RESEARCH ARTICLE

On $S_p$-Closed Spaces

Alias B. Khalaf * and Hardi A. Shareef †

* Department of Mathematics, University of Duhok, Kurdistan-Region, Iraq.
† Department of Mathematics, Faculty of Science and Science Education, University of Sulaimani, Kurdistan-Region, Iraq.

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In this paper we introduce a new type of closed spaces called $S_p$-closed space. This class of spaces is strictly between $s$-closed spaces and $S_c$-closed spaces. We give some characterizations of this space and investigate relationships to other types of closed spaces. Moreover, we study the image and preimage of $S_p$-closed spaces under some special types of functions.

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1. Introduction

In the literature, many mathematicians have tried to generalize the notion of compactness in topological spaces. In the course of their attempts, several weaker and stronger versions of compactness have been studied in detail. It is seen that certain sets other than open sets have been employed for such investigations. In [1] Levine, defined a new type of sets called semi-open sets. Mashhour et. al [2], defined the notion of preopen sets. Di Maio et. al [3] have taken up an investigation of a sort of covering property, known as $s$-closedness with the help of the notion of semi-open sets and by using pre open sets Abo-Khadra [4] introduced the concept of p-closedness, the class of $S_p$-open sets and $S_c$-open sets were defined respectively in [5] and [6] as a special types of semi-open sets. In this paper we introduce the concept of $S_p$-closed spaces as a general form of the concept of $s$-closed spaces.

Throughout this paper, $X$ and $Y$ denote topological spaces on which no separation axioms are assumed. Let $A$ be a subset of the space $X$, then $int(A)$ and $cl(A)$ denote the interior and closure of $A$ respectively. $S_c$-closure (resp. $S_p$-closure, preclosure, semiclosure) of $S$ denoted by $S_ccl(S)$ (resp. $S_pcl(S)$, $pcl(S)$, $scl(S)$) is the intersection of all $S_c$-closed (resp. $S_p$-closed, pre closed, semi-closed) subsets of $X$ containing $S$. $S_p$-interior and $S_c$-interior of $S$ denoted by $S_pint(S)$ and $S_cint(S)$ respectively are defined as usual. We denote the classes of all $S_p$-open (resp. $S_c$-open, semi-open, preopen) sets in a space $X$ by $S_pO(X)$ (resp. $S_cO(X)$, $SO(X)$, $PO(X)$).

2. Preliminaries

In this section, we state some existing definitions and results as a prerequisite for the development of subsequent sections.

* Corresponding author
Email: aliasbkhala@gmail.com
Definition 2.1 A subset $A$ of a space $X$ is called:

1. Semi-open [1], if $A \subseteq cl(int(A))$,
2. Preopen [2], if $A \subseteq int(cl(A))$,
3. Regular closed [7], if $A = cl(int(A))$.

Definition 2.2 A semi-open set $A$ of a space $X$ is called $S_p$-open [5] (resp. $S_c$-open [6]) set if for each $x \in A$, there exists a preclosed (resp. closed) set $F$ such that $x \in F \subseteq A$.

Remark 2.4 From Lemma 2.3, we conclude that $x \in A$.

Remark 2.5 Since each $S_p$-open set is $S_c$-open set for each $A$.

Lemma 2.7 Let $X$ be a space and $A \subseteq X$, then $X \setminus S_p int(A) = S_p cl(X \setminus A)$.

Definition 2.8 A space $X$ is called:

1. Submaximal [8], if every dense subset of $X$ is open.
2. Extremally disconnected [3], if the closure of every open set is open set.
3. Locally indiscrete [9], if every open subset of $X$ is closed.

The following results are in [9].

Theorem 2.9 A space $X$ is extremally disconnected if and only if every semi-open subset of $X$ is preopen.

Lemma 2.10 If the space $X$ is extremally disconnected, then every $S_p$-open subset of $X$ is preopen.

Proof Follows from Theorem 2.9.

Theorem 2.11 A space $X$ is submaximal if and only if every preopen set is open.

The following results are in [10].

Theorem 2.12 Let $\Omega$ be a non-empty family of subsets of a space $X$. Then there exists a filterbase on $X$ containing $\Omega$ if and only if $\Omega$ has the finite intersection property (FIP).

