RESEARCH ARTICLE

Fine-irresolute Mappings

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By using the topology of the space $X$, a wide class of sets named as a class of fine-open sets has been defined in the present paper. This new collection contains open sets, $\alpha$-open sets, $\beta$-open sets, pre-open sets and semi-open sets. In view of this class of fine-open sets, the concept of a fine-irresolute mapping has been introduced which may be used in some branches of quantum physics where the homeomorphic images of the given sets are required with high accuracy. Using the idea of fine-open sets, a class of various continuous functions like fine-continuous, $\alpha f$-continuous, $\beta f$-continuous, $sf$-continuous, $pf$-continuous, $f\alpha f$-continuous, $f\beta f$-continuous, $fsf$-continuous, $fpf$-continuous, $sf$-irresolute, $pf$-irresolute are also defined and their properties have been studied. This new class of fine-irresolute functions contains several continuous functions which are already defined (cf. [Maheshwari S. N. and Thakur S. S., On $\alpha$-irresolute mappings, Tamkang J. Math., 11(1980), 209-214], [Beceren Y. and Noiri T., Strongly precontinuous functions. Acta. Math. Hung., 108(1-2)(2005), 47-53]).

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1. Introduction

Recently, general topology is playing a key role in many fields of applied sciences and several branches of mathematics such as data mining, computational topology for geometric design and molecular design, computer aided design, computer aided geometric design, digital topology, information systems, particle physics, quantum physics, high energy physics and superstring theory (cf. [14–20, 27–30, 39, 42, 44]). The concept of homeomorphism which is a very powerful concept in general topology has been used in the work of quantum physics, in particular refer Theorem 1 and 2 of [16] (see page 1341). The notions of several continuous functions in topological spaces and fuzzy topological spaces are widely developed which are used extensively in many practical and engineering problems.

Recently, generalization of continuity has been studied by many topologists (cf. [1, 13, 31, 32, 37, 38]). Balachandran et al. [4] defined a new class of mappings called generalized continuous mappings. The concept of $g$-closed sets was initiated by Levin in [33] and in last decade, the modifications of $g$-closed sets have been introduced by using $\alpha$-open sets, pre-open sets, semi-open sets, and $\beta$-open sets. The properties of these sets and several forms of continuous functions have been studies extensively. The concepts of closed sets and continuous functions have found to be useful in computer science and digital topology (cf. [27, 28]).

In the present paper, the authors have defined the most general class of sets by using the topology of the space $X$, which contains open sets, $\alpha$-open sets, $\beta$-open sets, pre-open sets and semi-open sets.

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These sets are named as fine-open sets in general and with the help of these sets the most general form of continuity has been defined which is identified as fine-irresolute continuity that covers almost all types of continuous functions defined earlier(cf. [1, 3–10, 13, 21–26, 31, 32, 37, 38, 41, 45]). The basic importance of this new class is that it gives the collection of all open, α-open, β-open, semi-open, and pre-open sets in a very simple manner which definitely enhances its applicational values. The notions of fine-closed sets and fine limit point of a set in a topological space are also introduced in this paper. The concept has been extended over the product space and the authors have discussed several results in the product space also.

Using the idea of fine-irresolute mapping, the different classes of continuous functions such as αf-irresolute, βf-irresolute, pf-irresolute, sf-irresolute, etc. are also defined. In view of this new class, the strong form of homeomorphism viz. fine-irresolute homeomorphism and fine-irresolute quotient space are also studied.

2. Preliminaries

Throughout this paper \( f : X \to Y \) denotes a single valued function of a topological space \((X, \tau)\) into a topological space \((Y, \tau')\). Let \( S \) be the subset of the space \((X, \tau)\). The closure and the interior of \( S \) are denoted by \( cl(S) \) and \( int(S) \) respectively.

Recall the following definitions, which are required later.

Definition 2.1 A subset \( S \) of a space \((X, \tau)\) is called:

1. \( \alpha \)-open [6] if \( S \subseteq int(cl(int(S))) \).
2. Semi-open [6] if \( S \subseteq cl(int(S)) \).
3. Pre-open [6] if \( S \subseteq int(cl(S)) \).
4. \( \beta \)-open [6] if \( S \subseteq cl(int(cl(S))) \).
5. Regular-open [6] if \( S = int(cl(S)) \).

Remark 2.2 Let \( S \) be subset of a space \((X, \tau)\) then, following may be verified easily.

1. If \( S \) is \( \alpha \)-open then, it is pre-open.
2. If \( S \) is semi-open then, it is \( \beta \)-open.
3. Intersection of semi-open and pre-open sets is an \( \alpha \)-open set.

Definition 2.3 A function \( f : (X, \tau) \to (Y, \tau') \) is called

1. Continuous [40] if \( f^{-1}(V) \) is open in \( X \) for every open set \( V \) of \( Y \).
2. \( \alpha \)-continuous [38] if \( f^{-1}(V) \) is \( \alpha \)-open in \( X \) for every open set \( V \) of \( Y \).
3. Pre-continuous [37] if \( f^{-1}(V) \) is pre-open in \( X \) for every open set \( V \) of \( Y \).
4. \( \beta \)-continuous [1] if \( f^{-1}(V) \) is \( \beta \)-open in \( X \) for every open set \( V \) of \( Y \).
5. Strongly \( \alpha \)-irresolute [34] if \( f^{-1}(V) \) is open in \( X \) for every \( \alpha \)-open set \( V \) of \( Y \).
6. \( \alpha \)-irresolute [35] if \( f^{-1}(V) \) is \( \alpha \)-open in \( X \) for every \( \alpha \)-open set \( V \) of \( Y \).
7. \( \alpha \)-pre-continuous [7] if \( f^{-1}(V) \) is pre-open in \( X \) for every \( \alpha \)-open set \( V \) of \( Y \).
8. Almost \( \alpha \)-irresolute [6] if \( f^{-1}(V) \) is \( \beta \)-open in \( X \) for every \( \alpha \)-open set \( V \) of \( Y \).
9. Strongly semi-continuous [2] if \( f^{-1}(V) \) is open in \( X \) for every semi-open set \( V \) of \( Y \).
10. Strongly \( \alpha \)-continuous [5] if \( f^{-1}(V) \) is \( \alpha \)-open in \( X \) for every semi-open set \( V \) of \( Y \).
11. Almost irresolute [11] if \( f^{-1}(V) \) is \( \beta \)-open in \( X \) for every semi-open set \( V \) of \( Y \).
12. Strongly \( \beta \)-irresolute [40] if \( f^{-1}(V) \) is open in \( X \) for every \( \beta \)-open set \( V \) of \( Y \).
13. Strongly \( \beta \)-irresolute [36] if \( f^{-1}(V) \) is \( \beta \)-open in \( X \) for every \( \beta \)-open set \( V \) of \( Y \).
14. \( \beta \)-irresolute [36] if \( f^{-1}(V) \) is \( \beta \)-open in \( X \) for every \( \beta \)-open set \( V \) of \( Y \).
15. Strongly \( \alpha \)-pre-continuous [8] if \( f^{-1}(V) \) is \( \alpha \)-open in \( X \) for every \( \beta \)-open set \( V \) of \( Y \).
16. Pre-continuous [43] if \( f^{-1}(V) \) is pre-open in \( X \) for every open set \( V \) of \( Y \).
17. Almost pre-continuous [43] if \( f^{-1}(V) \) is pre-open in \( X \) for every regular open set \( V \) of \( Y \).
3. Fine-open sets and fine-closed sets

