On Fixed Points of Weakly Commuting Mappings with Property (E.A)

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In this paper, we establish a common fixed point theorem for weakly commuting mappings in complete fuzzy metric spaces using the property (E.A).

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1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [1] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [2] and Kramosil and Michalek [3] have introduced the concept of fuzzy topological spaces induced by fuzzy metric. Many authors [4–14] have studied different properties, for e.g., topological, fixed point properties and applications of fuzzy (probabilistic) metric spaces and also its generalized and different versions. Recently, Kumar [15] proved a common fixed point theorem for a pair of weakly compatible maps under E.A. property and Wadhwa et al. [16] defined a E. A. like property and proved common fixed point theorems in fuzzy metric spaces.

Definition 1.1 [17] A binary operation \( * : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is a continuous \( t \)-norm if it satisfies the following conditions:

1. \( * \) is associative and commutative,
2. \( * \) is continuous,
3. \( a * 1 = a \) for all \( a \in [0, 1] \),
4. \( a * b \leq c * d \) whenever \( a \leq c \) and \( b \leq d \), for each \( a, b, c, d \in [0, 1] \).

Two typical examples of a continuous \( t \)-norm are \( a * b = ab \) and \( a * b = \min(a, b) \).

Definition 1.2 [2] A 3-tuple \((X, M, \ast)\) is called a fuzzy metric space if \( X \) is an arbitrary (non-empty) set, \( \ast \) is a continuous \( t \)-norm and \( M \) is a fuzzy set on \( X^2 \times (0, \infty) \), satisfying the following conditions for each \( x, y, z \in X \) and \( t, s > 0 \),

1. \( M(x, y, t) > 0 \),
(2) \( M(x, y, t) = 1 \) if and only if \( x = y \),
(3) \( M(x, y, t) = M(y, x, t) \),
(4) \( M(x, y, t) \cdot M(y, z, s) \leq M(x, z, t+s) \),
(5) \( M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1] \) is continuous.

Let \((X, M, \ast)\) be a fuzzy metric space. For \( t > 0 \), the open ball \( B(x, r, t) \) with center \( x \in X \) and radius \( 0 < r < 1 \) is defined by

\[
B(x, r, t) = \{ y \in X : M(x, y, t) > 1 - r \}.
\]

If \((X, M, \ast)\) is a fuzzy metric space, let \( \tau \) be the set of all \( A \subset X \) with \( x \in A \) if and only if there exist \( t > 0 \) and \( 0 < r < 1 \) such that \( B(x, r, t) \subset A \). Then \( \tau \) is a topology on \( X \) (induced by the fuzzy metric \( M \)). This topology is Hausdorff and first countable. A sequence \( \{x_n\} \) in \( X \) converges to \( x \) if and only if \( M(x_n, x, t) \rightarrow 1 \) as \( n \rightarrow \infty \), for each \( t > 0 \). It is called a Cauchy sequence if for each \( 0 < \varepsilon < 1 \) and \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x_m, t) > 1 - \varepsilon \) for each \( n, m \geq n_0 \). The fuzzy metric space \((X, M, \ast)\) is said to be complete if every Cauchy sequence is convergent. A subset \( A \) of \( X \) is said to be \( F \)-bounded if there exists \( t > 0 \) and \( 0 < r < 1 \) such that \( M(x, y, t) > 1 - r \) for all \( x, y \in A \).

Example 1.3 Let \( X = \mathbb{R} \). Put \( a \ast b = ab \) for all \( a, b \in [0, 1] \). For each \( t \in (0, \infty) \), define

\[
M(x, y, t) = \frac{t}{t + |x - y|}
\]

for all \( x, y \in X \).

Lemma 1.4 [18] Let \((X, M, \ast)\) be a fuzzy metric space. Then \( M(x, y, t) \) is non-decreasing with respect to \( t \), for all \( x, y \) in \( X \).

Definition 1.5 Let \((X, M, \ast)\) be a fuzzy metric space. Then \( M \) is said to be continuous on \( X^2 \times (0, \infty) \) if

\[
\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t),
\]

whenever a sequence \( \{(x_n, y_n, t_n)\} \) in \( X^2 \times (0, \infty) \) converges to a point \((x, y, t) \in X^2 \times (0, \infty) \). i.e.,

\[
\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1 \text{ and } \lim_{n \to \infty} M(x, y, t_n) = M(x, y, t).
\]

Lemma 1.6 Let \((X, M, \ast)\) be a fuzzy metric space. Then \( M \) is a continuous function on \( X^2 \times (0, \infty) \).

