RESEARCH ARTICLE

Pγ-open Sets and Pγ,β-Continuous Mappings in Topological Spaces

Alias B. Khalaf ∗ and Hariwan Z. Ibrahim †

∗ Department of Mathematics, Faculty of Science, University of Duhok, Duhok, Iraq.
† Department of Mathematics, Faculty of Science, University of Zakho, Zakho, Iraq.

(Received: 27 May 2012, Accepted: 30 June 2012)

In this paper we introduce the concept of Pγ-open sets as a generalization of γ-open sets. By using this set we introduce Pγ,T1 spaces and study some of its properties. Finally, we introduce Pγ,β-continuous mappings and give some properties of such mappings.

Keywords: Pγ-open; Pγ-g.closed; Pγ,T1; Pγ,β-continuous mapping.

AMS Subject Classification: 54A05, 54A10, 54C05.

1. Introduction

Mashhour et al. [1] introduced preopen sets. Kasahara [2] defined the concept of an operation on topological spaces and introduce the concept of α-closed graphs of an operation. Ogata [3] called the operation α (respectively α-closed set) as γ-operation (respectively γ-closed set) and introduced the notion of τγ which is the collection of all γ-open sets in a topological space. Also he introduced the concept of γ-Ti (i = 0, 1, 2) and characterized γ-Ti using the notion of γ-closed and γ-open sets.

In this paper, in Section 3, we introduce the concept of Pγ-open sets by using an operation γ and define the corresponding Pγ-closure and Pγ-interior operators. In Section 4, we introduce the concept of Pγ-generalized closed sets and Pγ,T1 spaces and characterize Pγ,T1 spaces using the notion of Pγ-closed or Pγ-open sets. In Section 5, we introduce Pγ,β-continuous mappings and study some of its properties.

2. Preliminaries

Let (X, τ) be a topological space and A be a subset of X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively. A subset A of a topological space (X, τ) is said to be preopen [1] if A ⊆ Int(Cl(A)). The complement of a preopen set is said to be preclosed. The family of all preopen (resp. preclosed) sets in a topological space (X, τ) is denoted by PO(X, τ) or PO(X)(resp. PC(X, τ) or PC(X)).

Definition 2.1 [1] Let (X, τ) be a topological space and A ⊆ X, then:

(1) pInt(A).

∗ Corresponding author
Email: aliasbkhalaf@gmail.com
(2) preclosure of $A$ is defined by intersection of all preclosed set containing $A$ and it is denoted by $pCl(A)$.

Definition 2.2 [2] Let $(X, \tau)$ be a topological space. An operation $\gamma$ on the topology $\tau$ is a mapping from $\tau$ to power set $P(X)$ of $X$ such that $V \subseteq \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of $\gamma$ at $V$. It is denoted by $\gamma : \tau \rightarrow P(X)$.

Definition 2.3 [3] A subset $A$ of a topological space $(X, \tau)$ is called $\gamma$-open set if for each $x \in A$ there exists an open set $U$ such that $x \in U$ and $\gamma(U) \subseteq A$.

Definition 2.4 [3] Let $(X, \tau)$ be a topological space and $A$ be subset of $X$, then $\gamma_{\tau}-Cl(A) = \bigcap\{F : A \subseteq F, X \setminus F \in \tau_{\gamma}\}$.

Definition 2.5 [4] Let $X, \tau$ be a topological space and $\gamma$ be any subset of $X$. The $\gamma_{\tau}-Int(A)$ is defined as $\gamma_{\tau}-Int(A) = \bigcup\{U : U$ is a $\gamma_{\tau}$-open set and $U \subseteq A\}$.

Definition 2.6 [3] A topological space $(X, \tau)$ is said to be $\gamma$-regular, where $\gamma$ is an operation on $\tau$, if for each $x \in X$ and for each open neighborhood $V$ of $x$, there exists an open neighborhood $U$ of $x$ such that $\gamma(U)$ contained in $V$.

Proposition 2.7 [3] If $(X, \tau)$ is $\gamma$-regular space, then $\tau = \tau_{\gamma}$.

3. $P_{\gamma}$-open sets

In this section, we introduce the class of $P_{\gamma}$-open sets which is an extension of $\gamma$-open sets in topological spaces and $\gamma$ is defined to be a mapping on $PO(X)$ in to $P(X)$ and $\gamma : PO(X) \rightarrow P(X)$ is called an operation on $PO(X)$, such that $V \subseteq \gamma(V)$ for each $V \in PO(X)$.

Definition 3.1 A subset $A$ of a space $X$ is called $P_{\gamma}$-open if for each $x \in A$, there exists a preopen set $U$ such that $x \in U$ and $\gamma(U) \subseteq A$.

Remark 3.2 The set of all $P_{\gamma}$-open set in a topological space $(X, \tau)$ is denoted as $PO(X)_{\gamma}$.

Remark 3.3 The concept of $P_{\gamma}$-open and open are independent.

Example 3.4 Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $PO(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Define an operation $\gamma$ on $PO(X)$ by

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a, c\} \\ X & \text{otherwise} \end{cases}$$

Then $P_{\gamma}$-open sets of $X$ are $\{\phi, X, \{a, c\}\}$.