Theorem 2.13 Every filter base on a set $X$ is contained in a maximal filter base.

Definition 2.14 A space $X$ is said to be $S$-closed [11] (resp. $s$-closed [3], $p$-closed [12], $S_c$-closed [6], quasi-H-closed [12] and compact [10]) if for every semi-open (resp. semi-open, preopen, $S_c$-open, open and open) cover $\{U_\alpha : \alpha \in \Delta\}$ of $X$, there exists a finite subfamily $\Delta_0$ of $\Delta$ such that $X = \bigcup_{\alpha \in \Delta_0} cl(U_\alpha)$ (resp. $X = \bigcup_{\alpha \in \Delta_0} scl(U_\alpha)$, $X = \bigcup_{\alpha \in \Delta_0} pcl(U_\alpha)$, $X = \bigcup_{\alpha \in \Delta_0} S_ccl(U_\alpha)$, $X = \bigcup_{\alpha \in \Delta_0} cl(U_\alpha)$, $X = \bigcup_{\alpha \in \Delta_0} U_\alpha$).

Definition 2.15 A Hausdorff space quasi H-closed space $X$ is called H-closed space [13].

Theorem 2.16 A space $X$ is $s$-regular if and only if for each open set $G$ and $x \in G$, there exists a semi-open set $V$ such that $x \in V$ and $scl(V) \subset G$ [14].

The following are in [5].

Definition 2.17 A function $f : X \to Y$ is $S_p$-continuous if and only if for every open subset $O$ of $Y$, $f^{-1}(O)$ is $S_p$-open set in $X$.

Theorem 2.18 A function $f : X \to Y$ is $S_p$-continuous if and only if for each $A \subseteq X$, $f(S_p cl(A)) \subseteq cl(f(A))$. 
Lemma 2.19 Let \( f : X \rightarrow Y \) be continuous and open function, then \( f^{-1}(B) \) is \( S_p \)-open set in \( X \) for any \( S_p \)-open set \( B \) in \( Y \).

Definition 2.20 A function \( f : X \rightarrow Y \) is said to be pre-semi-open if and only if for every semi-open subset \( O \) of \( X \), \( f(O) \) is semi-open set in \( Y \) [15].

3. \( S_p \)-Closed Spaces

In this section we define new types of convergence and accumulation to a point in a space \( X \) called \( S_p \)-convergence and \( S_p \)-accumulation and we start this section by the following two definitions.

Definition 3.1 A filterbase \( \beta \) on a space \( X \) is \( S_p \)-converges to a point \( x \in X \) if for each \( S_p \)-open set \( U \) containing \( x \), there exists \( B \in \beta \) such that \( B \subseteq \text{cl}(U) \).

Definition 3.2 A filterbase \( \beta \) on a space \( X \) is \( S_p \)-accumulates to a point \( x \in X \) if for every \( S_p \)-open set \( U \) containing \( x \) and every \( B \in \beta \), \( B \cap \text{cl}(U) \neq \phi \).

Proposition 3.3 If a filterbase \( \beta \) is \( S_p \)-converges to some point \( x \in X \), then \( \beta \) is \( S_p \)-accumulates to the same point.

Proof Obvious.

In general the converse of Proposition 3.3 is not necessarily true as shown in the following example:

Example 3.4 Let \( X = \{a,b,c,d\} \) and \( \tau = \{\phi, X, \{a\}, \{b\}, \{c,d\}, \{a,b\}, \{a,c,d\}, \{b,c,d\}\} \), then \( S_pO(X) = (X, \tau) \). Let \( \beta = \{X, \{a,b\}\} \) be a filterbase in \( (X, \tau) \), then \( \beta \) is \( S_p \)-accumulates to \( a \), but it is not \( S_p \)-converges to \( a \).

Proposition 3.5 Let \( \beta \) be a maximal filterbase on a space \( X \). Then \( \beta \) is \( S_p \)-accumulates to a point \( x_0 \in X \) if and only if \( \beta \) is \( S_p \)-converges to \( x_0 \).

Proof Obvious.