Definition 3.1 Let \((X, \tau)\) be a topological space we define \(\tau(A) = \tau_\alpha(\text{say}) = \{G_\alpha(\neq X) : G_\alpha \cap A \neq \phi, \) for \(A_\alpha \in \tau \) and \(A_\alpha \neq \phi, X, \) for some \(\alpha \in J, \) where \(J \) is the index set.\} Now, define \(\tau_f = \{\phi, X\} \cup \alpha \{\tau_\alpha\}.\) The above collection \(\tau_f\) of subsets of \(X\) is called the fine collection of subsets of \(X\) and \((X, \tau, \tau_f)\) is said to be the fine space \(X\) generated by the topology \(\tau\) on \(X.\)

Definition 3.2 A subset \(U\) of a fine space \(X,\) if \(U\) belongs to the collection \(\tau_f\) and the complement of every fine-open set of \(X\) is called the fine-closed set of \(X\) and we denote the collection by \(F_f.\)

The family of all \(\alpha\)-open sets (respectively \(\beta\)-open sets, \(\beta\)-open sets, \(\alpha\)-open sets, \(\alpha\)-open sets) is denoted by \(\tau^n\) (respectively \(\beta O(X), PO(X), SO(X)\)). Moreover, \(\tau \subset \tau^n \subset PO(X) \subset \beta O(X) \subset \tau_f\) and \(\tau \subset \tau^n \subset SO(X) \subset \beta O(X) \subset \tau_f\) (The last inclusion would be proved in Th. 4.2).

Example 3.3 Let \(X = \{a, b, c\}\) be a topological space with the topology \(\tau = \{\phi, X, \{a, b, \{a, b\}\}\} \cong \{\phi, X, A_\alpha, A_\beta, A_\gamma\}\) (say) where, \(A_\alpha = \{a\}, A_\beta = \{b\}, A_\gamma = \{a, b\}.\) In view of Definition 3.1, we have \(\tau_\alpha = \tau(\{a\}) = \{\{a\}, \{a, b\}, \{a, c\}\}, \tau_\beta = \tau(\{b\}) = \{\{b\}, \{a, b\}, \{b, c\}\}, \tau_\gamma = \tau(\{a, b\}) = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\},\) then, the fine-collection is \(\tau_f = \{\phi, X\} \cup \alpha \{\tau_\alpha\} = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} .\)

Example 3.4 Let \(R\) with the standard topology and let \(\{x_1, x_2, x_3, ..., x_n\}\) be any subset of \(R.\) Then, there exist \((a, b)\) such that \(\{x_1, x_2, x_3, ..., x_n\} \in (a, b),\) it implies that \(\{x_1, x_2, x_3, ..., x_n\} \setminus (a, b) \neq \phi.\) Hence, \(\{x_1, x_2, x_3, ..., x_n\}\) is an fine-open set of \(R\) with respect to the standard topology on \(R,\) in which each open interval of the type \((a, b)\) is an open set.

Definition 3.5 A fine-open set \(S\) of a space \((X, \tau, \tau_f)\) is called:

1. \(\alpha f\)-open if \(S\) is an \(\alpha\)-open subset of a topological space \((X, \tau).\)
2. \(sf\)-open if \(S\) is a semi open subset of a topological space \((X, \tau).\)
3. \(pf\)-open if \(S\) is a pre-open subset of a topological space \((X, \tau).\)
4. \(\beta f\)-open if \(S\) is a \(\beta\)-open subset of a topological space \((X, \tau).\)
5. \(rf\)-open if \(S\) is a regular-open subset of a topological space \((X, \tau).\)
6. \(f\)-clopen\((f\)-clopen\)) if \(S\) is both fine-open and fine-closed subset of \(X.\)

Remark 3.6 Basically \(\alpha f, \beta f, pf,\) and \(sf\) open sets are respectively \(\alpha\)-open, \(\beta\)-open, pre-open, and semi-open sets described in Definition 2.1.

Remark 3.7

1. Intersection of two \(\alpha f\)-open sets is an \(\alpha f\)-open set (cf. [10]).
2. Intersection of \(sf\)-open set and \(pf\)-open set is an \(\alpha f\)-open set (cf. [10]).

Remark 3.8 Let \((X, \tau, \tau_f)\) be a fine space. If the union of all \(A_\alpha(\neq X)\) is \(X\) then, \(\tau_f\) will be the discrete topology on \(X.\)

Example 3.9 Let \(X = \{a, b, c\}\) be a topological space with the topology \(\tau = \{\phi, X, \{a, b, \{a, b\}\}\}\) then, the collection of fine-open sets \(\tau_f\) is given by \(\tau_f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}\) and the collection of fine-closed sets is given by \(F_f = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b, a\}\}\)

Definition 3.10 Let \(A\) be a subset of a fine space \(X,\) we say that a point \(x \in X\) is a fine limit point of \(A\) if every fine-open set of \(X\) containing \(x\) must contains at least one point of \(A\) other than \(x.\)

Remark 3.11 If \(x\) is a fine-limit point of the subset \(A\) of \((X, \tau, \tau_f),\) then essentially, it is a limit point of \(A\) but if \(x\) is a limit point of \(A,\) then it is not necessary that it would be a fine-limit point of \(A.\)

Example 3.12 Let \(X = \{a, b, c\}\) be a topological space with the topology \(\tau = \{\phi, X, \{b, \{b, c\}\}\}\) then the fine-collection is given by \(\tau_f = \{\phi, X, \{b\}, \{b, c\}, \{a, b\}, \{c\}, \{a, c\}\}\). Let \(A = \{b, c\} \subset X,\) then \(\text{‘a’}\) is the fine-limit point of the set \(A,\) which is also a limit point of the set

\(G_f.\)
A. Again consider $B = \{c\} \subset X$, then it may be checked that $'a'$ is the limit point of the set $B$ which is not a fine-limit point of $B$.

Definition 3.13 Let $A$ be the subset of a fine space $X$, the fine interior of $A$ is defined as the union of all fine-open sets contained in the set $A$ i.e., the largest fine-open set contained in the set $A$ and is denoted by $f_{Int}$.

Example 3.14 Let $X = \{a, b, c\}$ be a topological space with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ then, the fine-collection is $\tau_f = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. We can see that $Int\{a, c\} = \{a\} ~and~ f_{Int}\{a, c\} = \{a, c\}$.

Definition 3.15 Let $A$ be the subset of a fine space $X$, the fine closure of $A$ is defined as the intersection of all fine-closed sets containing the set $A$ i.e the smallest fine-closed set containing the set $A$ and is denoted by $f_{cl}$.

Example 3.16 Let $X = \{a, b, c\}$ be a topological space with the topology $\tau = \{\phi, X, \{b\}\}$ then, the fine-collection is $\tau_f = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$, $F_f = \{\phi, X, \{a, c\}, \{c\}, \{a\}\}$. We can easily check that $cl\{a\} = \{a, c\} ~and~ f_{cl}\{a\} = \{a\}$.

Definition 3.17 Let $(X, \tau, \tau_f)$ be a fine-space and let $x \in X$, then a fine-open set $U$ of $X$ containing $x$ is called a fine-neighborhood of $X$.

4. Properties of fine-open and fine-closed sets

Theorem 4.1 Let $(X, \tau, \tau_f)$ be the fine space with respect to the topological space $(X, \tau)$, then $\tau_f$ contains all semi-open and $\alpha$-open sets.

Proof Consider, $F \subset X$ and $F \notin \tau_f$.

Claim: $F$ is not semi-open in $X$.

Since, it is given that $F \notin \tau_f \Rightarrow A_\alpha \cap F = \phi \forall \alpha \in J \Rightarrow Int(F) = \phi \Rightarrow cl(Int(F)) = \phi \Rightarrow F \nsubseteq cl(Int(F))$.

Hence, $F$ is not semi-open and hence $F \nsubseteq Int(cl(Int(F)))$ and therefore $F$ is not $\alpha$-open.