Remark 3.5 It is clear from the definition that every $P_{\gamma}$-open subset of a space $X$ is preopen, but the converse need not be true as shown in the following example.

Example 3.6 Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, X\}$ and $PO(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Define an operation $\gamma$ on $PO(X)$ by

$$\gamma(A) = \begin{cases} A & \text{if } b \in A \\ X & \text{if } b \notin A \end{cases}$$

Then $PO(X)_{\gamma} = \{\phi, \{a, b\}, X\}$ and $\{a\} \in PO(X)$, but $\{a\} \notin PO(X)_{\gamma}$.

Proposition 3.7 If $A$ is a $\gamma$-open set in $(X, \tau)$, then $A$ is a $P_{\gamma}$-open set.

Proof Follows from that every open set is preopen set.

The converse of the above proposition need not be true in general as it is shown below.
Example 3.11 Consider $X = \{a, b, c\}$ with the discrete topology on $X$. Define an operation $\gamma$ on $PO(X)$ by

$$\gamma(A) = \begin{cases} \{a, b\} & \text{if } A = \{a\} \text{ or } \{b\} \\ A & \text{otherwise} \end{cases}$$

Then $A = \{a, b\}$ and $B = \{a, c\}$ are $P_\gamma$-open sets but $A \cap B = \{a\}$ is not a $P_\gamma$-open set.

From the above example we notice that the family of all $P_\gamma$-open subsets of a space $X$ is a supratopology and need not be a topology in general.

Proposition 3.12 The set $A$ is $P_\gamma$-open in the space $(X, \tau)$ if and only if for each $x \in A$, there exists a $P_\gamma$-open set $B$ such that $x \in B \subseteq A$.

Proof Suppose that $A$ is $P_\gamma$-open set in the space $(X, \tau)$. Then for each $x \in A$, put $B = A$ is a $P_\gamma$-open set such that $x \in B \subseteq A$.

Conversely, suppose that for each $x \in A$, there exists a $P_\gamma$-open set $B$ such that $x \in B \subseteq A$, thus $A = \bigcup B_x$ where $B_x \in PO(X)_\gamma$ for each $x$. Therefore, $A$ is a $P_\gamma$-open set. ■

Definition 3.13 An operation $\gamma$ on $PO(X)$ is said to be pre regular if for every preopen sets $U$ and $V$ of each $x \in X$, there exists a preopen set $W$ of $x$ such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

Definition 3.14 An operation $\gamma$ on $PO(X)$ is said to be pre open if for every preopen set $U$ of each $x \in X$, there exists a $P_\gamma$-open set $V$ such that $x \in V$ and $V \subseteq \gamma(U)$.

In the following two examples, we show that pre regular operation is incomparable with the pre open operation.

Example 3.15 Consider $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Define an operation $\gamma$ on $PO(X)$ by

$$\gamma(A) = \begin{cases} \{a, b\} & \text{if } A = \{a\} \\ X & \text{if } A \neq \{a\} \end{cases}$$

Then $\gamma$ is pre regular but not pre open.

Example 3.16 Consider $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Define an operation $\gamma$ on $PO(X)$ by

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\ X & \text{otherwise} \end{cases}$$

Then $\gamma$ is not pre regular but $\gamma$ is pre open.
In the following proposition the intersection of two $P_\gamma$-open sets is also a $P_\gamma$-open set.

**Proposition 3.17** Let $\gamma$ be a pre regular operation on $PO(X)$. If $A$ and $B$ are $P_\gamma$-open sets in $X$, then $A \cap B$ is also a $P_\gamma$-open set.

**Proof** Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $A$ and $B$ are $P_\gamma$-open sets, there exist preopen sets $U$ and $V$ such that $x \in U$ and $\gamma(U) \subseteq A$, $x \in V$ and $\gamma(V) \subseteq B$. Since $\gamma$ is a pre regular operation, then there exists a preopen set $W$ of $x$ such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V) \subseteq A \cap B$. This implies that $A \cap B$ is $P_\gamma$-open set.

**Remark 3.18** By the above proposition, if $\gamma$ is a pre regular operation on $PO(X)$. Then $PO(X)_\gamma$ form a topology on $X$.

**Definition 3.19** Let $(X, \tau)$ be a topological space, a subset $F$ of $X$ is called $P_\gamma$-closed if $X \setminus F$ is $P_\gamma$-open.

**Remark 3.20** The family of all $P_\gamma$-closed in $(X, \tau)$ is denoted by $PC(X)_\gamma$.

**Proposition 3.21** Let $\{F_\alpha\}_{\alpha \in J}$ be a collection of $P_\gamma$-closed subsets of $X$. Then $\cap_{\alpha \in J} F_\alpha$ is also $P_\gamma$-closed.

**Proof** Follows from Proposition 3.10.

**Definition 3.22** A point $x \in X$ is said to be $P_\gamma$-limit point of a set $A$ if for each $P_\gamma$-open set $U$ containing $x$, then $U \cap (A \setminus \{x\}) \neq \emptyset$. The set of all $P_\gamma$-limit points of $A$ is called a $P_\gamma$-derivative set of $A$ and is denoted by $p_\gamma D(A)$.