Proposition 3.6 Let \( F_1 \) and \( F_2 \) be two filterbases on a space \( X \) such that \( F_1 \subseteq F_2 \), if \( F_2 \) is \( S_p \)-accumulates to a point \( x_0 \in X \), then \( F_1 \) is also \( S_p \)-accumulates to \( x_0 \).

Proof Obvious.

Definition 3.7 A space \( X \) is said to be \( S_p \)-closed space if for every \( S_p \)-open cover \( \{V_\alpha : \alpha \in \Delta\} \) of \( X \) there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( X = \bigcup_{\alpha \in \Delta_0} S_p \text{cl}(V_\alpha) \).

Proposition 3.8 For a space \( X \) the following are equivalents:

1. \( X \) is \( S_p \)-closed space,
2. For each family of \( S_p \)-closed sets \( \{F_\alpha : \alpha \in \Delta\} \) such that \( \bigcap_{\alpha \in \Delta} F_\alpha = \phi \), there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( \bigcap_{\alpha \in \Delta_0} S_p \text{int}(F_\alpha) = \phi \),
3. Each filterbase \( \beta \) \( S_p \)-accumulates to a point \( x_0 \in X \),
4. Every maximal filterbase \( \beta \) \( S_p \)-converges to a point \( x_0 \in X \),

Proof (1)\( \Rightarrow \) (2): Let \( X \) be \( S_p \)-closed space and let \( \{F_\alpha : \alpha \in \Delta\} \) be a family of \( S_p \)-closed sets in \( X \) such that \( \bigcap_{\alpha \in \Delta} F_\alpha = \phi \). Now \( \{X \setminus F_\alpha : \alpha \in \Delta\} \) is family of \( S_p \)-open sets in \( X \) and since \( X = \bigcup_{\alpha \in \Delta} (X \setminus F_\alpha) \) implies that \( \{X \setminus F_\alpha : \alpha \in \Delta\} \) is an \( S_p \)-open cover of \( X \) and since \( X \) is \( S_p \)-closed so there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( X = \bigcup_{\alpha \in \Delta_0} S_p \text{cl}(X \setminus F_\alpha) \) now by Lemma 2.7 \( X = \bigcup_{\alpha \in \Delta} [X \setminus S_p \text{int}(F_\alpha)] \) this implies that

\[ \bigcap_{\alpha \in \Delta_0} S_p \text{int}(F_\alpha) = \phi. \]
(2)⇒(1): Let (2) be satisfied and to show $X$ is $S_p$-closed space, let $\{G_\alpha : \alpha \in \Delta\}$ be any $S_p$-open cover of $X \Rightarrow X = \bigcup_{\alpha \in \Delta} G_\alpha$ implies that $\bigcap_{\alpha \in \Delta} (X \setminus G_\alpha) = \phi$, then $X \setminus G_\alpha : \alpha \in \Delta$ is a family of $S_p$-closed sets in $X$ such that $\bigcap_{\alpha \in \Delta} (X \setminus G_\alpha) = \phi$ so by (2) there exists a finite subset $\Delta_0$ of $\Delta$ such that
$$\bigcap_{\alpha \in \Delta_0} S_p\text{int}(X \setminus G_\alpha) = \phi.$$ And by Lemma 2.7 for each $\alpha \in \Delta S_p\text{int}(X \setminus G_\alpha) = X \setminus S_p\text{cl}(G_\alpha)$ implies that $\bigcap_{\alpha \in \Delta} (X \setminus S_p\text{cl}(G_\alpha)) = \phi \Rightarrow X = \bigcup_{\alpha \in \Delta_0} S_p\text{cl}(G_\alpha)$. Thus $X$ is $S_p$-closed space.