Proof See [19, Proposition 1].

Definition 1.7 [20] Let \( A \) and \( S \) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the mappings are said to be weak compatible if they commute at a coincidence point, that is, \( Ax = Sx \) implies that \( ASx = SAx \).

Definition 1.8 [21] Let \( A \) and \( S \) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the mappings are said to be weakly commuting if

\[
M(ASx, SAx, t) \geq M(Ax, Sx, t), \quad \text{for all } t > 0, \ x \in X.
\]

Example 1.9 Let \((X, M, \ast)\) be a fuzzy metric space, where \( X = [0, 1] \), with t-norm defined \( a \ast b = \min\{a, b\} \), for all \( a, b \in [0, 1] \) and \( M(x, y, t) = \frac{t}{t + |x - y|} \) for all \( t > 0 \) and \( x, y \in X \). Define self-maps \( A \) and \( S \) on \( X \) as follows:

\[
Ax = \frac{x}{8}, \quad Sx = \frac{x}{x + 16}
\]
for any \( x \in X \).

\[
M(ASx, SAx, t) = M\left(\frac{x}{x+128}, \frac{x}{8x+128}, t\right) = \frac{t}{t + \left|\frac{x}{8} - \frac{x}{x+128}\right|} = M(Ax, Sx, t), \quad \text{for all } t > 0.
\]

Thus \( S \) and \( A \) satisfy in above definition but do not commute for any \( x \neq 0 \).

### 2. The Main Results

Let \( \Psi \) denote a family of mappings such that for each \( \psi \in \Psi, \psi : [0,1]^3 \rightarrow [0,1] \),

1. \( \psi(x, y, z) \) is continuous in each co-ordinate variable and,
2. \( \psi(x, y, z) = 1 \) only if \( x = y = z = 1 \) and \( \psi(u, 1, 1) \) or \( \psi(1, u, 1) \) or \( \psi(1, 1, u) > u \) for all \( u \neq 1 \).

Examples of \( \psi \) are \( \psi(x, y, z) = \max\{x, y, z\}, \psi(x, y, z) = a(t)x + b(t)y + c(t)z \), where \( a, b, c : \mathbb{R}^+ \rightarrow [0,1] \), are function such that \( a(t) + b(t) + c(t) = 1 \) for every \( t > 0 \). Other examples may also be constructed.

**Definition 2.1** Let \( A, T \) and \( S \) be mappings from a fuzzy metric space \((X, M, \ast)\) into itself. Then the mappings are said have property (E.A) if there exists a sequence \( \{x_n\} \in X \) such that

\[
\lim_{n \to \infty} M(Ax_n, Sx_n, t) = \lim_{m \to \infty} M(Ax_m, Tx_m, t) = 1, \quad \text{for all } t > 0.
\]

**Example 2.2** Let \((X, M, \ast)\) be a fuzzy metric space, where \( X = [0,1] \), with t-norm defined \( a \ast b = \min\{a, b\} \), for all \( a, b \in [0,1] \) and \( M(x, y, t) = \frac{t}{t + |x - y|} \) for all \( t > 0 \) and \( x, y \in X \). Define self-maps \( A \) and \( S \) on \( X \) as follows:

\[
Ax = x, \quad Sx = \begin{cases} 
1 & \text{if } x \in \mathbb{Q}, \\
0 & \text{otherwise, }\end{cases} \quad Tx = x^2
\]

for any \( x \in X \). If define \( x_n = 1 - \frac{1}{n} \), then

\[
\lim_{n \to \infty} M(Ax_n, Sx_n, t) = \lim_{m \to \infty} M(Ax_m, Tx_m, t) = 1, \quad \text{for all } t > 0.
\]

**Theorem 2.3** Let \( A, T \) and \( S \) be self-mappings of a complete fuzzy metric space \((X, M, \ast)\) satisfying:

