RESEARCH ARTICLE

Reciprocally Continuous Maps in a Fuzzy Metric Space involving Implicit Relations

Pooja Sharma * and R. S. Chandel †

* MIG-9, Sector 4-B, Saketnagar, Bhopal (M.P.), India.
† Department of Mathematics, Geetanjali College, Barkatullah University, Bhopal (M.P.), India.

(Received: 2 April 2012, Accepted: 23 May 2012)

The aim of this paper is to prove common fixed point theorems for three mappings through implicit relation in fuzzy metric space. In this paper, we established common fixed point theorems using notion of semi compatibility and reciprocal continuity of maps.

Keywords: Fuzzy metric space; Semi compatibility; reciprocally continuous mapping; Common fixed point; implicit relation.

AMS Subject Classification: 54H25, 47H10.

1. Introduction

The foundation of fuzzy mathematics is laid by L. A. Zadeh [1] with the introduction of fuzzy sets in 1965. There have been a number of generalizations of metric space. The fuzzy theory has become an area of active research for the last forty years. Several authors have applied various form from general topology of fuzzy sets and developed the concept of fuzzy space. It has a wide range of applications in the field of science and engineering, for example, computer programming, medicine, population and so forth. A number of fixed theorems have been obtained by various authors in fuzzy metric space by using the concept of compatible map, weakly compatible map, etc. D. Mihet [2] proved common fixed point theorems in fuzzy metric spaces. The concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [3], and George and Veeramani [4] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Grabiec’s [5] followed Kramosil and Michalek and obtained the fuzzy version of Banach contraction principle.


The main objective of this paper is to obtain some common fixed point theorem in fuzzy metric spaces using weakly compatibility, semi compatibility, an implicit and reciprocal continuity. Our result established common fixed point theorems using notion of semi compatibility and reciprocal continuity of maps.

* Corresponding author
Email: poojاشरما020283@gmail.com
2. Preliminaries

Definition 2.1 A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is continuous $t$-norm if $*$ is satisfies the following conditions:

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0, 1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2 The 3-tuple $(X, M, *)$ is said to be fuzzy metric space if $X$ is an arbitrary set, $*$ is continuous $t$-norm and $M$ is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous,
(FM-6) $\lim_{n \to \infty} M(x, y, t) = 1$, for all $x, y \in X$.

Example 2.3 Let $X = R$ with metric space. Define $d(x, y) = |x - y|$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is fuzzy metric space.

Definition 2.4 Let $(X, M, *)$ be a fuzzy metric space.

(a) The sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$, if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for all $t > 0$.
(b) The sequence $\{x_n\}$ in $X$ is said to be a Cauchy sequence in $X$ if $\lim_{n \to \infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p > 0$.

(c) The space is said to be complete if every Cauchy sequence in it converges to a point of it.

Definition 2.5 Let $A$ and $B$ be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then, the mappings are said to be compatible if $\lim_{n \to \infty} M(ABx_n, Bx_n, t) = 1$ for all $t > 0$.

whenever $\{x_n\}$ is sequence in $X$ such that

$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x \in X$.

Remark 2.6 Compatibility implies weak compatibility. The converse is not true.

Definition 2.7 Let $A$ and $B$ be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then, the mappings are said to be weakly compatible if they commute at their coincidence points, that is, $Ax = Bx$ implies that $ABx = BAx$.

Definition 2.8 Let $A$ and $B$ be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then, the
mappings are said to be semi compatible if
\[\lim_{n \to \infty} M(ABx_n, Bx, t) = 1 \quad \text{for all} \quad t > 0.\]
whenever \(\{x_n\}\) is sequence in \(X\) such that
\[\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x \in X.\]

Remark 2.9 Semi compatibility implies weak compatibility. The converse is not true.

Definition 2.10 Let \(A\) and \(B\) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then, the mappings are said to be reciprocally continuous if
\[\lim_{n \to \infty} ABx_n = Ax, \quad \lim_{n \to \infty} BAx_n = Bx\]
whenever \(\{x_n\}\) is sequence in \(X\) such that
\[\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x \in X.\]

If \(A\) and \(B\) are both continuous, then they are obviously reciprocally continuous, but the converse is not true.