(1)⇒(3): Let $X$ be $S_p$-closed space and let $F = \{A_\alpha : \alpha \in \Delta\}$ be a filterbase on $X$ that does not $S_p$-$\theta$-accumulate in $X$. Implies that for each $x \in X$, there exists an $S_p$-open set $V_x$ containing $x$ and some $A_{\alpha_x} \in F$ such that $A_{\alpha_x} \cap S_p\text{cl}(V_x) = \phi$. And now $X = \bigcup_{x \in X} V_x$ this implies that $\{V_x : x \in X\}$ is an $S_p$-open cover of $X$, but $X$ is $S_p$-closed space, then there exists a finite points of $X$ say $x_1, x_2, \ldots, x_n$ such that $X = \bigcup_{i=1}^n S_p\text{cl}(V_{x_i})$. Also since $F$ is a filterbase and $A_{\alpha_{x_i}} \in F$ for $i = 1, 2, \ldots, n$, then there exists $A_{\alpha_0} \in F$ such that $A_{\alpha_0} \subseteq \bigcap_{i=1}^n A_{\alpha_{x_i}}$ and $A_{\alpha_0} \neq \phi$ implies that for some $1 \leq j \leq n,$
$$A_{\alpha_0} \cap S_p\text{cl}(V_{x_j}) \neq \phi \Rightarrow A_{\alpha_0} \cap S_p\text{cl}(V_{x_j}) \neq \phi$$
which is a contradiction of the fact that $A_{\alpha_0} \cap S_p\text{cl}(V_{x_j}) = \phi$. So $F = \{A_\alpha : \alpha \in \Delta\}$ is a filterbase on $X$ that $S_p$-$\theta$-accumulates to some point in $X$.

(3)⇒(2): Let (3) be satisfied and let $\{F_\alpha : \alpha \in \Delta\}$ be a family of $S_p$-closed sets such that $\bigcap_{\alpha \in \Delta} F_\alpha = \phi$, and let for each finite subcollection $\{F_{\alpha_i} : \alpha \in \Delta\}$ we have $\bigcap_{i=1}^n S_p\text{int}(F_{\alpha_i}) \neq \phi$ then by Theorem 2.12 the family $\beta = \{S_p\text{int}(F_\alpha) : \alpha \in \Delta\}$ with finite intersection property (FIP) form a filterbase, this implies that by hypothesis this filterbase is $S_p$-$\theta$-accumulate to some $x_\alpha$ in $X \Rightarrow$ for every $S_p$-open set $V$ containing $x_\alpha$ and every $\alpha \in \Delta$, $S_p\text{int}(F_\alpha) \cap S_p\text{cl}(V) = \phi$, then $F_\alpha \cap S_p\text{cl}(V) = \phi$ for each $\alpha \in \Delta$. Now, since $\bigcap_{\alpha \in \Delta} F_\alpha = \phi$ implies that $X = \bigcup_{\alpha \in \Delta} (X \setminus F_\alpha)$. So the family $\{X \setminus F_\alpha : \alpha \in \Delta\}$ is $S_p$-open cover of $X$ and since $x_\alpha \in X \Rightarrow$ there exists $\gamma \in \Delta$ such that $x_\alpha \in X \setminus F_\gamma$, but
$$S_p\text{int}(F_\gamma) \cap S_p\text{cl}(X \setminus F_\gamma) = S_p\text{int}(F_\gamma) \cap (X \setminus S_p\text{int}(F_\gamma)) = \phi$$
which is a contradiction to the fact that $S_p\text{int}(F_\alpha) \cap S_p\text{cl}(V) = \phi$ for every $S_p$-open set $V$ containing $x_\alpha$ and every $\alpha \in \Delta$. Thus there exists a finite subcollection $\{F_{\alpha_i} : i = 1, 2, \ldots, n\}$ such that $\bigcap_{i=1}^n S_p\text{int}(F_{\alpha_i}) = \phi$.

(3)⇒(4): Let (3) be satisfied and let $F = \{A_\alpha : \alpha \in \Delta\}$ be a maximal filterbase on $X$. Then by (3), $F$ is $S_p$-$\theta$-accumulates to some $x_\alpha \in X$ implies that by Proposition 3.5 $F$ is $S_p$-$\theta$-converges to $x_\alpha \in X$.

(4)⇒(3): Let (4) be satisfied and let $F$ be a filterbase on $X$, then by Theorem 2.13 $F$ is contained in a maximal filterbase $\beta$ and by (3) $\beta$ is $S_p$-$\theta$-accumulates to $x_\alpha$; therefore by Proposition 3.6 $F$ is $S_p$-$\theta$-accumulates to $x_\alpha$.

Proposition 3.9 Every s-closed space $X$ is $S_p$-closed space.