**Definition 3.23** Let $A$ be a subset of $(X, \tau)$, and $\gamma : PO(X) \to P(X)$ be an operation. Then the $P_\gamma$-closure (resp., $P_\gamma$-interior) of $A$ is denoted by $p_\gamma Cl(A)$ (resp., $p_\gamma Int(A)$) and defined as follows:

1. $p_\gamma Cl(A) = \bigcap \{F : F$ is $P_\gamma$-closed and $A \subseteq F\}$.
2. $p_\gamma Int(A) = \bigcup \{U : U$ is $P_\gamma$-open and $U \subseteq A\}$.

**Theorem 3.24** Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$, then

1. $p_\gamma Int(A)$ is $P_\gamma$-open set contained in $A$.
2. $p_\gamma Cl(A)$ is $P_\gamma$-closed set containing $A$.
3. $A$ is $P_\gamma$-closed if and only if $p_\gamma Cl(A) = A$.
4. $A$ is $P_\gamma$-open if and only if $p_\gamma Int(A) = A$.

**Proof** Follows from the Definition 3.23 and Theorem 3.10.

**Theorem 3.25** For a point $x \in X$, $x \in p_\gamma Cl(A)$ if and only if for every $P_\gamma$-open set $V$ of $X$ containing $x$, $A \cap V \neq \emptyset$.

**Proof** Let $x \in p_\gamma Cl(A)$ and suppose that $V \cap A = \emptyset$ for some $P_\gamma$-open set $V$ which contains $x$. Then $(X \setminus V)$ is $P_\gamma$-open and $A \subseteq (X \setminus V)$, thus $p_\gamma Cl(A) \subseteq (X \setminus V)$. But this implies that $x \in (X \setminus V)$, a contradiction. Therefore $V \cap A \neq \emptyset$.

Conversely, let $A \subseteq X$ and $x \in X$ such that for each $P_\gamma$-open set $U$ which contains $x$, $U \cap A \neq \emptyset$. If $x \notin p_\gamma Cl(A)$, there is a $P_\gamma$-closed set $F$ such that $A \subseteq F$ and $x \notin F$. Then $(X \setminus F)$ is a $P_\gamma$-open set with $x \in (X \setminus F)$, and thus $(X \setminus F) \cap A \neq \emptyset$, which is a contradiction.

**Proposition 3.26** Let $A$ be any subset of a topological space $(X, \tau)$ and $\gamma$ be an operation on $PO(X)$, then $p_\gamma Cl(A) = A \cup p_\gamma D(A)$.

**Proof** Since $p_\gamma D(A) \subseteq p_\gamma Cl(A)$ and $A \subseteq p_\gamma Cl(A)$, then $A \cup p_\gamma D(A) \subseteq p_\gamma Cl(A)$.

On the other hand, to show that $p_\gamma Cl(A) \subseteq A \cup p_\gamma D(A)$, since $p_\gamma Cl(A)$ is the smallest $P_\gamma$-closed set containing $A$, so it is enough to prove that $A \cup p_\gamma D(A)$ is $P_\gamma$-closed. Let $x \notin A \cup p_\gamma D(A)$. This implies that $x \notin A$ and $x \notin p_\gamma D(A)$. Since $x \notin p_\gamma D(A)$, there exists a $P_\gamma$-open set $U$ of $x$ which contains no point of $A$ other than $x$ but $x \notin A$. So $U$ contains no point of $A$, which implies $U \subseteq X \setminus A$. Again, $U$ is a $P_\gamma$-open set of each of its points. But as $U$ does not contain any point of $A$, no point
of $U$ can be a $P_\gamma$-limit point of $A$. Therefore, no point of $U$ can belong to $p_\gamma D(A)$. This implies that $U \subseteq X \setminus p_\gamma D(A)$. Hence, it follows that $x \in U \subseteq X \setminus A \cap X \setminus p_\gamma D(A) = X \setminus (A \cup p_\gamma D(A))$. Therefore, $A \cup p_\gamma D(A)$ is $P_\gamma$-closed. Hence $p_\gamma Cl(A) \subseteq A \cup p_\gamma D(A)$. Thus $p_\gamma Cl(A) = A \cup p_\gamma D(A)$. ■

Proposition 3.27 Let $A$ be any subset of a topological space $(X, \tau)$ and $\gamma$ be an operation on $PO(X)$. Then $p_\gamma Int(A) = A \setminus p_\gamma D(X \setminus A)$.

Proof If $x \in A \setminus p_\gamma D(X \setminus A)$, then $x \notin p_\gamma D(X \setminus A)$ and so there exists a $P_\gamma$-open set $U$ containing $x$ such that $U \cap (X \setminus A) = \emptyset$. Then $x \in U \subseteq A$ and hence $x \in p_\gamma Int(A)$, i.e., $A \setminus p_\gamma D(X \setminus A) \subseteq p_\gamma Int(A)$. On the other hand, if $x \in p_\gamma Int(A)$, then $x \notin p_\gamma D(X \setminus A)$ since $p_\gamma Int(A)$ is $P_\gamma$-open and $p_\gamma Int(A) \cap (X \setminus A) = \emptyset$. Hence $p_\gamma Int(A) = A \setminus p_\gamma D(X \setminus A)$.