Theorem 4.2 Let $(X, \tau, \tau_f)$ be the fine space with respect to the topological space $(X, \tau)$, then $\tau_f$ contains all pre-open and $\beta$-open sets.

Proof Consider $F \subset X$ and $F \notin \tau_f$.

Claim: $F$ is not pre-open and not $\beta$-open in $X$.

Since, it is given that $F \notin \tau_f \Rightarrow A_\alpha \cap F = \phi \forall \alpha \in J \Rightarrow F \subseteq C(A_\alpha)$(where $C$ stands for the complement) and $cl(F) \subseteq C(A_\alpha)$ (by the definition of the closure of $F$ in $(X, \tau)$) and $C(A_\alpha)$ is a closed set containing $F$. Since, $A_\alpha \cap C(A_\alpha) = \phi$ and $cl(F) \subseteq C(A_\alpha) \Rightarrow (A_\alpha) \cap cl(F) = \phi \Rightarrow Int(cl(F)) = \phi$ and hence, $F \nsubseteq Int(cl(F))$. Therefore, $F$ is not pre-open and thus $F \nsubseteq cl(Int(cl(F)))$. Hence, $F$ is not $\beta$-open.

Example 4.3 Let $X = \{a, b, c\}$ be a topological space with the topology $\tau = \{\phi, X, \{a\}\}$, then the fine-collection is $\tau_f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$, $F_f = \{\phi, X, \{b, c\}, \{c\}, \{b\}\}$. It may be easily checked that, the sets $\{b, c\}, \{c\}, \{b\}$ are not the member of $\tau_f$, they are not $\alpha$-open, $\beta$-open, semi-open and pre-open, but they are fine-closed sets.

Theorem 4.4 Let $(X, \tau, \tau_f)$ be the fine space. Consider $A(\neq \phi) \subset X$, then $x \in A_\alpha(\neq X)$ for each $\alpha \in J$, can not be a fine limit point of $A$ for $A_\alpha \in \tau$.

Proof Consider $A_\alpha \in \tau$ and $x \in A_\alpha$, then by the definition of $\tau_f$, the set $\{x\}$ is fine open in $(X, \tau, \tau_f)$.

Hence, $\{x\}$ is the fine-open set containing $x$ which does not contain any point of $A$ other than $x$. Thus, $x(x \in A_\alpha)$ is not a fine limit point of $A$. ■
Theorem 4.5 Let \((X, \tau, \tau_f)\) be the fine space and let \(A(\neq \emptyset, X) \subseteq X\), then if \(x \in X\) is fine limit point of \(A\), then it is a limit point of \(A\) but the converse is not necessarily true.

**Proof** The proof of this result follows by the fact that every open set is a fine open set but the converse is not true in general. \(\blacksquare\)

Remark 4.6 For the converse of the Theorem 4.5, please refer Example 3.12.

Remark 4.7 Let \((X, \tau, \tau_f)\) be a fine space. If union of all \(A_{\alpha(a \in J)}(\neq X)\) is \(X\) then, each subset of \(X\) does not have a limit point as in this case \(\tau\) would generate the collection of power sets of \(X\).

Theorem 4.8 Let \((X, \tau, \tau_f)\) be the fine space, then arbitrary union of fine-open sets in \(X\) is fine open set in \(X\).

**Proof** Let \(\{G_{\alpha}\}_{\alpha \in J}\) be the collection of fine-open sets of \(X\). \(\Rightarrow G_{\alpha} \cap A_{\alpha} \neq \emptyset, \forall \alpha \in J\) and \(A_{\alpha}(\neq \emptyset, X) \in \tau\).

Claim: \(\bigcup_{\alpha \in J} G_{\alpha} = G\) is fine-open.

It is enough to show that \(G \cap A_{\beta} \neq \emptyset\) for \(A_{\beta}(\neq \emptyset, X) \in \tau\). Now, \((\bigcup_{\alpha \in J} G_{\alpha} \cap A_{\beta}) = (G_{\alpha} \cap A_{\beta} \cup (G_{\beta} \cap A_{\beta})) \Rightarrow\) there exists an index \(\beta \in J\) such that \(G_{\beta} \cap A_{\beta} \neq \emptyset\) (since \(G_{\beta} \in \tau_f\)). Hence, \((\bigcup_{\alpha \in J} G_{\alpha}) \cap A_{\beta} \neq \emptyset\) \(\Rightarrow G\) is fine-open. \(\blacksquare\)

Remark 4.9 The intersection of two fine-open sets need not be a fine-open set as the following example shows. Thus, the collection of fine-open sets in a space \(X\) does not form a topology but forms a generalized topology on \(X\) (cf. [12]).

Example 4.10 Let \(X = \{a, b, c\}\) be a topological space with the topology \(\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\), then \(\tau_f = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}\). It may be easily checked that the above collection \(\tau_f\) is not a topology since, \(\{a, c\} \cap \{b, c\} = \{c\}\) which is not a member of \(\tau_f\).

Theorem 4.11 Let \((X, \tau, \tau_f)\) be the fine space, then arbitrary intersection of fine-closed sets in \(X\) is a fine-closed set.

**Proof** Let \(\{F_{\alpha}\}_{\alpha \in J}\) be the collection of fine-closed sets of \(X\).

Claim: \(\bigcap_{\alpha \in J} F_{\alpha} = F\) (4.1)

is fine-closed.

It is enough to show that \(CF\) is fine-open. It is a consequence of De Morgan’s law to get \(CF = \bigcup CF_{\alpha}\) in view of (4.1). By using Theorem 4.8, it may be seen that the union of fine-open sets is a fine-open set which implies that \(CF = \bigcup_{\alpha \in J} CF_{\alpha}\) is a fine-open set. Hence, \(F\) is fine-closed. \(\blacksquare\)

Remark 4.12 The union of two fine-closed sets need not be a fine-closed set as the following example shows.

Example 4.13 Let \(X = \{a, b, c\}\) be a topological space with the topology \(\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\), then \(\tau_f = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}\). \(F_f = \{\emptyset, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}\). We can easily check that union of two fine-closed sets \(\{a\} \cup \{b\} = \{a, b\}\) is not a fine-closed set.

The following result is a direct consequence of definitions of closure and interior in fine space.

Theorem 4.14 Let \((X, \tau, \tau_f)\) be a fine space and \(A\) be any arbitrary subset of \(X\). Then:

(1) \(Int(A) \subseteq fInt(A)\).

(2) \(fcl(A) \subseteq cl(A)\).

**Proof** The proof of this theorem is a direct-consequence of the definitions. \(\blacksquare\)

Example 4.15
Definition 5.1 A function \( f : (X, \tau) \to (Y, \tau') \) is fine-continuous if \( f^{-1}(V) \) is fine-open in \( X \) for every fine-open set \( V \) of \( Y \).

Example 6.2 Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}\} \) and \( \tau_f = \{\Phi, X, \{a, b\}, \{a, c\}, \{b, c\}\} \). Let \( A = \{a, b\} \), then \( \text{Int}(A) = \{b\} \) and \( f_{\text{Int}}(A) = \{a, b\} \).

Proof As given in [24](see Theorem 17.5, page 96).

5. Fine-irresolute mapping

Definition 5.1 A function \( f : (X, \tau) \to (Y, \tau') \) is called fine-irresolute (or f-irresolute) if \( f^{-1}(V) \) is fine-open in \( X \) for every fine-open set \( V \) of \( Y \).

Example 5.2 Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\} \) and \( Y = \{1, 2, 3\} \) with the topology \( \tau' = \{\Phi, Y, \{1\}, \{2\}, \{1, 2\}\} \).

Theorem 5.3 Let \( X \) and \( Y \) be fine spaces and \( f : X \to Y \). Then, the following are equivalent(cf. [40]).