**Proof** Let $X$ be any s-closed space and $\{V_\alpha : \alpha \in \Delta\}$ be any $S_p$-open cover of $X$. Since every $S_p$-open set is semi-open set implies that $\{V_\alpha : \alpha \in \Delta\}$ is semi-open cover of $X$ but $X$ is s-closed space so there exists a finite subset $\Delta_0$ of $\Delta$ such that $X = \bigcup_{\alpha \in \Delta_0} S_p\text{cl}(V_\alpha)$, then by Remark 2.5 $X = \bigcup_{\alpha \in \Delta_0} S_p\text{cl}(V_\alpha)$ this implies that $X$ is $S_p$-closed space.

The converse of Proposition 3.9 is not true in general as shown in the following example:
Example 3.10 Let \( X = \mathbb{R} \) and \( \tau = \{ (a, \infty) : a \in \mathbb{R} \} \cup \{ (a, \infty) : a \in \mathbb{R} \} \), then \( \text{SO}(\mathbb{R}) = \{ \phi, R, (a, \infty), [a, \infty) : a \in \mathbb{R} \} \), \( \text{PO}(\mathbb{R}) = \{ \phi, R, (a, \infty) : a \in \mathbb{R} \} \), \( \text{PC}(\mathbb{R}) = \{ \phi, R, (-\infty, a) : a \in \mathbb{R} \} \) and \( S_p\text{O}(\mathbb{R}) = \{ \phi, R \} \) then \( R \) is \( S_p \)-closed space but not \( s \)-closed space.

Proposition 3.11 Every \( S_p \)-closed space is \( S_c \)-closed space.

**Proof** Let \( X \) be an \( S_p \)-closed space and \( \{ V_\alpha : \alpha \in \Delta \} \) be any \( S_c \)-open cover of \( X \) implies that by Lemma 2.3 \( \{ V_\alpha : \alpha \in \Delta \} \) is an \( S_p \)-open cover of \( X \) and since \( X \) is \( S_p \)-closed space so there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( X = \bigcup_{\alpha \in \Delta_0} S_p\text{cl}(V_\alpha) \) and then by Remark 2.4 \( X = \bigcup_{\alpha \in \Delta_0} S_c\text{cl}(V_\alpha) \). Thus \( X \) is \( S_c \)-closed space.

Theorem 3.12 If for every family \( \{ F_\alpha : \alpha \in \Delta \} \) of semi-closed sets in \( X \) such that \( \bigcap_{\alpha \in \Delta} F_\alpha = \phi \), there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( \bigcap_{\alpha \in \Delta_0} F_\alpha = \phi \), then \( X \) is \( S_p \)-closed space.

**Proof** Let \( \{ V_\alpha : \alpha \in \Delta \} \) be any \( S_p \)-open cover of \( X \), then \( X = \bigcup_{\alpha \in \Delta} V_\alpha \) implies that \( \bigcap_{\alpha \in \Delta} (X \setminus V_\alpha) = \phi \). Then \( \{ X \setminus V_\alpha : \alpha \in \Delta \} \) is a family of \( S_p \)-closed sets in \( X \) and since every \( S_p \)-closed set is semi-closed set so \( \{ X \setminus V_\alpha : \alpha \in \Delta \} \) is a family of semi-closed sets such that \( \bigcap_{\alpha \in \Delta} (X \setminus V_\alpha) = \phi \), then by hypothesis there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( \bigcap_{\alpha \in \Delta_0} (X \setminus V_\alpha) = \phi \) implies that \( X = \bigcup_{\alpha \in \Delta_0} V_\alpha \), therefore \( X = \bigcup_{\alpha \in \Delta_0} S_p\text{cl}(V_\alpha) \). Thus \( X \) is \( S_p \)-closed space.

Theorem 3.13 Every extremally disconnected \( p \)-closed space is \( S_p \)-closed space.