Proposition 3.28 Let $A$ be any subset of a topological space $(X, \tau)$ and $\gamma$ be an operation on $PO(X)$, then $X \setminus p_\gamma Int(A) = p_\gamma Cl(X \setminus A)$.

Proposition 3.29 Let $(X, \tau)$ be any subset of a topological space and $\gamma$ be a pre regular operation on $PO(X)$. Then

1. For every $P_\gamma$-open set $G$ and every subset $A \subseteq X$ we have $p_\gamma Cl(A) \cap G \subseteq p_\gamma Cl(A \cap G)$.
2. For every $P_\gamma$-closed set $F$ and every subset $A \subseteq X$ we have $p_\gamma Int(A \cup F) \subseteq p_\gamma Int(A) \cup F$.

Proof

1. Let $x \in p_\gamma Cl(A) \cap G$, then $x \in p_\gamma Cl(A)$ and $x \in G$. Let $V$ be the $P_\gamma$-open set containing $x$. Then by Proposition 3.17, $V \cap G$ is also $P_\gamma$-open set containing $x$. Since $x \in p_\gamma Cl(A)$, implies $(V \cap G) \cap A \neq \emptyset$. This implies $V \cap (A \cap G) \neq \emptyset$. This is true for every $V$ containing $x$, hence by Proposition 3.25, $x \in p_\gamma Cl(A \cap G)$. Therefore $p_\gamma Cl(A) \cap G \subseteq p_\gamma Cl(A \cap G)$.

2. Proof follows from (1) and Proposition 3.28.

Definition 3.30 A point $x \in X$ is in pre closure-$\gamma$ of a set $A \subseteq X$, if $\gamma(U) \cap A \neq \emptyset$ for each preopen set $U$ of $x$. The pre closure-$\gamma$ of a set $A$ is denoted by $pCl_\gamma(A)$.

Remark 3.31 Let $A$ be any subset of a topological space $(X, \tau)$ and $\gamma$ be an operation on $PO(X)$. Then the following relation holds.

$$\tau \gamma - Int(A) \subseteq p_\gamma Int(A) \subseteq p\gamma Int(A) \subseteq p\gamma Cl(A) \subseteq p\gamma Cl(A) \subseteq \tau \gamma - Cl(A).$$

Proof Follows from Remark 3.5 and Proposition 3.7.

Proposition 3.32 Let $\gamma : PO(X) \to P(X)$ be an operation on $PO(X)$ and $A$ be a subset of $X$, then

1. A subset $pCl_\gamma(A)$ is a preclosed set in $(X, \tau)$.
2. If $\gamma$ is pre open, then $pCl_\gamma(A) = p_\gamma Cl(A)$, and $pCl_\gamma(pCl_\gamma(A)) = pCl_\gamma(A)$, and $pCl_\gamma(A)$ is $P_\gamma$-closed.

Proof

1. To prove that $pCl_\gamma(A)$ is preclosed. Let $x \in pCl(pCl_\gamma(A))$, then $U \cap pCl_\gamma(A) \neq \emptyset$ for every preopen set $U$ of $x$. Let $y \in U \cap pCl_\gamma(A)$, $y \in U$ and $y \in pCl_\gamma(A)$. Since $U$ is preopen set containing $y$, implies $\gamma(U) \cap A \neq \emptyset$. Therefore $x \in pCl_\gamma(A)$. Hence $pCl(pCl_\gamma(A)) \subseteq pCl_\gamma(A)$. This implies $pCl_\gamma(A)$ is a preclosed set.

2. By Remark 3.31, we have $pCl_\gamma(A) \subseteq p_\gamma Cl(A)$. Now to prove that $p_\gamma Cl(A) \subseteq pCl_\gamma(A)$. Let $x \notin pCl_\gamma(A)$, then there exists a preopen set $U$ such that $\gamma(U) \cap A = \emptyset$. Since $\gamma$ is pre open, there exists a $P_\gamma$-open set $V$ such that $x \in V \subseteq \gamma(U)$. Therefore $V \cap A = \emptyset$. This implies $x \notin p_\gamma Cl(A)$. Hence $p_\gamma Cl(A) \subseteq pCl_\gamma(A)$. Therefore $pCl_\gamma(A) = p_\gamma Cl(A)$. Now, $pCl_\gamma(pCl_\gamma(A)) = p_\gamma Cl(pCl_\gamma(A)) = p_\gamma Cl(A) = pCl_\gamma(A)$. ■
4. \(P_{\gamma-g}.closed\) sets and \(P_{\gamma-T_2}\) spaces

In this section we define \(P_{\gamma-g}.closed\) sets and investigate a general operation approaches on \(T_2\) spaces.

Definition 4.1 A subset \(A\) of the space \((X, \tau)\) is said to be \(P_{\gamma-g}.closed\) (briefly, \(P_{\gamma-g}.closed\)) if \(p_\gamma Cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is a \(P_{\gamma-g}\)-open set in \((X, \tau)\).

It is clear that every \(P_{\gamma-g}\)-closed subset of \(X\) is also a \(P_{\gamma-g}\)-closed set. The following example shows that a \(P_{\gamma-g}\)-closed set need not be \(P_{\gamma-g}\)-closed.