(1) \( f \) is fine-irresolute.

(2) For every subset \( A \) of \( X \), one has \( f(f_{\text{cl}}(A)) \subseteq f_{\text{cl}}(f(A)) \).

(3) For every fine closed set \( B \) in \( Y \), the set \( f^{-1}(B) \) is fine closed in \( X \).

Proof (1) \( \Rightarrow \) (2) : Assume that \( f \) is fine-irresolute. Let \( A \) be a subset of \( X \). We show that if \( x \in f_{\text{cl}}(A) \), then \( f(x) \in f(f_{\text{cl}}(A)) \Rightarrow f(x) \in f_{\text{cl}}(f(A)) \). Let \( V \) be a fine neighborhood of \( f(x) \). Then \( f^{-1}(V) \) is fine open set of \( X \) containing \( x \); it must intersect \( A \) in some point \( y \)(cf. Theorem 4.8 of section 4 ). Then \( V \) intersects \( f(A) \) in the point \( y \), so that \( f(x) \in f_{\text{cl}}(f(A)) \), as desired.

(2) \( \Rightarrow \) (3) : Let \( B \) be fine-closed in \( Y \) and \( A = f^{-1}(B) \). We wish to prove that \( A \) is fine-closed in \( X \); we show that \( f_{\text{cl}}(A) = A \). By elementary set theory, we have \( f(A) = f(f^{-1}(B)) \subseteq B \). Therefore, if \( x \in f_{\text{cl}}(A) \), then \( f(x) \in f(f_{\text{cl}}(A)) \subseteq f_{\text{cl}}(f(A)) \subseteq f_{\text{cl}}(B) = B \) (since, \( B \) is fine-closed), so that \( x \in f^{-1}(B) = A \). Thus \( f_{\text{cl}}(A) \subseteq A \), so that \( f_{\text{cl}}(A) = A \), as desired.

(3) \( \Rightarrow \) (1) : Let \( B \) be a fine-open set of \( Y \). Set \( B = Y - V \). Then \( f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V) \). Now, \( B \) is a fine-closed set of \( Y \). Then \( f^{-1}(B) \) is fine-closed in \( X \) by hypothesis, so that \( f^{-1}(V) \) is fine-open in \( X \), as desired.

6. Some general class of fine-continuous functions

Definition 6.1 A function \( f : (X, \tau, \tau_f) \to (Y, \tau', \tau'_f) \) is called fine-continuous if \( f^{-1}(V) \) is open in \( X \) for every fine-open set \( V \) of \( Y \).

Example 6.2 Refer an example of a function which is a fine-continuous. Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}\} \) and \( Y = \{1, 2, 3\} \) with the topology \( \tau' = \{\Phi, Y, \{1\}\} \),
\[
\tau_f^\prime = \{\Phi, Y, \{1\}, \{1, 2\}, \{1, 3\}\}. \text{ We define a mapping } f : X \to Y \text{ such that } f(a) = 1, f(b) = 2 \text{ and } f(c) = 3.
\]

(1) It may be seen that for the given fine-open sets \(\{1\}, \{1, 2\}, \{1, 3\}\) of \(Y\), their respective pre-images \(\{a\}, \{a, b\}, \{a, c\}\) are open in \(X\). Therefore, \(f\) is fine-continuous.

Definition 6.3 A function \(f : (X, \tau, \tau_f) \to (Y, \tau', \tau_f^\prime)\) is called fine semi-continuous if \(f^{-1}(V)\) is fine-open in \(X\) for every open set \(V\) of \(Y\).

Example 6.4 We construct an example of a function which is a fine-semi continuous but not continuous. Let \(X = \{a, b, c\}\) with the topology \(\tau = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}\}\) and \(Y = \{1, 2, 3\}\) with the topology \(\tau' = \{\Phi, Y, \{1\}, \{1, 2\}\}\). Define a mapping \(f : X \to Y\) such that \(f(a) = 1, f(b) = 2\) and \(f(c) = 3\).

(1) It may be verified easily that \(f\) is not a continuous map.

(2) It may be seen that for the given open sets \(\{1\}, \{1, 2\}\) of \(Y\), their respective pre-images \(\{a\}, \{a, b\}\) are fine-open in \(X\). Therefore, \(f\) is fine-continuous.

Definition 6.5 A function \(f : (X, \tau, \tau_f) \to (Y, \tau', \tau_f^\prime)\) is called

(1) \(\alpha f\)-continuous if \(f^{-1}(V)\) is \(\alpha f\)-open in \(X\) for every fine-open set \(V\) of \(Y\).

(2) \(\beta f\)-continuous if \(f^{-1}(V)\) is \(\beta f\)-open in \(X\) for every fine-open set \(V\) of \(Y\).

(3) \(pf\)-continuous if \(f^{-1}(V)\) is \(pf\)-open in \(X\) for every fine-open set \(V\) of \(Y\).

(4) \(sf\)-continuous if \(f^{-1}(V)\) is \(sf\)-open in \(X\) for every fine-open set \(V\) of \(Y\).

Example 6.6 Thought our discussion in this example, we consider \(X = \{a, b, c\}\).

(1) An example of a function which is a \(\alpha f\)-continuous. Define topology \(\tau = \{\Phi, X, \{a\}, \{a, b\}\}, \tau_f = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}\}\) and \(Y = \{1, 2, 3\}\) with the topology \(\tau' = \{\Phi, Y, \{1\}\}\), \(\tau_f' = \{\Phi, Y, \{1\}, \{1, 2\}\}\). Define a mapping \(f : X \to Y\) such that \(f(a) = 1, f(b) = 2\) and \(f(c) = 3\).

(2) An example of a function which is a \(\beta f\)-continuous. Define topology \(\tau = \{\Phi, X, \{a, b\}\}, \tau_f = \{\Phi, X, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}\}\) and \(Y = \{1, 2, 3\}\) with the topology \(\tau' = \{\Phi, Y, \{1\}\}\), \(\tau_f' = \{\Phi, Y, \{1\}, \{1, 2\}\}\). Consider a mapping \(f : X \to Y\) such that \(f(a) = 1, f(b) = 2\) and \(f(c) = 3\).

(3) We give an example of a function which is a \(pf\)-continuous. Define topology \(\tau = \{\Phi, X, \{a\}\}, \tau_f = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}\}\) and \(Y = \{1, 2, 3\}\) with the topology \(\tau' = \{\Phi, Y, \{1\}\}\), \(\tau_f' = \{\Phi, Y, \{1\}, \{1, 2\}\}\). Consider a mapping \(f : X \to Y\) such that \(f(a) = 1, f(b) = 2\) and \(f(c) = 3\).

(4) Please refer an example of a function which is a \(sf\)-continuous. Define topology \(\tau = \{\Phi, X, \{b\}, \{a, b\}\}, \tau_f = \{\Phi, X, \{b\}, \{b, c\}, \{a, b\}, \{a, c\}\}\) and \(Y = \{1, 2, 3\}\) with the topology \(\tau' = \{\Phi, Y, \{2\}\}\), \(\tau_f' = \{\Phi, Y, \{2\}, \{2, 3\}, \{1, 2\}\}\). Define a mapping \(f : X \to Y\) such that \(f(a) = 1, f(b) = 2\) and \(f(c) = 3\).

(1) The pre-images of fine-open sets \(\{2\}, \{2, 3\}\) of \(Y\) are \(\{b\}, \{b, c\}, \{a, b\}\), which are \(sf\)-open in \(X\). Therefore, \(f\) is \(sf\)-continuous.

Definition 6.7 A function \(f : (X, \tau, \tau_f) \to (Y, \tau', \tau_f^\prime)\) is called:

(1) \(\alpha f\)-continuous if \(f^{-1}(V)\) is \(\alpha f\)-open in \(X\) for every \(\alpha f\)-open set \(V\) of \(Y\).