Lemma 2.11 Let \(\{y_n\}\) be a sequence in a fuzzy metric space \((X, M, \ast)\) with the condition (FM-6). If there exists a number \(k \in (0, 1)\) such that
\[M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)\]
for all \(t > 0\), then \(\{y_n\}\) is a Cauchy sequence in \(X\).

A Class of Implicit Relations: Let \(K_4\) be the set of all real continuous functions \(\varphi : \mathbb{R}^4 \to \mathbb{R}\), non decreasing in first argument and satisfying the following conditions:
(i) for \(u, v \geq 0\), \(\varphi(u, v, v, u) \geq 0\) or \(\varphi(u, v, u, v) \geq 0\) implies that \(u \geq v\),
(ii) \(\varphi(u, u, 1, 1) \geq 0\) implies that \(u \geq 1\).

Example 2.12 \(\varphi(t_1, t_2, t_3, t_4) = t_1 - \min\{t_2, t_3, t_4\}\).

3. Main Result

Theorem 3.1 Let \(A, B\) and \(S\) be self mappings of a complete fuzzy metric space \((X, M, \ast)\) such that:
(a) \(A(X) \cap B(X) \subset S(X)\).
(b) The pair \((A, S)\) is semi compatible and \((B, S)\) is weakly compatible.
(c) The pair \((A, S)\) is reciprocally continuous.

For some \(\varphi \in K_4\), there exists \(k \in (0, 1)\) such that for all \(x, y \in X\) and \(t > 0\),
\[\varphi[M(A^2x, B^2y, kt), M(S^2x, S^2y, t), M(A^2x, S^2x, t), M(B^2y, S^2y, kt)] \geq 0, \quad (1)\]
\[\varphi[M(A^2x, B^2y, kt), M(S^2x, S^2y, t), M(A^2x, S^2x, kt), M(B^2y, S^2y, t)] \geq 0. \quad (2)\]

Then \(A, B\) and \(S\) have a unique common fixed point.
Proof Let $x_0$ be an arbitrary point in $X$. Since $A(X) \cap B(X) \subset S(X)$, there exists $x_1, x_2 \in X$ such that $Ax_0 = Sx_1$ and $Bx_1 = Sx_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in $X$ such that

\begin{align*}
y_{2n+1} &= A^2x_{2n} = S^2x_{2n+1}, \quad y_{2n+2} = B^2x_{2n+1} = S^2x_{2n+2} \quad (3)
\end{align*}

for $n = 0, 1, 2, \ldots$

Now putting $x = x_{2n}$, $y = x_{2n+1}$ in (1), we obtain

\begin{align*}
\varphi[M(A^2x_{2n}, B^2x_{2n+1}, kt), M(S^2x_{2n}, S^2x_{2n+1}, t), M(A^2x_{2n}, S^2x_{2n}, t), M(B^2x_{2n+1}, S^2x_{2n+1}, kt)] &\geq 0, \quad (4)
\end{align*}

that is,

\begin{align*}
\varphi[M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n+1}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+2}, y_{2n+1}, kt)] &\geq 0. \quad (5)
\end{align*}

Using (i), we get

\begin{align*}
M(y_{2n+2}, y_{2n+1}, kt) &\geq M(y_{2n+1}, y_{2n}, t). \quad (6)
\end{align*}

Similarly, putting $x = x_{2n+2}$, $y = x_{2n+1}$ in (2), we have

\begin{align*}
\varphi[M(y_{2n+3}, y_{2n+2}, kt), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+3}, y_{2n+2}, kt), M(y_{2n+2}, y_{2n+2}, t)] &\geq 0. \quad (7)
\end{align*}

Using (i), we get

\begin{align*}
M(y_{2n+3}, y_{2n+2}, kt) &\geq M(y_{2n+2}, y_{2n+2}, t). \quad (8)
\end{align*}

Thus, from (6) and (8), for $n$ and $t$, we have

\begin{align*}
M(y_n, y_{n+1}, kt) &\geq M(y_{n+1}, y_{n}, t). \quad (9)
\end{align*}

