)} \right)
\]
\[
= (a_1 + 2a_2 + 2a_3 + a_4 + a_5)d(u, u)
\]
Which is true only if \( d(u, u) = 0 \). Since \( 0 \leq (a_1 + 2a_2 + 2a_3 + a_4 + a_5) < 1 \) and \( d(u, u) = 0 \). Thus \( d(u, u) = 0 \) for fixed point \( u \) of \( T \). Similarly \( d(v, v) = 0 \). Let \( u, v \) be fixed of \( T \). Then by (3), we have
\[
d(u, v) = d(Tu, Tv) \leq a_1 d(u, v) + a_2 d(u, v) + d(v, v) + a_3 d(u, v) + d(v, v)
\]
\[
+ a_4 \left( \frac{d(u, Tu)d(v, Tv)}{d(u, v)} \right) + a_5 \left( \frac{d(u, Tu)(1 + d(u, Tu))}{1 + d(u, v)} \right)
\]
\[
= (a_1 + a_3)d(u, v) + a_3 d(v, u)
\]
Similarly, we have \( d(v, u) = (a_1 + a_3)d(v, u) + a_3 d(u, v) \), hence \( |d(u, v) - d(v, u)| \leq a_1|d(u, v) - d(v, u)| \) which implies \( d(u, v) = d(v, u) \), since \( 0 < a_1 < 1 \).

Again from (1),
\[
d(u, v) = d(Tu, Tv) \leq a_1 d(u, v) + a_2 d(u, v) + d(v, v) + a_3 d(u, v) + d(v, v)
\]
\[
+ a_4 \left( \frac{d(u, Tu)d(v, Tv)}{d(u, v)} \right) + a_5 \left( \frac{d(u, Tu)(1 + d(u, Tu))}{1 + d(u, v)} \right)
\]
\[
= (a_1 + 2a_3)d(u, v)
\]
Which gives \( d(u, v) = 0 \), since \( 0 \leq a_1 + 2a_3 < 1 \). Further \( d(u, v) = d(v, u) = 0 \) gives \( u = v \). Hence the fixed point is unique. This completes the proof.

**Theorem 2.2** Let \((X, d)\) be a complete dislocated metric space and \( T : X \to X \) be a continuous mapping satisfying the following condition:
\[
d(Tx, Ty) + a_1 \max \{d(x, Ty), d(y, Tx)\} \geq a_2 d(x, y) + a_3 d(x, Tx)
\]
\[
+ a_4 (\{d(Ty, y) + d(Ty, x)\}) + a_5 \left( \frac{d(x, Tx)d(y, Ty)}{d(x, y)} \right) + a_6 \left( \frac{d(x, Tx)(1 + d(y, Ty))}{1 + d(x, y)} \right),
\]
for all \( x, y \in X \), where \( a_i \geq 0, i = 1, 2, \ldots, 6 \) and \( a_2 + a_3 + 2a_4 + a_5 + a_6 > 1 + 2a_1, a_2 + a_4 > a_1 \). Then \( T \) has a unique fixed point.
Proof: Let $x_0$ be an arbitrary point in $X$. Define a sequence $\{x_n\}$ in $X$ such that $Tx_n = x_{n-1}$ for $n = 1, 2, 3, \ldots$

Using (1), we obtain

$$d(Tx_{n+1},Tx_{n+2}) + a_1 \max \{d(x_{n+1},Tx_{n+2}), d(x_{n+2},Tx_{n+1})\} \geq a_2 d(x_{n+1},x_{n+2}) + a_3 d(x_{n+1},Tx_{n+1})$$
$$+ a_4 (d(Tx_{n+2},x_{n+2}) + d(Tx_{n+2},x_{n+1})) + a_5 \left(\frac{d(x_{n+1},Tx_{n+1})d(x_{n+2},Tx_{n+2})}{d(x_{n+1},x_{n+2})}\right)$$
$$+ a_6 \left(\frac{d(x_{n+1},Tx_{n+1})(1 + d(x_{n+2},Tx_{n+2}))}{1 + d(x_{n+1},x_{n+2})}\right),$$