Example 4.2 Consider \(X = \{a, b, c\}\) with the topology \(\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}\). Define an operation \(\gamma\) on \(PO(X)\) by

\[
\gamma(A) = \begin{cases} 
A & \text{if } A = \{b\} \text{ or } \{a, c\} \\
X & \text{otherwise}
\end{cases}
\]

Now, if we let \(A = \{a\}\), since the only \(P_{\gamma-g}\)-open supersets of \(A\) are \(\{a, c\}\) and \(X\), then \(A\) is \(P_{\gamma-g}\)-closed. But it is easy to see that \(A\) is not \(P_{\gamma-g}\)-closed.

Proposition 4.3 A subset \(A\) of \((X, \tau)\) is \(P_{\gamma-g}.closed\) if and only if \(p_\gamma Cl(\{x\}) \cap A \neq \phi\), holds for every \(x \in p_\gamma Cl(A)\).

Proof Let \(U\) be a \(P_{\gamma-g}\)-open set such that \(A \subseteq U\) and let \(x \in p_\gamma Cl(A)\). By assumption, there exists a \(z \in p_\gamma Cl(\{x\})\) and \(z \in A \subseteq U\). It follows from Proposition 3.25, that \(U \cap \{x\} \neq \phi\), hence \(x \in U\), this implies \(p_\gamma Cl(A) \subseteq U\). Therefore \(A\) is \(P_{\gamma-g}\)-closed.

Conversely, suppose that \(x \in p_\gamma Cl(A)\) such that \(p_\gamma Cl(\{x\}) \cap A = \phi\). Since, \(p_\gamma Cl(\{x\})\) is \(P_{\gamma-g}\)-closed, therefore \(X \setminus p_\gamma Cl(\{x\})\) is a \(P_{\gamma-g}\)-open set in \(X\). Since \(A \subseteq X \setminus (p_\gamma Cl(\{x\}))\) and \(A\) is \(P_{\gamma-g}\)-closed, this implies \(p_\gamma Cl(A) \subseteq X \setminus p_\gamma Cl(\{x\})\) holds, and hence \(x \notin p_\gamma Cl(A)\). This is a contradiction. Therefore \(p_\gamma Cl(\{x\}) \cap A \neq \phi\). \(\blacksquare\)

Proposition 4.4 A set \(A\) of a space \(X\) is \(P_{\gamma-g}.closed\) if and only if \(p_\gamma Cl(A) \setminus A\) does not contain any non-empty \(P_{\gamma-g}\)-closed set.

Proof Necessity. Suppose that \(A\) is \(P_{\gamma-g}\)-closed set in \(X\). We prove the result by contradiction. Let \(F\) be a \(P_{\gamma-g}\)-closed set such that \(F \subseteq p_\gamma Cl(A) \setminus A\) and \(F \neq \phi\). Then \(F \subseteq X \setminus A\) which implies \(A \subseteq X \setminus F\). Since \(A\) is \(P_{\gamma-g}\)-closed, and \(X \setminus F\) is \(P_{\gamma-g}\)-open, therefore \(p_\gamma Cl(A) \subseteq X \setminus F\), that is \(F \subseteq X \setminus p_\gamma Cl(A)\). Hence \(F \subseteq p_\gamma Cl(A) \cap (X \setminus p_\gamma Cl(A)) = \phi\). This shows that, \(F = \phi\) which is a contradiction. Hence \(p_\gamma Cl(A) \setminus A\) does not contains any non-empty \(P_{\gamma-g}\)-closed set in \(X\).

Sufficiency. Let \(A \subseteq U\), where \(U\) is \(P_{\gamma-g}\)-open in \((X, \tau)\). If \(p_\gamma Cl(A)\) is not contained in \(U\), then \(p_\gamma Cl(A) \cap X \setminus U \neq \phi\). Now, since \(p_\gamma Cl(A) \cap X \setminus U \subseteq p_\gamma Cl(A) \cap X \setminus U\) and \(p_\gamma Cl(A) \cap X \setminus U\) is a non-empty \(P_{\gamma-g}\)-closed set, then we obtain a contradiction and therefore \(A\) is \(P_{\gamma-g}\)-closed. \(\blacksquare\)

Corollary 4.1 If a subset \(A\) of \(X\) is \(P_{\gamma-g}\)-closed set in \(X\), then \(p_\gamma Cl(A) \setminus A\) dose not contain any non-empty \(\gamma\)-closed set in \(X\).

Proof Follows from the Proposition 3.7. \(\blacksquare\)

The converse of the above corollary is not true in general as it is shown in the following example.

Example 4.5 Consider \(X = \{a, b, c\}\) with the topology \(\tau = \{\phi, \{c\}, X\}\). Define an operation \(\gamma\) on \(PO(X)\) by \(\gamma(A) = A\). If we let \(A = \{a, c\}\) then \(A\) is not \(P_{\gamma-g}\)-closed, since \(A \subseteq \{a, c\} \in PO(X)\) and \(p_\gamma Cl(A) = X \not\subseteq \{a, c\}\), where \(p_\gamma Cl(A) \setminus A = \{b\}\) does not contain any non-empty \(\gamma\)-closed set in \(X\).