Hence, by Lemma 2.11, $\{y_n\}$ is a Cauchy sequence in $X$, which is complete. Therefore, $\{y_n\}$ converges to $p \in X$. The sequences $\{A^2x_{2n}\}$, $\{B^2x_{2n+1}\}$ and $\{S^2x_{2n+1}\}$, being subsequences of $\{y_n\}$, also converge to $p$, that is,

\begin{align*}
\{A^2x_{2n}\} \rightarrow p, \quad \{B^2x_{2n+1}\} \rightarrow p. \quad (10)
\end{align*}

The reciprocal continuity of the pair $(A, S)$ gives

\begin{align*}
A^2S^2x_{2n} &\rightarrow A^2p, \quad S^2A^2x_{2n} \rightarrow S^2p. \quad (11)
\end{align*}

The semi compatibility of the pair $(A, S)$ gives

\begin{align*}
\lim_{n \to \infty} A^2S^2x_{2n} = S^2p. \quad (12)
\end{align*}

From the uniqueness of the limit in fuzzy metric space, we obtain that

\begin{align*}
A^2p = S^2p. \quad (13)
\end{align*}

By putting $x = p$, $y = x_{2n+1}$ in (1), we obtain

\begin{align*}
\varphi[M(A^2p, B^2x_{2n+1}, kt), M(S^2p, S^2x_{2n+1}, t), M(A^2p, S^2p, t), M(B^2x_{2n+1}, S^2x_{2n+1}, kt)] &\geq 0. \quad (14)
\end{align*}
Letting \( n \to \infty \) and using (10) and (13), we get
\[
\varphi[M(S^2p, p, kt), M(S^2p, p, t), M(S^2p, p, t), M(p, p, kt)] \geq 0. \tag{15}
\]
As \( \varphi \) is non decreasing in first argument, we have
\[
\varphi[M(S^2p, p, kt), M(S^2p, p, t), 1, 1] \geq 0. \tag{16}
\]
Using (ii), we have \( M(S^2p, p, t) \geq 1 \) for all \( t > 0 \), which gives \( M(S^2p, p, t) = 1 \) that is,
\[
S^2p = p = A^2p. \tag{17}
\]

As \( A(X) \subset S(X) \), there exist \( u \in X \) such that \( A^2p = S^2p = p = S^2u \).

Now putting \( x = x_{2n}, y = u \) in (1), we get
\[
\varphi[M(A^2x_{2n}, B^2u, kt), M(S^2x_{2n}, S^2u, t), M(A^2x_{2n}, S^2x_{2n}, t), M(B^2u, S^2u, kt)] \geq 0. \tag{18}
\]
Letting \( n \to \infty \) and using (10), we get
\[
\varphi[M(p, B^2u, kt), 1, 1, M(B^2u, p, kt)] \geq 0. \tag{19}
\]
Using (ii), we have \( M(p, B^2u, kt) \geq 1 \) for all \( t > 0 \), which gives \( M(p, B^2u, kt) = 1 \). Thus \( p = B^2u \).

Therefore, \( B^2u = S^2u = p \). Since \( (B, S) \) is weak compatible, we get \( S^2B^2u = B^2S^2u \), that is
\[
B^2p = S^2p. \tag{20}
\]

By putting \( x = p, y = p \) in (1) and using (17) and (20), we obtain
\[
\varphi[M(A^2p, B^2p, kt), M(S^2p, S^2p, t), M(A^2p, S^2p, t), M(B^2p, S^2p, kt)] \geq 0. \tag{21}
\]
That is, \( \varphi[M(A^2p, B^2p, kt), M(A^2p, B^2p, t), 1, 1] \geq 0. \) As \( \varphi \) is non decreasing in first argument, we have
\[
\varphi[M(A^2p, B^2p, t), M(A^2p, B^2p, t), 1, 1] \geq 0. \tag{22}
\]
Using (ii), we have \( M(A^2p, B^2u, t) \geq 1 \) for all \( t > 0 \), which gives \( M(A^2p, B^2u, t) = 1 \). Thus
\[
A^2p = B^2p. \tag{23}
\]

Therefore, \( p = A^2p = B^2p = S^2p \), that is \( p \) is common fixed point of \( A, B \) and \( S \).