$$d(x_n,x_{n+1}) + a_1 \max \{d(x_{n+1},x_{n+1}), d(x_{n+2},x_n)\} \geq a_2 d(x_{n+1},x_{n+2}) + a_3 d(x_{n+1},x_n)$$
$$+ a_4 (d(x_{n+1},x_{n+2}) + d(x_{n+1},x_{n+1})) + a_5 d(x_{n+1},x_n) + a_6 d(x_n,x_{n+1}),$$

$$d(x_n,x_{n+1}) + a_1 d(x_{n+2},x_{n+1}) + a_1 d(x_{n+1},x_n) \geq a_2 d(x_{n+1},x_{n+2}) + a_3 d(x_{n+1},x_n)$$
$$+ a_4 d(x_{n+1},x_{n+2}) + a_5 d(x_{n+1},x_n) + a_6 d(x_n,x_{n+1}),$$

$$(1 + a_1 - a_3 - a_5 - a_6)d(x_n,x_{n+1}) \geq (a_2 + a_4 - a_1)d(x_{n+1},x_{n+2}),$$

$$d(x_{n+1},x_{n+2}) \leq \left(\frac{1 + a_1 - a_3 - a_5 - a_6}{a_2 + a_4 - a_1}\right) d(x_n,x_{n+1})$$
$$= kd(x_n,x_{n+1}),$$
Thus such that $x$ is a Cauchy sequence in $X$. Now, 

\[ d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \quad \text{and} \quad d(x_{n+1}, x_{n+2}) \leq k^2d(x_{n-1}, x_n). \]

By induction, we obtain

\[ d(x_{n+1}, x_{n+2}) \leq k^{n+1}d(x_0, x_1). \]

Note that for $m, n \in \mathbb{N}$ such that $m > n$, we have

\[
d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\
\leq [k^{m-1} + k^{m-2} + \ldots + k^n]d(x_0, x_1) \\
= k^n[1 + k + k^2 + \ldots + k^{m-n-1}]d(x_0, x_1) \\
\leq k^n \sum_{r=0}^{\infty} k^r d(x_0, x_1) = (\frac{k^n}{1-k})d(x_0, x_1).
\]

Since $0 \leq k < 1$, then as $k \to \infty$, $k^n(1-k)^{-1} \to 0$ and $d(x_m, x_n) \to 0$ as $m, n \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in $X$. But $X$ is a complete dislocated metric space. There exists a point $u \in X$ such that $x_n \to u$. By continuity of $T$,

\[ T(u) = T(\lim x_n) = \lim T(x_n) = \lim x_{n+1} = u. \]

Thus $u$ is a fixed point $T$.

**Uniqueness:** Let $u, v$ be fixed points of $T$ in $X$. By condition (2), we obtain

\[
d(Tu, Tv) + a_1 \max\{d(u, Tv), d(v, Tu)\} \\
\geq a_2d(u, v) + a_3d(u, Tu) + a_4(d(Tv, v) + d(Tv, u)) \\
+ a_5\left(\frac{d(u, Tu)d(v, Tv)}{d(u, v)}\right) + a_6\left(\frac{d(u, Tu)(1 + d(v, Tv))}{1 + d(u, v)}\right),
\]

\[
d(u, v) + a_1d(u, v) \geq a_2d(u, v) + a_3d(u, u) + a_4(d(v, v) + d(u, v)) \\
+ a_5\left(\frac{d(u, u)d(v, v)}{d(u, v)}\right) + a_6\left(\frac{d(u, u)(1 + d(v, v))}{1 + d(u, v)}\right).
\]