Proposition 4.6 If \(A\) is a \(P_{\gamma-g}\)-closed set of a space \(X\), then the following are equivalent:

1. \(A\) is \(P_{\gamma-g}\)-closed.
2. \(p_\gamma Cl(A) \setminus A\) is \(P_{\gamma-g}\)-closed.
Proposition 4.7 For a space \((X, \tau)\), the following are equivalent:

1. Every subset of \(X\) is \(P_{\gamma}\)-g.closed.
2. \(PO(X, \tau)_{\gamma} = PC(X, \tau)_{\gamma}\).

Proof (1) \(\Rightarrow\) (2): Let \(U \in PO(X, \tau)_{\gamma}\). Then by hypothesis, \(U\) is \(P_{\gamma}\)-g.closed which implies that \(p_{\gamma}Cl(U) \subseteq U\), so, \(p_{\gamma}Cl(U) = U\), therefore \(U \in PC(X, \tau)_{\gamma}\). Also let \(V \in PC(X, \tau)_{\gamma}\). Then \(X \setminus V \in PO(X, \tau)_{\gamma}\), hence by hypothesis \(X \setminus V\) is \(P_{\gamma}\)-g.closed and then \(X \setminus V \in PC(X, \tau)_{\gamma}\), thus \(V \in PO(X, \tau)_{\gamma}\) according above we have \(PO(X, \tau)_{\gamma} = PC(X, \tau)_{\gamma}\).

(2) \(\Rightarrow\) (1): If \(A\) is a subset of a space \(X\) such that \(A \subseteq U\) where \(U \in PO(X, \tau)_{\gamma}\), then \(U \in PC(X, \tau)_{\gamma}\) and therefore \(p_{\gamma}Cl(U) \subseteq U\) which shows that \(A\) is \(P_{\gamma}\)-g.closed.

Proposition 4.8 If \(A\) is \(\gamma\)-open and \(P_{\gamma}\)-g.closed then \(A\) is \(P_{\gamma}\)-closed.

Proof Suppose that \(A\) is \(\gamma\)-open and \(P_{\gamma}\)-g.closed. As every \(\gamma\)-open is \(P_{\gamma}\)-open and \(A \subseteq A\), we have \(p_{\gamma}Cl(A) \subseteq A\), also \(A \subseteq p_{\gamma}Cl(A)\), therefore \(p_{\gamma}Cl(A) = A\). That is \(A\) is \(P_{\gamma}\)-closed.

Proposition 4.9 If a subset \(A\) of \(X\) is \(P_{\gamma}\)-g.closed and \(A \subseteq B \subseteq p_{\gamma}Cl(A)\), then \(B\) is a \(P_{\gamma}\)-g.closed set in \(X\).

Proof Let \(A\) be \(P_{\gamma}\)-g.closed set such that \(A \subseteq B \subseteq p_{\gamma}Cl(A)\). Let \(U\) be a \(P_{\gamma}\)-open set of \(X\) such that \(B \subseteq U\). Since \(A\) is \(P_{\gamma}\)-g.closed, we have \(p_{\gamma}Cl(A) \subseteq U\). Now \(p_{\gamma}Cl(A) \subseteq p_{\gamma}Cl(B) \subseteq p_{\gamma}Cl[p_{\gamma}Cl(A)] = p_{\gamma}Cl(A) \subseteq U\). That is \(p_{\gamma}Cl(B) \subseteq U\), where \(U\) is \(P_{\gamma}\)-open. Therefore \(B\) is a \(P_{\gamma}\)-g.closed set in \(X\).

The converse of the above proposition need not be true as seen from the following example.

Example 4.10 Consider \(X = \{a, b, c\}\) with the topology \(\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}\). Define an operation \(\gamma\) on \(PO(X)\) by \(\gamma(A) = A\). Let \(A = \{b\}\) and \(B = \{b, c\}\). Then \(A\) and \(B\) are \(P_{\gamma}\)-g.closed sets in \((X, \tau)\). But \(A \subseteq B \not\subseteq p_{\gamma}Cl(A)\).

Proposition 4.11 Let \(\gamma\) be an operation on \(PO(X)\). Then for each \(x \in X\), \(\{x\}\) is \(P_{\gamma}\)-closed or \(X \setminus \{x\}\) is \(P_{\gamma}\)-g.closed in \((X, \tau)\).

Proof Suppose that \(\{x\}\) is not \(P_{\gamma}\)-closed, then \(X \setminus \{x\}\) is not \(P_{\gamma}\)-open. Let \(U\) be any \(P_{\gamma}\)-open set such that \(X \setminus \{x\}\) \(\subseteq U\), implies \(U = X\). Therefore \(p_{\gamma}Cl(X \setminus \{x\}\) \(\subseteq U\). Hence \(X \setminus \{x\}\) is \(P_{\gamma}\)-g.closed.

Definition 4.12 A space \((X, \tau)\) is said to be \(P_{\gamma}\)-\(\mathcal{T}\) if every \(P_{\gamma}\)-g.closed set is \(P_{\gamma}\)-closed.