**Uniqueness:** Let \( q \) be another common fixed point of \( A, B \) and \( S \). Then \( q = A^2q = B^2q = S^2q \). By putting \( x = p \) and \( y = q \) in (1), we get
\[
\varphi[M(A^2p, B^2q, kt), M(S^2p, S^2q, t), M(A^2p, S^2p, t), M(B^2q, S^2q, kt)] \geq 0. \tag{24}
\]
That is, \( \varphi[M(p, q, kt), M(p, q, t), 1, 1] \geq 0. \) As \( \varphi \) is non decreasing in first argument, we have
\[
\varphi[M(p, q, t), M(p, q, t), 1, 1] \geq 0. \tag{25}
\]
Using (ii), we have \( M(p, q, t) \geq 1 \) for all \( t > 0 \), which gives \( M(p, q, t) = 1 \), that is \( p = q \). Therefore, \( p \) is the common fixed point of the self maps \( A, B \) and \( S \).
Corollary 3.1 Let $A$ and $S$ be self-mappings of a complete fuzzy space $(X, M, \ast)$ such that for some $\varphi \in K_4$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

\[ \varphi[M(A^2 x, S^2 y, kt), M(x, y, t), M(A^2 x, x, t), M(S^2 y, y, kt)] \geq 0, \]
\[ \varphi[M(A^2 x, S^2 y, kt), M(x, y, t), M(A^2 x, x, kt), M(S^2 y, y, t)] \geq 0, \] (26)

Then, $A$ and $S$ have a unique fixed point in $X$.

Since semi compatibility implies weak compatibility, we have the following.

Corollary 3.2 Let $A, B$ and $S$ be self mappings of a complete fuzzy metric space $(X, M, \ast)$ satisfying the conditions (a), (1) and (2) of Theorem 3.1, and $(A, S)$ is compatible, and $(B, S)$ is weak compatible. Then $A, B$ and $S$ have a unique common fixed point.

Theorem 3.2 Let $A, B$ and $S$ be self mappings of a complete fuzzy metric space $(X, M, \ast)$ satisfying the conditions (a), (c), (1) and (2) of Theorem 3.1, and $(A, S)$ is compatible, and $(B, S)$ is weak compatible. Then $A, B$ and $S$ have a unique common fixed point.

**Proof** As in the proof of Theorem 3.1, the sequence $\{y_n\}$ converges to $p \in X$ and (10) is satisfied. The reciprocal continuity of the pair $(A, S)$ gives

\[ A^2 S^2 x_{2n} \to A^2 p, \quad S^2 A^2 x_{2n} \to S^2 p. \] (27)

The compatibility of the pair $(A, S)$ gives

\[ \lim_{n \to \infty} (A^2 S^2 x_{2n}) = A^2 p = \lim_{n \to \infty} S^2 A^2 x_{2n}, \] (28)

From the uniqueness of the limit in a fuzzy metric space, we obtain that $A^2 p \to S^2 p$. By putting $x = A^2 x_{2n}$, $y = x_{2n+1}$ in (1), we obtain

\[ \varphi[M(A^2 A^2 x_{2n}, B^2 x_{2n+1}, kt), M(S^2 A^2 x_{2n}, S^2 x_{2n+1}, t), M(A^2 A^2 x_{2n}, S^2 A^2 x_{2n}, t), M(B^2 x_{2n+1}, S^2 x_{2n+1}, kt)] \geq 0. \] (29)

Letting $n \to \infty$ and using (10) and (13), we get

\[ \varphi[M(A^2 p, p, kt), M(A^2 p, p, t), M(A^2 p, A^2 p, t), M(p, p, kt)] \geq 0, \]
\[ \varphi[M(A^2 p, p, kt), M(A^2 p, p, t), M(A^2 p, A^2 p, t), M(p, p, kt)] \geq 0 \] (30)

As $\varphi$ is non decreasing in first argument, we get

\[ \varphi[M(A^2 p, p, kt), M(A^2 p, p, t), 1, 1] \geq 0. \] (31)

Using (ii), we have $M(A^2 p, p, t) \geq 1$ for all $t > 0$, which gives $M(A^2 p, p, t) = 1$. That is,

\[ S^2 p = p = A^2 p. \] (32)