(2) \(\beta f\)-continuous if \(f^{-1}(V)\) is \(\beta f\)-open in \(X\) for every \(\beta f\)-open set \(V\) of \(Y\).

(3) \(pf\)-continuous if \(f^{-1}(V)\) is \(pf\)-open in \(X\) for every \(pf\)-open set \(V\) of \(Y\).
Example 6.8 We now discuss some illustrative examples of several continuous functions given in Definition 6.7 with $X = \{a, b, c\}$.

(1) Define topology $\tau = \{\Phi, X, \{a, b\}, \{a, b, c\}\}, \tau_f = \{\Phi, X, \{a, b\}, \{a, b, c\}, \{a, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\Phi, Y, \{1\}\}, \tau'_f = \{\Phi, Y, \{1\}, \{1, 2\}, \{1, 3\}\}$. Consider a mapping $f : X \rightarrow Y$ such that $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$.

• Considering the $\alpha$-open sets $\{1\}, \{1, 2\}, \{1, 3\}$ of $Y$, it can be seen that their respective pre-images $\{a\}, \{a, b\}, \{a, c\}$ are fine-open in $X$. Therefore, $f$ is $f\alpha$-continuous but not continuous.

(2) Define topology $\tau = \{\Phi, X, \{b\}, \{b, c\}\}, \tau_f = \{\Phi, X, \{b\}, \{b, c\}, \{b, a\}, \{c\}, \{a, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\Phi, Y, \{2\}\}, \tau'_f = \{\Phi, Y, \{2\}, \{2, 3\}, \{2, 1\}\}$. Define a mapping $f : X \rightarrow Y$ such that $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$.

• It may be checked that the subsets $\{2\}, \{2, 3\}, \{2, 1\}$ of $Y$ are the only $\beta_f$-open sets of $Y$ and their respective pre-images $\{b\}, \{b, c\}, \{b, a\}$ are fine-open in $X$. Therefore, $f$ is $f\beta_f$-continuous function.

(3) Define topology $\tau = \{\Phi, X, \{a, b\}\}, \tau_f = \{\Phi, X, \{a, b\}, \{a, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\Phi, Y, \{1\}\}, \tau'_f = \{\Phi, Y, \{1\}, \{1, 2\}, \{1, 3\}\}$. For this case consider a mapping $f : X \rightarrow Y$ such that $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$.

• It may be checked that the subsets $\{1\}, \{1, 2\}, \{1, 3\}$ of $Y$ are the only $pf$-open sets of $Y$ and their respective pre-images $\{a\}, \{a, b\}, \{a, c\}$ are fine-open in $X$. Therefore, $f$ is $fpf$-continuous.

(4) Please refer an example of $f\alpha f$-continuous function. Define topology $\tau = \{\Phi, X, \{a\}\}, \tau_f = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\Phi, Y, \{1\}, \{1, 2\}\}, \tau'_f = \{\Phi, Y, \{1\}, \{1, 2\}, \{1, 3\}, \{2\}, \{2, 3\}\}$. Define a mapping $f : X \rightarrow Y$ such that $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$.

• It may be checked that the subsets $\{1\}, \{1, 2\}, \{1, 3\}$ of $Y$ are the only $sf$-open sets of $Y$ and their respective pre-images $\{a\}, \{a, b\}, \{a, c\}$ are fine-open in $X$. Therefore, $f$ is $f\alpha f$-continuous.

Definition 6.9 A function $f : (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ is called

(1) Almost fine-continuous if $f^{-1}(V)$ is fine-open in $X$ for every $rf$-open set $V$ of $Y$.

(2) $\alpha f$ (resp. $\beta f$)-irresolute if $f^{-1}(V)$ is $\alpha f$ (resp. $\beta f$)-open in $X$ for every $\alpha f$ (resp. $\beta f$)-open set $V$ of $Y$.

(3) $pf f$-irresolute if $f^{-1}(V)$ is $pf$-open in $X$ for every $pf$-open set $V$ of $Y$.

(4) $sf f$-irresolute if $f^{-1}(V)$ is $sf$-open in $X$ for every $sf$-open set $V$ of $Y$.

Example 6.10 In view of Definition 6.9, we discuss following examples with $X = \{a, b, c\}$.

(1) Define topology $\tau = \{\Phi, X, \{a, b\}\}, \tau_f = \{\Phi, X, \{a, b\}, \{a, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\Phi, Y, \{1\}, \{2, 3\}\}, \tau'_f = \{\Phi, Y, \{1\}, \{1, 2\}, \{1, 3\}, \{2\}, \{2, 3\}\}$. Consider a mapping $f : X \rightarrow Y$ such that $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$. By construction $f$ is one-one and onto.

• It may be seen that the subsets $\{1\}, \{2, 3\}$ of $Y$ are the only $rf$-open sets of $Y$ and their respective pre-images $\{a\}, \{b\}$ are fine-open in $X$. Therefore, $f$ is almost fine-continuous.

(2) Define topology $\tau = \{\Phi, X, \{b\}\}, \tau_f = \{\Phi, X, \{b\}, \{b, c\}, \{c\}, \{a, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\Phi, Y, \{2\}\}, \tau'_f = \{\Phi, Y, \{2\}, \{2, 3\}\}$. Define a mapping $f : (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ such that $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$.

• Considering all the $pf f$-open sets $\{2\}, \{1, 2\}, \{2, 3\}$ of $Y$, it can be seen that their respective pre-images $\{b\}, \{a, b\}, \{b, c\}$ are $pf f$-open in $X$. Therefore, $f$ is $pf f$-irresolute.

(3) Define topology $\tau = \{\Phi, X, \{b\}\}, \tau_f = \{\Phi, X, \{b\}, \{a, b\}, \{b, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\Phi, Y, \{2\}, \{2, 3\}\}, \tau'_f = \{\Phi, Y, \{2\}, \{1, 2\}, \{2, 3\}, \{3\}, \{1, 3\}\}$. Define a mapping $f : X \rightarrow Y$ such that $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$. 

(4) $f\alpha f$-continuous if $f^{-1}(V)$ is fine-open in $X$ for every $sf$-open set $V$ of $Y$. 
• It may be checked that the pre-images of $sf$-open sets of $Y$ viz. $\{2\}, \{1,2\}, \{2,3\}$ are $\{b\}, \{a,b\}, \{b,c\}$ respectively, which are $sf$-open in $X$. Therefore, $f$ is $sf$-irresolute.

Remark 6.11 Basically $\alpha f(\beta f)$-irresolute functions are $\alpha(\beta)$-irresolute, described in Definition 2.3.