Theorem 4.13 The following statements are equivalent for a topological space \((X, \tau)\) with an operation \(\gamma\) on \(PO(X)\):

1. \((X, \tau)\) is \(P_{\gamma}\)-\(\mathcal{T}\).
2. Each singleton \(\{x\}\) of \(X\) is either \(P_{\gamma}\)-closed or \(P_{\gamma}\)-open.

Proof (1) \(\Rightarrow\) (2): Suppose \(\{x\}\) is not \(P_{\gamma}\)-closed. Then by Proposition 4.11, \(X \setminus \{x\}\) is \(P_{\gamma}\)-g.closed. Now since \((X, \tau)\) is \(P_{\gamma}\)-\(\mathcal{T}\), \(X \setminus \{x\}\) is \(P_{\gamma}\)-closed, that is \(\{x\}\) is \(P_{\gamma}\)-open.

(2) \(\Rightarrow\) (1): Let \(A\) be any \(P_{\gamma}\)-g.closed set in \((X, \tau)\) and \(x \in p_{\gamma}Cl(A)\). By (2), we have \(\{x\}\) is \(P_{\gamma}\)-closed or \(P_{\gamma}\)-open. If \(\{x\}\) is \(P_{\gamma}\)-closed then \(x \notin A\) will imply \(x \in p_{\gamma}Cl(A) \setminus A\), which is not possible by Proposition 4.4. Hence \(x \in A\). Therefore, \(p_{\gamma}Cl(A) = A\), that is \(A\) is \(P_{\gamma}\)-closed. So, \((X, \tau)\) is \(P_{\gamma}\)-\(\mathcal{T}\). On the other hand, if \(\{x\}\) is \(P_{\gamma}\)-open then as \(x \in p_{\gamma}Cl(A), \{x\} \cap A \neq \emptyset\). Hence \(x \in A\). So \(A\) is \(P_{\gamma}\)-closed.
5. \( P_{\gamma,\beta}\)-Continuous Mappings

In this section we introduce the concept of \( P_{\gamma,\beta}\)-continuous mappings and study some of its basic properties. Throughout this section let \((X, \tau)\) and \((Y, \sigma)\) be two topological spaces and let \( \gamma : PO(X) \to P(X) \) and \( \beta : PO(Y) \to P(Y) \) be the operations on \( PO(X) \) and \( PO(Y) \), respectively.

Definition 5.1 A mapping \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( P_{\gamma,\beta}\)-continuous if for each \( x \in X \) and each \( P_{\gamma}\)-open set \( V \) containing \( f(x) \), there exists a \( P_{\gamma}\)-open set \( U \) such that \( x \in U \) and \( f(U) \subseteq V \).

Theorem 5.2 Let \( f : (X, \tau) \to (Y, \sigma) \) be a \( P_{\gamma,\beta}\)-continuous mapping. Then

1. \( f(p_{\gamma}Cl(A)) \subseteq p_{\beta}Cl(f(A)) \) holds for every subset \( A \) of \( X \).
2. For every \( P_{\beta}\)-closed set \( B \) of \( (Y, \sigma) \), \( f^{-1}(B) \) is \( P_{\gamma}\)-closed in \((X, \tau)\).

Proof

1. Let \( y \in f(p_{\gamma}Cl(A)) \) and \( V \) be the \( P_{\gamma}\)-open set containing \( y \), then there exists a point \( x \in X \) and a \( P_{\gamma}\)-open set \( U \) such that \( f(x) = y \), \( x \in U \) and \( f(U) \subseteq V \). Since \( x \in p_{\gamma}Cl(A) \), we have \( U \cap A \neq \emptyset \), and hence \( f(U \cap A) \subseteq f(U) \subseteq V \cap f(A) \). This implies \( y \in p_{\beta}Cl(f(A)) \).
2. It is sufficient to prove that (1) implies (2). Let \( B \) be the \( P_{\beta}\)-closed set in \((Y, \sigma)\). That is \( p_{\beta}Cl(B) = B \). By using (1) we have \( f(p_{\gamma}Cl(f^{-1}(B))) \subseteq p_{\beta}Cl(f(f^{-1}(B))) \subseteq p_{\beta}Cl(B) = B \) holds. Therefore \( p_{\gamma}Cl(f^{-1}(B)) \subseteq f^{-1}(B) \), and hence \( f^{-1}(B) = p_{\beta}Cl(f^{-1}(B)) \). Hence \( f^{-1}(B) \) is \( P_{\gamma}\)-closed in \((X, \tau)\).

Definition 5.3 A mapping \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( P_{\gamma,\beta}\)-closed if for any \( P_{\gamma}\)-closed set \( A \) of \((X, \tau)\), \( f(A) \) is \( P_{\beta}\)-closed.

Definition 5.4 Let \( id : PO(X) \to P(X) \) be the identity operation. If \( f \) is \( P_{id,\beta}\)-closed, then \( f(F) \) is \( P_{\beta}\)-closed for any preclosed set \( F \) of \((X, \tau)\).

Remark 5.5 If \( f \) is bijective mapping and \( f^{-1} : (Y, \sigma) \to (X, \tau) \) is \( P_{\beta, id}\)-continuous, then \( f \) is \( P_{id,\beta}\)-closed.