Now,

\[
d(Tu, Tu) + a_1d(u, u) \geq a_2d(u, u) + a_3d(u, u) + a_4(d(u, u) + d(u, u)) \\
+ a_5\left(\frac{d(u, u)d(u, u)}{d(u, u)}\right) + a_6\left(\frac{d(u, u)(1 + d(u, u))}{1 + d(u, u)}\right),
\]

where

\[ k = \frac{1 + a_1 - a_3 - a_5 - a_6}{a_2 + a_4 - a_1}. \]
\[ d(Tu, Tu) + a_1 d(u, u) \geq a_2 d(u, u) + a_3 d(u, u) + 2a_4 d(u, u) \\
+ a_5 d(u, u) + a_6 d(u, u), \]

\[ d(u, u) \geq (a_2 + a_3 + 2a_4 + a_5 + a_6 - a_1) d(u, u). \]

We have \( d(u, u) = 0 \), since \((a_2 + a_3 + 2a_4 + a_5 + a_6 > 1 + 2a_1)\) and similarly \( d(v, v) = 0 \). Hence the fixed points \( u, v \), we get

\[ d(u, v) + a_1 d(u, v) \geq a_2 d(u, v) + a_3 d(u, v) \]

\[ d(u, v) \geq (a_2 + a_4 - a_1) d(u, v). \]

So we must have \( d(u, v) = 0 \) and similarly we get \( d(v, u) = 0 \). Therefore, \( u = v \) and \( T \) has unique fixed point.

Theorem 2.3 Let \((X, d)\) be a complete dislocated metric space and \( T : X \to X \) be a continuous mapping satisfying the following condition:

\[ d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) \]

\[ + d(y, Tx) + a_5 \frac{d(x, Tx)d(y, Ty)}{d(x, y)} \]

\[ + a_6 \frac{d(x, Tx)(1 + d(y, Ty))}{1 + d(x, y)}, \]

for all \( x, y \in X \) and \( a_i \geq 0, i = 1, 2, ..., 6 \) with \( a_1 + a_2 + 2a_4 + a_5 + a_6 < 1 \). Then \( T \) has a unique fixed point.

Proof Let \( \{x_n\} \) be a sequence in \( X \), defined as sequence as follows. Let \( x_0 \in X, x_1 = T(x_0), x_2 = T(x_1), \ldots, x_n = T(x_{n-1}), x_{n+1} = T(x_n), \ldots \)

Using (3), we obtain

\[ d(x_{n+1}, x_n) = d(Tx_{2n}, Tx_{n-1}) \leq a_1 d(x_n, x_{n-1}) + a_2 d(x_n, Tx_n) + a_3 d(x_{n-1}, Tx_{n-1}) \]

\[ + a_4 d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n) + a_5 \frac{d(x_n, Tx_n)d(x_{n-1}, Tx_{n-1})}{d(x_{n-1}, x_n)} \]

\[ + a_6 \frac{d(x_n, Tx_n)(1 + d(x_{n-1}, Tx_{n-1}))}{1 + d(x_{n-1}, x_n)} \]

\[ = a_1 d(x_n, x_{n-1}) + a_2 d(x_n, x_{n-1}) + a_3 d(x_{n-1}, x_n) + a_4 d(x_n, x_n) + d(x_{n-1}, x_{n+1}) \]

\[ + a_5 \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_{n-1}, x_n)} + a_6 \frac{d(x_n, x_{n+1})(1 + d(x_{n-1}, x_n))}{1 + d(x_{n-1}, x_n)} \]

\[ \leq a_1 d(x_n, x_{n-1}) + a_2 d(x_n, x_{n-1}) + a_3 d(x_{n-1}, x_n) + a_4 d(x_{n-1}, x_{n+1}) \]

\[ + a_5 \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{d(x_{n-1}, x_n)} + a_6 \frac{d(x_n, x_{n+1})(1 + d(x_{n-1}, x_n))}{1 + d(x_{n-1}, x_n)} \]

\[ \leq a_1 d(x_n, x_{n-1}) + a_2 d(x_n, x_{n-1}) + a_3 d(x_{n-1}, x_n) + a_4 d(x_{n-1}, x_n) + a_4 d(x_n, x_{n+1}) \]

\[ + a_5 d(x_n, x_{n+1}) + a_6 d(x_n, x_{n+1}). \]
Therefore
\[ d(x_{n+1}, x_n) \leq \left( \frac{a_1 + a_3 + a_4}{1 - (a_2 + a_4 + a_5 + a_6)} \right) d(x_n, x_{n-1}) = kd(x_n, x_{n-1}). \]