**Proof** Let \( X \) be extremally disconnected and \( p \)-closed space and let \( \{ V_\alpha : \alpha \in \Delta \} \) be any \( S_p \)-open cover of \( X \), then by Lemma 2.10 \( \{ V_\alpha : \alpha \in \Delta \} \) is a \( p \)-open cover of \( X \). Since \( X \) is \( p \)-closed so there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( X = \bigcup_{\alpha \in \Delta_0} \text{pcl}(V_\alpha) \). But for each \( \alpha \in \Delta_0 \), \( \text{pcl}(V_\alpha) \subseteq S_p\text{cl}(V_\alpha) \) implies that \( X = \bigcup_{\alpha \in \Delta_0} S_p\text{cl}(V_\alpha) \). Hence \( X \) is \( S_p \)-closed space.

Proposition 3.14 If \( X \) is an extremally disconnected, submaximal and quasi-\( H \)-closed space, then it is \( S_p \)-closed space.

**Proof** \( X \) be extremally disconnected and submaximal space and let \( \{ V_\alpha : \alpha \in \Delta \} \) be any \( S_p \)-open cover of \( X \). Since \( X \) is extremally disconnected then by Lemma 2.10 for each \( \alpha \in \Delta \), \( V_\alpha \) is \( p \)-open set and since \( X \) is submaximal then by Theorem 2.11 for each \( \alpha \in \Delta \), \( V_\alpha \) is open set implies that \( \{ V_\alpha : \alpha \in \Delta \} \) is an open cover of \( X \) and since \( X \) is quasi-\( H \)-closed space so there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( X = \bigcup_{\alpha \in \Delta_0} \text{cl}(V_\alpha) \). But \( X \) is extremally disconnected so \( S_p\text{cl}(V_\alpha) = \text{cl}(V_\alpha) \), for each \( \alpha \in \Delta \). Thus \( X = \bigcup_{\alpha \in \Delta_0} S_p\text{cl}(V_\alpha) \), therefore; \( X \) is \( S_p \)-closed space.

Lemma 3.15 Every regular closed set of a space \( X \) is \( S_p \)-open set.

Lemma 3.16 If a space \( X \) is locally indiscrete, then every semi-open set is \( S_p \)-open set.

**Proof** Let \( A \) be a \( p \)-open set in \( X \), then \( A \subseteq \text{clint} A \). Since \( X \) is locally indiscrete, then \( \text{int} A \) is closed and hence \( \text{int} A = \text{clint} A \), which implies that \( A \) is regular closed. Therefore by Lemma 3.15, \( A \) is \( S_p \)-open set.

Theorem 3.17 Every locally indiscrete \( S_p \)-closed space is \( S \)-closed space.

**Proof** \( \{ V_\alpha : \alpha \in \Delta \} \) be any \( p \)-open cover of \( X \), then by Lemma 3.16 \( \{ V_\alpha : \alpha \in \Delta \} \) is \( S_p \)-open cover of \( X \). But \( X \) is \( S_p \)-closed space so there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( X = \bigcup_{\alpha \in \Delta_0} S_p\text{cl}(V_\alpha) \) and since \( X \) is locally indiscrete so from Lemma 3.16 we conclude that every open subset of \( X \) is \( S_p \)-open set this implies that for each \( \alpha \in \Delta \), \( S_p\text{cl}(V_\alpha) \subseteq \text{cl}(V_\alpha) \), then \( X = \bigcup_{\alpha \in \Delta_0} \text{cl}(V_\alpha) \).
Thus $X$ is $S$-closed space.

Theorem 3.18 If a space $X$ is $S_p$-closed, locally indiscrete and s-regular space, then it is compact space.

Proof Let $\{V_{\alpha} : \alpha \in \Delta\}$ be any open cover of $X$, then for each $x \in X$ there exists an $\alpha_x \in \Delta$ such that $x \in V_{\alpha_x}$. Since $X$ is s-regular space, so by Theorem 2.16 there exists a semi-open set $U_x$ such that $x \in U_x \subseteq \text{scl}(U_x) \subseteq V_{\alpha_x}$, implies that $\{U_x : x \in X\}$ is semi-open cover of $X$. But $X$ is locally indiscrete space, hence by Lemma 3.16 $\{U_x : x \in X\}$ is $S_p$-open cover of $X$. Since $X$ is $S_p$-closed space, so there exist finite points $\{x_1, x_2, \ldots, x_n\}$ of $X$ such that $X = \bigcup_{i=1}^{n} S_p\text{cl}(U_{x_i})$ also from Lemma 3.16 we conclude that $\text{scl}(U_x) = S_p\text{cl}(U_x)$ for each $x \in X$. Thus

$$X = \bigcup_{i=1}^{n} S_p\text{cl}(U_{x_i}) \subseteq \bigcup_{i=1}^{n} \text{scl}(U_{x_i}) = \bigcup_{i=1}^{n} (V_{\alpha_{x_i}})$$

this implies that $X$ is compact.