From the definitions given in this section, the following implications hold for different mapping, which have been displayed in the tabular form:

<table>
<thead>
<tr>
<th>$f$-continuity $\to$</th>
<th>strong $\beta$-continuity $\to$</th>
<th>strong semi-continuity $\to$</th>
<th>strong $\alpha$-irresoluteness $\to$</th>
<th>continuity $\to$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha f$-continuity $\to$</td>
<td>$\alpha$ - pre-irresoluteness $\to$</td>
<td>$\alpha$-continuity $\to$</td>
<td>$\alpha$-irresoluteness $\to$</td>
<td>$\alpha$-continuity $\to$</td>
</tr>
<tr>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
<td></td>
</tr>
<tr>
<td>$pj$-continuity $\to$</td>
<td>strong $\beta$-pre-irresoluteness $\to$</td>
<td>strong pre-continuity $\to$</td>
<td>$\alpha$-pre-continuity $\to$</td>
<td>pre-continuity $\to$</td>
</tr>
<tr>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
<td></td>
</tr>
<tr>
<td>$\beta j$-continuity $\to$</td>
<td>$\beta$-irresoluteness $\to$</td>
<td>almost-irresoluteness $\to$</td>
<td>almost $\alpha$-irresoluteness $\to$</td>
<td>$\beta$-continuity $\to$</td>
</tr>
</tbody>
</table>

Example 6.12 An example of a function which is $f$-irresolute then, it is strong $\beta$-irresolute, strong semi-continuous, strong $\alpha$-irresolute, continuous. Let $X = \{a,b,c\}$ with the topology $\tau = \{\Phi, X, \{a\}, \{a,b\}, \{a,c\}\}$, $\tau_f = \{\Phi, X, \{a\}, \{a,b\}, \{a,c\}, \{b\}, \{c\}, \{b,c\}\}$ and $Y = \{1,2,3\}$ with the topology $\tau' = \{\Phi, Y, \{1\}, \{1,2\}\}$, $\tau'_f = \{\Phi, Y, \{1\}, \{1,2\}, \{1,3\}, \{2\}, \{2,3\}\}$. Define a mapping $f : X \to Y$ such that $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$.

• It may be checked that the subsets $\{1\}, \{1,2\}, \{1,3\}, \{2\}, \{2,3\}$ of $Y$ are the only fine-open sets of $Y$ and their respective pre-images $\{a\}, \{a,b\}, \{a,c\}, \{b\}, \{b,c\}$ are fine-open in $X$. Therefore, $f$ is fine-irresolute continuous function.

• It may be checked that the subsets $\{1\}, \{1,2\}, \{1,3\}$ of $Y$ are the only $\beta$-open sets of $Y$ and their respective pre-images $\{a\}, \{a,b\}, \{a,c\}$ are $\beta$-open in $X$. Therefore, $f$ is strong $\beta$-irresolute.

• It may be verified that the subsets $\{1\}, \{1,2\}, \{1,3\}$ of $Y$ are the only semi-open sets of $Y$ and their respective pre-images $\{a\}, \{a,b\}, \{a,c\}$ are semi-open in $X$. Therefore, $f$ is strong semi-continuous function.

• It may be observed that the subsets $\{1\}, \{1,2\}, \{1,3\}$ of $Y$ are the only $\alpha$-open sets of $Y$ and their respective pre-images $\{a\}, \{a,b\}, \{a,c\}$ are $\alpha$-open in $X$. Therefore, $f$ is strong $\alpha$-irresolute and continuous.

• It may be seen that the subsets $\{1\}, \{1,2\}$ of $Y$ are the open sets of $Y$ and their respective pre-images $\{a\}, \{a,b\}$ are open in $X$. Therefore, $f$ is continuous.

Theorem 6.13 If $f : (X, \tau, \tau_f) \to (Y, \tau', \tau'_f)$ is $\alpha f$-continuous and if $A$ is an $\alpha$-open subset of $X$, then the restriction $f|_A : A \to Y$ is $\alpha f$-continuous.

Proof Let $V$ be any fine-open set of $Y$. Since, $f$ is $\alpha f$-continuous, $f^{-1}(V)$ is $\alpha f$-open in $X$. Also by hypothesis, $A$ is $\alpha f$-open in $X$, therefore by using the property of $\alpha f$-open set (viz. the intersection of two $\alpha f$-open sets is $\alpha f$-open), then

$$f|_A^{-1}(V) = A \cap f^{-1}(V) \quad (6.1)$$
is αf-open in A and hence f|A is αf-continuous (cf. Definition 6.5).

Theorem 6.14 If f: (X, τ, τf) → (Y, τ', τ'f) is sf-continuous and if A is pf-open subset of X, then the restriction f|A: A → Y is αf-continuous.

Proof Let V be any fine-open set of Y, then by the definition of sf-continuous function, f−1(V) is sf-open in X. Since,

\[ f|^{-1}_A(V) = A \cap f^{-1}(V) \]  

(6.2)

which is αf-open in X for given A, pf-open (by hypothesis) and f−1(V) is sf-open (cf. Remark 3.7). Thus, f|A is αf-continuous.

Theorem 6.15 If f: (X, τ, τf) → (Y, τ', τ'f) is pf-continuous and if A is an sf-open subset of X, then the restriction f|A: A → Y is αf-continuous.

Proof Similar to that of Theorem 6.14.

Theorem 6.16 Let f: (X, τ, τf) → (Y, τ', τ'f) and g: (Y, τ', τ'f) → (Z, τ'', τ''f) be functions. Then the composition g ◦ f: X → Z is fine-irresolute if f is fαf-continuous and g is αf-continuous.

Proof Let V be any fine-open subset of Z. Since g is αf-continuous, g−1(V) is αf-open in Y. Since g−1(V) is αf-open in Y and f is fαf-continuous, f−1(g−1(V)) is fine-open in X. But

\[ f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \]  

(6.3)

Thus, g ◦ f is fine-irresolute.

Theorem 6.17 Let f: (X, τ, τf) → (Y, τ', τ'f) and g: (Y, τ', τ'f) → (Z, τ'', τ''f) be functions. Then, the composition g ◦ f: X → Z may be studied in the following several cases for different choices of f and g:

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Mapping f</th>
<th>Mapping g</th>
<th>g ◦ f</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>fβf-continuous</td>
<td>βf-continuous</td>
<td>fine-irresolute</td>
</tr>
<tr>
<td>2.</td>
<td>fpf-continuous</td>
<td>pf-continuous</td>
<td>fine-irresolute</td>
</tr>
<tr>
<td>3.</td>
<td>fsf-continuous</td>
<td>sf-continuous</td>
<td>fine-irresolute</td>
</tr>
<tr>
<td>4.</td>
<td>αf-continuous</td>
<td>αf-continuous</td>
<td>αf-irresolute</td>
</tr>
<tr>
<td>5.</td>
<td>βf-continuous</td>
<td>fβf-continuous</td>
<td>βf-irresolute</td>
</tr>
<tr>
<td>6.</td>
<td>pf-continuous</td>
<td>fpf-continuous</td>
<td>pf-irresolute</td>
</tr>
<tr>
<td>7.</td>
<td>sf-continuous</td>
<td>fsf-continuous</td>
<td>sf-irresolute</td>
</tr>
<tr>
<td>8.</td>
<td>fαf-continuous</td>
<td>αf-irresolute</td>
<td>fαf-continuous</td>
</tr>
<tr>
<td>9.</td>
<td>fβf-continuous</td>
<td>βf-irresolute</td>
<td>fβf-continuous</td>
</tr>
<tr>
<td>10.</td>
<td>fsf-continuous</td>
<td>sf-irresolute</td>
<td>fsf-continuous</td>
</tr>
<tr>
<td>11.</td>
<td>fpf-continuous</td>
<td>pf-irresolute</td>
<td>fpf-continuous</td>
</tr>
</tbody>
</table>

Proof Similar to that of Theorem 6.16.

7. Fine-irresolute homeomorphism

Definition 7.1 A function f: (X, τ, τf) → (Y, τ', τ'f) is f-irresolute homeomorphism if
Example 7.2 Let $X = \{a, b, c\}$ with the topology $\tau = \{\Phi, X, \{a, b\}\}$, $\tau_f = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\Phi, Y, \{1\}, \{2\}\}$. We define a mapping $f : X \to Y$ such that $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$. By construction $f$ is one-one and onto.

- It may be verified that the function $f$ is not a continuous function. Hence, it not homeomorphism.
- It may be seen that the subsets $\{1\}, \{2\}, \{1, 2\}$ of $Y$ and their respective pre-images $\{a\}, \{a, b\}, \{a, c\}, \{b, c\}$ are the fine-open sets of $Y$ and their respective pre-images $\{a\}, \{a, b\}, \{a, c\}, \{b, c\}$ are fine-open in $X$. Therefore $f$ is $f$-irresolute. Similarly, it may be checked that the inverse function $f^{-1} : Y \to X$ is also $f$-irresolute. Thus, $f$ is $f$-irresolute homeomorphism.