Where \( k = \left( \frac{a_1 + a_3 + a_4}{1 - (a_2 + a_4 + a_5 + a_6)} \right) \) with \( 0 \leq k < 1. \)

Similarly we show that \( d(x_n, x_{n-1}) \leq kd(x_{n-1}, x_{n-2}) \). Continuing in this way we have
\[ d(x_{n+1}, x_n) \leq k^n d(x_1, x_0), \]

since \( 0 \leq k < 1, k^n \to 0 \) as \( n \to \infty \). Hence \( \{x_n\} \) is a Cauchy sequence in complete dislocated metric space \( X \). There exists a point \( u \in X \) such that \( x_n \to u \). By continuity of \( T \),
\[ T(u) = T(\lim x_n) = \lim T(x_n) = \lim x_{n+1} = u. \]

Thus \( u \) is a fixed point \( T \).

**Uniqueness:** Let \( u, v \) be fixed points of \( T \).
\[ d(u, v) = d(Tu, Tv) \leq a_1 d(u, v) + a_2 d(u, u) + a_3 d(v, v) + a_4 (d(u, v) + d(u, v)) \]
\[ + a_5 \left( \frac{d(u, u) d(v, v)}{d(u, v)} \right) + a_6 \left( \frac{d(u, u)(1 + d(v, v))}{1 + d(u, v)} \right) \]

Now
\[ d(u, u) = d(Tu, Tu) \leq a_1 d(u, u) + a_2 d(u, u) + a_3 d(u, u) + a_4 (d(u, u) + d(u, u)) + a_5 d(u, u) + a_6 d(u, u) \]
\[ = (a_1 + a_2 + a_3 + 2a_4 + a_5 + a_6) d(u, u). \]

Since \( 0 \leq (a_1 + a_2 + a_3 + 2a_4 + a_5 + a_6) < 1 \), we have \( d(u, u) = 0 \). Similarly \( d(v, v) = 0 \). Hence the fixed points \( u, v \), we get \( d(u, v) = (a_1 + 2a_4) d(u, v) \). Since \( 0 \leq a_1 + 2a_4 < 1 \), we get \( d(u, v) = 0 \). Similarly we have \( d(v, u) = 0 \). Hence \( u = v. \)

**Theorem 2.4** Let \( (X, d) \) be a complete dislocated metric space. Let \( f, g : X \to X \) be a continuous mapping satisfying the following condition:
\[ d(fx, gy) \leq \alpha d(x, y) + \beta (d(x, fx) + d(y, gy)) + \gamma (d(x, gy) + d(y, fx)), \] (4)

for all \( x, y \in X \) and \( \alpha, \beta, \gamma \geq 0 \) with \( 0 < \alpha + 2\beta + 2\gamma < 1, \alpha + 2\gamma < 1 \). Then \( f \) and \( g \) have a unique common fixed point.

**Proof** Let \( x_0 \) be an arbitrary point in \( X \). Define the sequence \( \{x_n\} \) by \( x_1 = f(x_0), x_2 = g(x_1), \ldots, x_{2n} = g(x_{2n-1}), x_{2n+1} = f(x_{2n}), \ldots \)
Consider,

\[ d(x_{2n+1}, x_{2n+2}) = d(fx_{2n}, gx_{2n+1}) \leq \alpha d(x_{2n}, x_{2n+1}) + \beta (d(x_{2n}, fx_{2n}) + d(x_{2n+1}, gx_{2n+1})) + \gamma (d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})) \]