4. Image and Preimage of $S_p$-Closed Spaces

In this section we give some results about the image and preimage of $S_p$-closed spaces.

Theorem 4.1 If $f : X \to Y$ is $S_p$-continuous function from $S_p$-closed space $X$ into a space $Y$, then $f(X)$ is quasi-H-closed relative to $Y$.

Proof Let $\{V_{\alpha} : \alpha \in \Delta\}$ be any cover of $f(X)$ by open sets in $Y$, then $f(X) \subseteq \bigcup_{\alpha \in \Delta} V_{\alpha}$ implies that $X \subseteq f^{-1}(\bigcup_{\alpha \in \Delta} V_{\alpha}) = \bigcup_{\alpha \in \Delta} f^{-1}(V_{\alpha})$. Now since $f$ is $S_p$-continuous so by Definition 2.17 $f^{-1}(V_{\alpha}) \subseteq S_pO(X)$ for each $\alpha \in \Delta$, this implies that $\{f^{-1}(V_{\alpha}) : \alpha \in \Delta\}$ is $S_p$-open cover of $X$. But $X$ is $S_p$-closed space so there exists a finite subset $\Delta_0$ of $\Delta$ such that $X \subseteq \bigcup_{\alpha \in \Delta_0} S_p\text{cl}(f^{-1}(V_{\alpha}))$. Since $f$ is $S_p$-continuous so by Theorem 2.18 $X \subseteq \bigcup_{\alpha \in \Delta_0} f^{-1}(\text{cl}(V_{\alpha}))$, therefore $f(X) \subseteq \bigcup_{\alpha \in \Delta_0} \text{cl}(V_{\alpha})$. Thus by Definition 2.14 $f(X)$ is quasi-H-closed relative to $Y$.

Corollary 4.1 If $f : X \to Y$ be $S_p$-continuous function from $S_p$-closed space $X$ onto a space $Y$, then $Y$ is quasi-H-closed space.

Proof Follows from Theorem 4.1.

Corollary 4.2 If $f : X \to Y$ be $S_p$-continuous function from $S_p$-closed space $X$ onto a Hausdorff space $Y$, then $Y$ is H-closed space.

Proof Directly proved by Theorem 4.1 and Definition 2.15.

Lemma 4.2 Let $f : X \to Y$ be an open continuous function. If $F$ be $S_p$-closed set in $Y$, then $f^{-1}(F)$ is $S_p$-closed set in $X$.

Proof Obviously by Lemma 2.19 and the fact that $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$.

Lemma 4.3 Let $X$ be a space and $A$, $B$ be two subsets of $X$, then:

1. $S_p\text{cl}(A)$ is the smallest $S_p$-closed set which containing $A$.
2. $A$ is $S_p$-closed if and only if $S_p\text{cl}(A) = A$.
3. $S_p\text{cl}(Spc(A)) = S_p\text{cl}(A)$.
4. If $A \subseteq B$, then $S_p\text{cl}(A) \subseteq S_p\text{cl}(B)$.

Proof The proof of them is obvious.
Lemma 4.4 Let \( f : X \to Y \) be an open continuous function. Then for any \( B \subseteq Y \), \( S_{p}cl(f^{-1}(B)) \subseteq f^{-1}(S_{p}cl(B)) \).

**Proof** Let \( B \subseteq Y \), then by Lemma 4.3 \( S_{p}cl(B) \) is \( S_{p} \)-closed set in \( Y \). Now since \( f \) is open and continuous function so by Lemma 4.2 \( f^{-1}(S_{p}cl(B)) \) is \( S_{p} \)-closed in \( X \) and by Lemma 4.3 \( f^{-1}(S_{p}cl(B)) = S_{p}cl(f^{-1}(S_{p}cl(B))) \). But \( B \subseteq S_{p}cl(B) \) implies that \( f^{-1}(B) \subseteq f^{-1}(S_{p}cl(B)) \) and by Lemma 4.3 \( S_{p}cl(f^{-1}(B)) \subseteq f^{-1}(S_{p}cl(B)) \). ■

**Theorem 4.5** Let \( f : X \to Y \) be an onto continuous and open function. If \( X \) is \( S_{p} \)-closed space, then \( Y \) is also \( S_{p} \)-closed space.