Definition 7.3 A function $f : (X, \tau, \tau_f) \to (Y, \tau', \tau'_f)$ is $pf$-irresolute homeomorphism if

1. $f$ is one-one and onto.
2. Both the function $f$ and the inverse function. $f^{-1} : Y \to X$ is $pf$-irresolute.

Example 7.4 The following is an example of $pf$-irresolute homeomorphism. Let $X = \{a, b, c\}$ with the topology $\tau = \{\Phi, X, \{a, b\}\}$, $\tau_f = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\Phi, Y, \{1\}\}$. Define a mapping $f : X \to Y$ such that $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$.

- Considering the $pf$-open sets $\{1\}, \{2\}, \{1, 2\}$ of $Y$, it can be seen that their respective pre-images $\{a\}, \{a, b\}, \{a, c\}$ are $pf$-open in $X$. Therefore $f$ is $pf$-irresolute. Similarly, we may check that the inverse function $f^{-1} : Y \to X$ is also $pf$-irresolute. Hence, $f$ is $pf$-irresolute homeomorphism.

Definition 7.5 A function $f : (X, \tau, \tau_f) \to (Y, \tau', \tau'_f)$ is $sf$-irresolute homeomorphism if

1. $f$ is one-one and onto.
2. Both the function $f$ and the inverse function. $f^{-1} : Y \to X$ are $sf$-irresolute.

Example 7.6 An example of $sf$-irresolute homeomorphism. Let $X = \{a, b, c\}$ with the topology $\tau = \{\Phi, X, \{b, c\}\}$, $\tau_f = \{\Phi, X, \{b\}, \{b, c\}, \{b, a\}, \{c\}, \{a, c\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\Phi, Y, \{2\}\}$. For this example consider, a mapping $f : X \to Y$ such that $f(a) = 1$, $f(b) = 2$, and $f(c) = 3$. By construction, $f$ is one-one and onto.

- It may be verified that the function $f$ is not homeomorphism.
- It may be checked that the subsets $\{2\}, \{2, 3\}, \{2, 1\}$ of $Y$ are the only $sf$-open sets of $Y$ and their respective pre-images $\{b\}, \{b, c\}, \{b, a\}$ are $sf$-open in $X$. Therefore $f$ is $sf$-irresolute. Similarly, we can easily check that the inverse function $f^{-1} : Y \to X$ is also $sf$-irresolute hence, the function $f$ is $sf$-irresolute homeomorphism.

8. Fine-Hausdorff space

Definition 8.1 A space $(X, \tau, \tau_f)$ is said to be fine-Hausdorff space if for each pair $x_1$, $x_2$ of distinct points of $X$, there exist fine-open sets $U_1$ and $U_2$ containing $x_1$ and $x_2$, respectively, that are disjoint.

Example 8.2 A space which is fine-Hausdorff but not Hausdorff. Let $X = \{a, b, c\}$ with the topology $\tau = \{\Phi, X, \{a\}, \{a, b\}\}$, $\tau_f = \{\Phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. It may be verified that this space is not Hausdorff but it is a fine-Hausdorff space.

Theorem 8.3 Every singleton set in a fine-Hausdorff space $X$ is fine-closed.

Proof It is sufficient to show that every one-point set $\{x_0\}$ is closed. If $x$ is a point of $X$ different from $x_0$, then, $x$ and $x_0$ have disjoint fine-neighborhoods $U$ and $V$, respectively. Since $U$ does not intersect $\{x_0\}$, the point $x$ cannot belong to the fine-closure of the set $\{x_0\}$. As a result, the fine closure of the
set \( \{x_0\} \) is \( \{x_0\} \) itself, so that it is fine-closed.

Remark 8.4 In fine-Hausdorff space a finite point set need not be fine-closed (cf. Remark 4.12).

9. Fine-quotient map

Definition 9.1 Let \( X \) and \( Y \) be a topological spaces; let \( p : X \to Y \) be onto map. The map \( p \) is said to be \((f,pf,sf)\)-irresolute quotient map provided a subset \( U \) of \( Y \) is \((f,pf,sf)\)-open in \( Y \) if and only if \( p^{-1}(U) \) is \((f,pf,sf)\)-open in \( X \).

Definition 9.2 A map \( f : X \to A \) is said to be an \((f,pf,sf)\)-open map if for each \((f,pf,sf)\)-open set \( U \) of \( X \), the set \( f(U) \) is \((f,pf,sf)\)-open in \( Y \).

Definition 9.3 A map \( f : X \to A \) is said to be an \((f,pf,sf)\)-closed map if for each \((f,pf,sf)\)-closed set \( U \) of \( X \), the set \( f(U) \) is \((f,pf,sf)\)-closed in \( Y \).

Remark 9.4 If \( p : X \to A \) is an onto continuous map that is either \( f \)-open or \((f,pf,sf)\)-closed, then \( p \) is a \((f,pf,sf)\)-irresolute quotient map.

Theorem 9.5 Let \( p : (X,\tau,\tau_f) \to (Y,\tau',\tau'_f) \) be a \( f \)-irresolute quotient map. Let \( Z \) be a space and let \( g : (X,\tau,\tau_f) \to (Z,\tau'',\tau''_f) \) be a \( f \)-irresolute map that is constant on each set \( p^{-1}\{y\} \) for \( y \in Y \). Then \( g \) induces \( f \)-irresolute map \( f : (Y,\tau',\tau'_f) \to (Z,\tau'',\tau''_f) \) such that \( fop = g \) (see Theorem 22.2 of [24]).

\[ \begin{array}{ccc} X & \xrightarrow{p} & Y \\ & / \searrow & f \\ Z \end{array} \]

**Proof** For each \( y \in Y \), the set \( g\{p^{-1}(y)\} \) is one point set in \( Z \) (Since \( g \) is constant on \( p^{-1}(y) \)). If we let \( f(y) \) denote this point then we have defined a map \( f : Y \to Z \) such that for each \( x \in X \),

\[ f(p(x)) = g(x) \]

Claim : \( f \) is \( f \)-irresolute. Let \( V \) be \( f \)-open set in \( Z \). \( f \)-irresoluteness of \( g \) implies that \( g^{-1}(V) = p^{-1}(f^{-1}(V)) \) (cf. (4.4)) is \( f \)-open set in \( X \). Since \( p \) is \( f \)-irresolute quotient map, \( p^{-1}(f^{-1}(V)) \) is \( f \)-open set in \( X \) if and only if \( f^{-1}(V) \) is \( f \)-open set in \( Y \). Thus, \( f \) is \( f \)-irresolute.

Remark 9.6 Theorem 9.5 also holds with the following choice of the mappings \( p \) and \( g \):

1. If \( p \) is \( \alpha f \) (resp. \( \beta f \))-irresolute quotient map and \( g \) is \( \alpha f \) (resp. \( \beta f \))-continuous map, then \( f \) is \( \alpha f \) (resp. \( \beta f \))-continuous.
2. If \( p \) is \( sf \) (resp. \( pf \))-irresolute quotient map and \( g \) is \( sf \) (resp. \( pf \))-continuous map, then \( f \) is \( sf \) (resp. \( pf \))-continuous.

10. Fine-continuity in product space

In this section, we study the extension of our concept of fine-open sets on the product space.