\[ = \alpha d(x_{2n}, x_{2n+1}) + \beta (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) + \gamma (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) \]

\[ \leq \alpha d(x_{2n}, x_{2n+1}) + \beta (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) + \gamma d(x_{2n}, x_{2n+2}) + \gamma (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) \]

\[ \leq \alpha d(x_{2n}, x_{2n+1}) + \beta (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) \]

\[ + \gamma (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) \leq \alpha d(x_{2n}, x_{2n+1}) + \beta (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) + \gamma d(x_{2n}, x_{2n+2}) \]

\[ \leq \alpha d(x_{2n}, x_{2n+1}) + \beta (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) + \gamma d(x_{2n}, x_{2n+2}) \]

\[ + \gamma (d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})) \]

\[ d(x_{2n+1}, x_{2n+2}) \leq \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} d(x_{2n}, x_{2n+1}). \quad (5) \]

Again,

\[ d(x_{2n+1}, x_{2n}) = d(fx_{2n}, gx_{2n-1}) \leq \alpha d(x_{2n}, x_{2n-1}) + \beta (d(x_{2n}, fx_{2n}) + d(x_{2n-1}, gx_{2n-1})) + \gamma (d(x_{2n}, gx_{2n-1}) + d(x_{2n-1}, fx_{2n})) \]

\[ = \alpha d(x_{2n}, x_{2n-1}) + \beta (d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n})) + \gamma (d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n})) \]

\[ \leq \alpha d(x_{2n}, x_{2n-1}) + \beta (d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n})) + \gamma d(x_{2n-1}, x_{2n-1}) \]

\[ \leq \alpha d(x_{2n}, x_{2n-1}) + \beta (d(x_{2n}, x_{2n-1}) + d(x_{2n-1}, x_{2n})) \]

\[ + \gamma (d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n})) \]

\[ d(x_{2n+1}, x_{2n}) \leq \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} d(x_{2n}, x_{2n-1}). \quad (6) \]

Thus,

\[ d(x_{2n+2}, x_{2n+1}) \leq \left( \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} \right)^2 d(x_{2n}, x_{2n-1}). \]

In this way we have

\[ d(x_{2n+1}, x_{2n+2}) \leq k^{2n} d(x_0, x_1). \quad (7) \]

Where \( k = \left( \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} \right) \). Since \( k < 1 \), \( k^{2n} \to 0 \) as \( n \to \infty \).

Thus \( \{x_n\} \) is a Cauchy sequence in a complete dislocated metric space \( X \). There exists a point \( u \in X \) such that \( x_n \to u \). Therefore the subsequences \( \{f_{2n}\} \to u \) and \( \{g_{2n+1}\} \to u \). Since \( f \) and \( g \) are continuous function, we have \( fu = u \) and \( gu = u \).
**Uniqueness:** Let $u$ and $v$ be a common fixed point of $f$ and $g$. Then,

\[
d(u, v) = d(fu, gv) \leq \alpha d(u, v) + \beta (d(u, fu) + d(v, gv)) + \gamma (d(u, gv) + d(v, fu))
\]

\[
= \alpha d(u, v) + \beta (d(u, u) + d(v, v)) + \gamma (d(u, v) + d(u, v))
\]

\[
= (\alpha + 2\gamma) d(u, v) + \beta (d(u, u) + d(v, v)).
\]

Replacing $v$ by $u$, we get $d(u, u) \leq (\alpha + 2\beta + 2\gamma) d(u, u)$. Then $d(u, u) = 0$. Similarly we have $d(v, v) = 0$. In this way we have $d(u, v) \leq (\alpha + 2\gamma) d(u, v)$. Therefore $d(u, v) = 0$. Similarly we have $d(v, u) = 0$. Then $u = v$. Hence $f$ and $g$ have a unique common fixed point.

3. **Acknowledgment**

The authors would like to thank the reviewers for their valuable comments and helpful suggestions for improvement of the original manuscript.

References