**Proof** Let \( \{V_{\alpha} : \alpha \in \Delta \} \) be any \( S_{p} \)-open cover of \( Y \), then \( Y = \bigcup_{\alpha \in \Delta} V_{\alpha} \) this implies that \( X = \bigcup_{\alpha \in \Delta} f^{-1}(V_{\alpha}) \).

Now since \( f \) is continuous and open function so by Lemma 2.19 \( f^{-1}(V_{\alpha}) \in S_{p}O(X) \) for each \( \alpha \in \Delta \), then \( \{V_{\alpha} : \alpha \in \Delta \} \) is an \( S_{p} \)-open cover of \( X \) and since \( X \) is \( S_{p} \)-closed space there exists a finite subset \( \Delta_{0} \) of \( \Delta \) such that \( X = \bigcup_{\alpha \in \Delta_{0}} S_{p}cl(f^{-1}(V_{\alpha})) \), since \( f \) is open and continuous function so by Lemma 4.4

\[
\bigcup_{\alpha \in \Delta_{0}} S_{p}cl(f^{-1}(V_{\alpha})) \subseteq \bigcup_{\alpha \in \Delta_{0}} f^{-1}(S_{p}cl(V_{\alpha}))
\]

this implies that

\[
Y = f(X) \subseteq f\left( \bigcup_{\alpha \in \Delta_{0}} f^{-1}(S_{p}cl(V_{\alpha})) \right) = f(f^{-1}(\bigcup_{\alpha \in \Delta_{0}} S_{p}cl(V_{\alpha}))) \subseteq \bigcup_{\alpha \in \Delta_{0}} S_{p}cl(V_{\alpha})
\]

therefore; \( Y \) is \( S_{p} \)-closed space. ■

**Lemma 4.6** A bijective function \( f : X \to Y \) is pre-semi-open if and only if the image of every semi-closed set in \( X \) is semi-closed in \( Y \).

**Proof** Followed from Definition 2.20. ■

**Lemma 4.7** A bijective function \( f : X \to Y \) is pre-semi-open if and only if for each \( A \subseteq X \), \( scl(f(A)) \subseteq f(scl(A)) \).

**Proof** Obviously by Lemma 4.6. ■

**Proposition 4.8** Let \( f : X \to Y \) be a bijective pre-semi-open function. If \( Y \) is \( s \)-closed space, then \( X \) is \( S_{p} \)-closed space.

**Proof** Let \( \{V_{\alpha} : \alpha \in \Delta \} \) be any \( S_{p} \)-open cover of \( X \), then \( X = \bigcup_{\alpha \in \Delta} V_{\alpha} \) implies that \( Y = f(X) = \bigcup_{\alpha \in \Delta} f(V_{\alpha}) \). Since \( f \) is pre-semi-open function so by Definition 2.20 \( f(V_{\alpha}) \) is semi-open set in \( Y \) for each \( \alpha \in \Delta \), then \( \{f(V_{\alpha}) : \alpha \in \Delta \} \) is semi-open cover of \( Y \) but \( Y \) is \( s \)-closed space so there exists a finite subset \( \Delta_{0} \) of \( \Delta \) such that \( Y = \bigcup_{\alpha \in \Delta_{0}} scl(f(V_{\alpha})) \) and by Lemma 4.7 \( Y = \bigcup_{\alpha \in \Delta_{0}} f(scl(V_{\alpha})) \) implies that \( X = \bigcup_{\alpha \in \Delta_{0}} scl(V_{\alpha}) \) and by Remark 2.5 \( X = \bigcup_{\alpha \in \Delta_{0}} S_{p}cl(V_{\alpha}) \). Thus \( X \) is \( S_{p} \)-closed space. ■

**References**


REFERENCES