Proof Follows from Definitions 5.3 and 5.4.

Theorem 5.6 Suppose \( f : (X, \tau) \to (Y, \sigma) \) is \( P_{\gamma,\beta}\)-continuous and \( f \) is \( P_{\gamma,\beta}\)-closed, then

1. For every \( P_{\gamma, g}\)-closed set \( A \) of \((X, \tau)\) the image \( f(A) \) is \( P_{\beta, g}\)-closed.
2. For every \( P_{\gamma, g}\)-closed set \( B \) of \((Y, \sigma)\) the inverse set \( f^{-1}(B) \) is \( P_{\gamma, g}\)-closed.

Proof

1. Let \( V \) be any \( P_{\gamma}\)-open set in \((Y, \sigma)\) such that \( f(A) \subseteq V \), then by Theorem 5.2 (2) \( f^{-1}(V) \) is \( P_{\gamma}\)-open. Since \( A \) is \( P_{\gamma, g}\)-closed and \( A \subseteq f^{-1}(V) \), we have \( p_{\gamma}Cl(A) \subseteq f^{-1}(V) \), and hence \( f(p_{\gamma}Cl(A)) \subseteq V \). By assumption \( f(p_{\gamma}Cl(A)) \) is a \( P_{\beta}\)-closed set, therefore \( p_{\beta}Cl(f(A)) \subseteq p_{\beta}Cl(f(p_{\gamma}Cl(A))) = f(p_{\gamma}Cl(A)) \subseteq V \). This implies \( f(A) \) is \( P_{\beta, g}\)-closed.
2. Let \( U \) be any \( P_{\gamma}\)-open set such that \( f^{-1}(B) \subseteq U \). Let \( F = p_{\gamma}Cl(f^{-1}(B)) \cap (X \setminus U) \), then \( F \) is \( P_{\gamma}\)-closed in \((X, \tau)\). This implies \( f(F) \) is \( P_{\beta}\)-closed set in \((Y, \sigma)\). Since \( f(F) = f(p_{\gamma}Cl(f^{-1}(B)) \cap (X \setminus U)) \subseteq p_{\beta}Cl(B) \cap f(X \setminus U) \subseteq p_{\beta}Cl(B) \cap (Y \setminus B) \). This implies \( f(F) = \emptyset \) and hence \( F = \emptyset \). Therefore \( p_{\gamma}Cl(f^{-1}(B)) \subseteq U \). This implies \( f^{-1}(B) \) is \( P_{\gamma, g}\)-closed.

Theorem 5.7 Suppose \( f : (X, \tau) \to (Y, \sigma) \) is \( P_{\gamma,\beta}\)-continuous and \( P_{\gamma,\beta}\)-closed, then

1. If \( f \) is injective and \((Y, \sigma)\) is \( P_{\beta, \frac{1}{2}}\), then \((X, \tau)\) is \( P_{\gamma, \frac{1}{2}}\).
2. If \( f \) is a surjective and \((X, \tau)\) is \( P_{\gamma, \frac{1}{2}}\), then \((Y, \sigma)\) is \( P_{\beta, \frac{1}{2}}\).

Proof
Let $A$ be a $P_{\gamma}$-g.closed set of $(X, \tau)$. Now to prove that $A$ is $P_{\gamma}$-closed. By Theorem 5.6 (1), $f(A)$ is $P_{\beta}$-g.closed. Since $(Y, \sigma)$ is $P_{\beta}-T_{\frac{1}{2}}$, this implies that $f(A)$ is $P_{\beta}$-closed. Since $f$ is $P_{\gamma,\beta}$-continuous, then by Theorem 5.2, we have $A = f^{-1}(f(A))$ is $P_{\gamma}$-closed. Hence $(X, \tau)$ is $P_{\gamma}-T_{\frac{1}{2}}$.

Let $B$ be a $P_{\beta}$-g.closed set in $(Y, \sigma)$. Then $f^{-1}(B)$ is $P_{\gamma}$-closed, since $(X, \tau)$ is $P_{\gamma}-T_{\frac{1}{2}}$ space. It follows from the assumption that $B$ is $P_{\beta}$-closed.

Definition 5.8 A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be $P_{\gamma,\beta}$-homeomorphism, if $f$ is bijective, $P_{\gamma,\beta}$-continuous and $f^{-1}$ is $P_{\beta,\gamma}$-continuous.

Theorem 5.9 Let $f : (X, \tau) \to (Y, \sigma)$ be $P_{\gamma,\beta}$-homeomorphism. If $(X, \tau)$ is $P_{\gamma}-T_{\frac{1}{2}}$ then $(Y, \sigma)$ is $P_{\beta}-T_{\frac{1}{2}}$.

Proof Let $\{y\}$ be a singleton set of $(Y, \sigma)$, then there exists a point $x$ of $X$ such that $y = f(x)$. It follows from the assumption and Theorem 4.13 that $\{x\}$ is $P_{\gamma}$-open or $P_{\gamma}$-closed. By using Theorem 5.2 (2), $\{y\}$ is $P_{\beta}$-open or $P_{\beta}$-closed. This implies $(Y, \sigma)$ is $P_{\beta}-T_{\frac{1}{2}}$ space.

References