As $A(X) \subset S(X)$, there exists $u \in X$ such that $A^2 p = S^2 p = p$. Putting $x = x_{2n}$, $y = u$ in (1), we obtain that

\[ \varphi[M(A^2 x_{2n}, B^2 u, kt), M(S^2 x_{2n}, S^2 u, t), M(A^2 x_{2n}, S^2 x_{2n}, t), M(B^2 u, S^2 u, kt)] \geq 0. \] (33)
Letting $n \to \infty$ and using (10), we get

$$\varphi[M(p, B^2 u, kt), 1, 1, M(B^2 u, p, kt)] \geq 0.$$  \hfill (34)

Using (i), we have $M(p, B^2 u, kt) \geq 1$ for all $t > 0$, which gives $M(p, B^2 u, kt) = 1$. Thus $p = B^2 u$. Therefore, $B^2 u = S^2 u = p$. Since $(B, S)$ is weak compatible, we get $S^2 B^2 u = B^2 S^2 u$, that is

$$B^2 p = S^2 p.$$  \hfill (35)

By putting $x = p$, $y = p$ in (1) and using (17) and (20), we obtain

$$\varphi[M(A^2 p, B^2 p, kt), M(S^2 p, S^2 p, t), M(A^2 p, S^2 p, t), M(B^2 p, S^2 p, kt)] \geq 0.$$  \hfill (36)

That is, $\varphi[M(A^2 p, B^2 p, kt), 1, M(A^2 p, B^2 p, t), 1] \geq 0$. As $\varphi$ is non decreasing in first argument, we get

$$\varphi[M(A^2 p, B^2 p, t), 1, M(A^2 p, B^2 p, t), 1] \geq 0.$$  \hfill (37)

Using (ii), we have $M(A^2 p, B^2 p, t) \geq 1$ for all $t > 0$, which gives $M(A^2 p, B^2 p, t) = 1$. Thus

$$A^2 p = B^2 p.$$  \hfill (38)

Therefore, $p = A^2 p = B^2 p = S^2 p$, that is, $p$ is a common fixed point of $A$, $B$ and $S$. The rest of the proof is the same as in theorem (1).

Since compatibility implies weak compatibility, we have the following.

**Corollary 3.3** Let $A$, $B$ and $S$ be self maps of a complete fuzzy metric space $(X, M, *)$ satisfying the conditions (a), (1) and (2) of Theorem 3.1, and the pair $(A, S)$ and $(B, S)$ are compatible, and one of the pair, $(A, S)$ or $(B, S)$ is reciprocally continuous. Then $A$, $B$ and $S$ have a unique common fixed point.

**Example 3.3** Let $(X, M, *)$ be a fuzzy metric space with $X = [0, 1]$, a $t$-norm $*$ be define by $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $M$ be a fuzzy set on $X^2 \times (0, \infty)$ define by

$$M(x, y, t) = \left[\exp\left(\frac{|x - y|}{t}\right)\right]^{-1}$$

for all $x, y \in X$ and $t > 0$. Define the mappings $A$, $B$, $S : X \to X$ by $Ax = x$, $Bx = \frac{x}{2}$, $Sx = \frac{x}{4}$, respectively. Then for some $k \in [\frac{1}{2}, 1)$, we have

$$M(A^2 p, B^2 p, kt) = \left[\exp\left(\frac{|x - y|}{kt}\right)\right]^{-1} \geq \left[\exp\left(\frac{|\frac{x}{2} - y|}{t}\right)\right]^{-1} = M(S^2 p, S^2 p, t) \geq \min\{M(S^2 p, S^2 p, t), M(A^2 p, S^2 p, t), M(B^2 p, S^2 p, kt)\}.$$

Thus the (1) of Theorem 3.1 is satisfied. Also, it is obvious that the other conditions of the theorem is satisfied and zero is the unique fixed point of $A$, $B$ and $S$.

**4. Acknowledgements**

The author would like to thank the referee for carefully reading the paper. Further, the same author would like to thank Dr. Sushil Sharma for his valuable guidance and support.
References