Lemma 10.1 (Lemma 3.1 of [9]) Let \( \{X_\lambda : \lambda \in \Lambda\} \) be a family of spaces and \( \bigcup_{\lambda_i} \) be non-empty subset of \( X_{\lambda_i} \) for each \( i = 1,2,...,n \). Then \( U = \Pi_{\lambda \neq \lambda_i} X_\lambda \times \Pi_{i=1}^n U_{\lambda_i} \) is a non empty fine-open [9] subset of \( \Pi_{\lambda \in J} X_\lambda \) if and only if \( U_{\lambda_i} \) is fine-open in \( X_{\lambda_i} \) for each \( i = 1,2,...,n \).

**Proof** It is given that \( U = \Pi_{\lambda \neq \lambda_i} X_\lambda \times \Pi_{i=1}^n U_{\lambda_i} \) is a non-empty fine-open subset of \( \Pi X_\lambda \). By the definition of fine-open sets, we have for open sets \( A_{\lambda_i} \) of \( X_{\lambda_i} \), \( (\Pi_{\lambda \neq \lambda_i} X_\lambda \times \Pi_{i=1}^n U_{\lambda_i}) \cap (\Pi_{\lambda \neq \lambda_i} X_\lambda \times \Pi_{i=1}^n U_{\lambda_i}) \)
fine-irresolute for each

\[ \Pi_{n=1}^\infty A_{\lambda_i} \neq \phi \Rightarrow \Pi_{\lambda \neq \lambda_i} X_{\lambda_i} \times (\Pi_{i=1}^n U_{\lambda_i} \cap \Pi_{i=1}^n A_{\lambda_i}) \neq \phi \Rightarrow \Pi_{\lambda \neq \lambda_i} X_{\lambda_i} \times (\Pi_{i=1}^n (U_{\lambda_i} \cap A_{\lambda_i})) \neq \phi. \]

It holds if \( U_{\lambda_i} \cap A_{\lambda_i} \neq \phi \) and by the definition of fine-open set, \( U_{\lambda_i} \) is fine-open in \( X_{\lambda_i} \).

Conversely, Let \( U_{\lambda_i} \) is fine-open in \( X_{\lambda_i} \), it implies that \( U_{\lambda_i} \cap A_{\lambda_i} \neq \phi \forall \lambda_i \) where, \( i = 1, 2, 3, \ldots, n. \)

\[ \Rightarrow \Pi_{\lambda \neq \lambda_i} X_{\lambda_i} \times (\Pi_{i=1}^n (U_{\lambda_i} \cap A_{\lambda_i})) \neq \phi. \Rightarrow \Pi_{\lambda \neq \lambda_i} X_{\lambda_i} \times (\Pi_{i=1}^n U_{\lambda_i} \cap \Pi_{i=1}^n A_{\lambda_i}) \neq \phi \Rightarrow (\Pi_{\lambda \neq \lambda_i} X_{\lambda_i} \times \Pi_{i=1}^n U_{\lambda_i}) \cap (\Pi_{\lambda \neq \lambda_i} X_{\lambda_i} \times \Pi_{i=1}^n A_{\lambda_i}) \neq \phi. \]

From above we conclude that \((\Pi_{\lambda \neq \lambda_i} X_{\lambda_i} \times \Pi_{i=1}^n U_{\lambda_i})\) is fine-open in \( \Pi X_{\lambda_i} \).

Theorem 10.2 Let \( f : (X, \tau, \tau_f) \rightarrow (X, \tau', \tau'_f) \) be a fine-irresolute function. Then, for each \( x \in X \) and each fine-open set \( V \) of \( Y \) containing \( f(x) \) there exist a fine-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq V \).

**Proof** Let \( x \in X \) and let \( V \) be any fine-open set of \( Y \) such that \( f(x) \in X \). By definition \( f^{-1}(V) \) is fine-open in \( X \) such that \( x \in f^{-1}(V) \). Set \( U = f^{-1}(V) \), since \( f \) is fine-irresolute, \( U \) is a fine-open subset of \( X \) containing \( x \) such that \( f(U) \subseteq V \).

Theorem 10.3 A function \( f : X \rightarrow Y \) is \( f \)-irresolute then the graph of the function \( g : X \rightarrow X \times Y \), defined by \( g(x) = (x, f(x)) \) for each \( x \in X \), is fine-irresolute.

**Proof** Let \( x \in X \) and \( V \) be any fine-open set of \( Y \) containing \( f(x) \). Then, \( U \times V \) is a fine-open set of \( X \times Y \) by lemma 10.1 and contains \( g(x) \). Since, \( g \) is fine-irresolute there exist a fine-open set \( U \) of \( X \) containing \( x \), then by Theorem 10.2, we have \( g(U) \subseteq V \). Thus, \( f \) is fine-irresolute.

Definition 10.4 A mapping \( P_\lambda : \Pi \lambda X_\lambda \rightarrow Y_\lambda \) is said to be projection map on \( X_\lambda \) fine-space if \( P_\lambda^{-1}(X_\lambda) \) is fine-open in \( \Pi X_\lambda \) for \( U_\lambda \) fine-open in \( Y_\lambda \) space.

Remark 10.5 The projection map is fine-irresolute (cf. Definition 5.1.).

Theorem 10.6 If a function \( f : X \rightarrow \Pi Y_\lambda \) is fine-irresolute, then \( P_\lambda f : X \rightarrow Y_\lambda \) is fine-irresolute for each \( \lambda \in \Lambda \), where \( P_\lambda \) be the fine-projection of \( \Pi Y_\lambda \) onto \( Y_\lambda \).

**Proof** Let \( V_\lambda \) be any fine-open set of \( Y_\lambda \). Since, \( P_\lambda \) is fine-irresolute, hence \( P_\lambda^{-1}(V_\lambda) \) is fine-open in \( \Pi Y_\lambda \). Since \( f \) is fine-irresolute, then \( f^{-1}(P_\lambda^{-1}(V_\lambda)) = (P_\lambda f)^{-1}(V_\lambda) \) is fine-open in \( X \). Hence, \( P_\lambda f \) is fine-irresolute for each \( \lambda \in \Lambda \).

Theorem 10.7 If the product function \( f : \Pi X_\lambda \rightarrow \Pi Y_\lambda \) is fine-irresolute, then \( f_\lambda : X_\lambda \rightarrow Y_\lambda \) is fine-irresolute for each \( \lambda \in \Lambda \).

**Proof** Let \( \lambda_0 \in \Lambda \) be an arbitrary fixed index and \( V_{\lambda_0} \) be any fine-open set of \( Y_{\lambda_0} \). Then, \( \Pi Y_\lambda \times V_{\lambda_0} \) is fine-open in \( \Pi Y_\lambda \) by lemma 10.1, where \( \lambda_0 \neq \gamma \in \Lambda \). Since \( f \) is fine-irresolute, then \( f^{-1}(\Pi Y_\lambda \times V_{\lambda_0}) = \Pi X_\gamma \times f_{\lambda_0}^{-1}(V_{\lambda_0}) \) is fine-open in \( \Pi X_\lambda \) and hence, by lemma 10.1 \( f_{\lambda_0}^{-1}(V_{\lambda_0}) \) is fine-open in \( X_{\lambda_0} \). This implies that \( f_{\lambda_0} \) is fine-irresolute.

11. Conclusion

The concept of fine-irresoluteness is of course very strong concept of continuity which assigns fine open sets of the domain to the fine open sets of the range. The ideas of fine-irresolute homeomorphism and fine-quotient mapping which have been studied in this paper are extensively used in quantum physics (see [14–20]). Our new definition of homeomorphism which associates some additional sets of domain and range would definitely help in finding a close homeomorphic image of the given set or the space. It has been observed that in some of the results (cf. Theorem 1 and Theorem 2 of [16]) of quantum physics instead of the original set its homeomorphic image has been studied and analyzed for some technical reasons. Our approach would definitely help in getting the homeomorphic images with more accuracy.
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References