1. \( M(Sx, Ty, t) \geq \psi(M(Ax, Sx, t), M(Ay, Ty, t), M(Ax, Tx, t)) \), for every \( x, y \in X \) and for every \( t > 0 \),
2. let \( A, T \) and \( S \) have property (E.A),
3. the pair \((A, S)\) and \((A, T)\) are weakly commuting and \( A \) is continuous, then \( A, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof** Since \( A, S \) and \( T \) have property (E.A), there exists sequence \( \{x_n\} \in X \) such that

\[
\lim_{n \to \infty} M(Ax_n, Sx_n, t) = \lim_{m \to \infty} M(Ax_m, Tx_m, t) = 1, \quad \text{for all } t > 0.
\]
Now, we prove \( \{Ax_n\} \) is a Cauchy sequence. By (1), we have
\[
M(Ax_n, Ax_m, t) \geq M(Ax_n, Sx_n, t/3) * M(Sx_n, Tx_m, t/3) * M(Tx_m, Ax_m, t/3)
\]
\[
\geq M(Ax_n, Sx_n, t/3) * \psi(M(Ax_n, Sx_n, t/3), M(Ax_m, Tx_m, t/3), M(Ax_n, Tx_n, t/3))
\]
\[
* M(Tx_m, Ax_m, t/3).
\]
On making \( n \to \infty \) in above inequality and by property (2) of \( \psi \), we have
\[
\lim_{n \to \infty} M(Ax_n, Ax_m, t) \to 1.
\]
Hence \( \{Ax_n\} \) is Cauchy and the completeness of \( X \), \( \{Ax_n\} \) converges to \( z \) in \( X \). That is, \( \lim_{n \to \infty} Ax_n = \psi 
\]
Since \( M(Sx_n, z, t) \geq M(Sx_n, Ax_n, t/2) * M(Ax_n, z, t/2) \),
By continuous \( M \), on making \( n \to \infty \) the above inequality, we get
\[
\lim_{n \to \infty} M(Sx_n, z, t) \geq \lim_{n \to \infty} M(Sx_n, Ax_n, t/2) * \lim_{n \to \infty} M(Ax_n, z, t/2) \to 1.
\]
Hence \( \lim_{n \to \infty} Sx_n = z \). Thus \( \lim_{n \to \infty} ASx_n = Az \).
Similarly we can prove that \( \lim_{m \to \infty} Tx_m = z \) and it implies \( \lim_{n \to \infty} ATx_m = Az \). On the other hand, by weakly commuting \( (A, S) \) we have,
\[
M(SAx_n, Az, t) \geq M(SAx_n, ASx_n, t/2) * M(ASx_n, Az, t/2)
\]
\[
\geq M(Sx_n, Ax_n, t/2) * M(ASx_n, Az, t/2).
\]
On making \( n \to \infty \) the above inequality, we get \( \lim_{n \to \infty} SAx_n = Az \). Also
\[
M(TAx_m, Az, t) \geq M(TAx_m, ATx_m, t/2) * M(ATx_m, Az, t/2)
\]
\[
\geq M(Tx_m, Ax_m, t/2) * M(ATx_m, Az, t/2).
\]
On making \( m \to \infty \) the above inequality, we get \( \lim_{m \to \infty} TAx_m = Az \). We prove \( Az = z \). By (i), we get
\[
M(Sx_n, TAx_n, t) \geq \psi(M(Ax_n, Sx_n, t), M(A^2x_n, TAx_n, t), M(Ax_n, Tx_n, t)).
\]
On making \( n \to \infty \) the above inequality, we get
\[
M(z, Az, t) \geq \psi(1, 1, 1) = 1,
\]
i.e., \( Az = z \). Similarly, we prove that \( Tz = z \).
\[
M(z, Tz, t) = M(Az, Tz, t) \geq M(Az, SAx_n, et) * M(SAx_n, Tz, (1 - \epsilon)t)
\]
\[
\geq M(Az, SAx_n, et) * \psi(M(A^2x_n, SAx_n, (1 - \epsilon)t), M(Az, Tz, (1 - \epsilon)t), M(A^2x_n, TAx_n, (1 - \epsilon)t)).
\]
For every $0 < \epsilon < 1$, on making $n \to \infty$ the above inequality, we get

$$M(z, Tz, t) \geq 1 * \psi(1, M(z, Tz, (1 - \epsilon)t), 1).$$

On making $\epsilon \to 0$ we have

$$M(z, Tz, t) \geq \psi(1, M(z, Tz, t), 1).$$

If $Tz \neq z$, by property of $\psi$ it follows that $M(z, Tz, t) > M(z, Tz, t)$, is a contradiction. Hence $Tz = z$. We prove that $Sz = z$. If $Sz \neq z$ then

$$M(Sz, z, t) = M(Sz, Tz, t) \geq \psi(M(Az, Sz, t), M(Az, Tz, t), M(Az, Tz, t))$$

$$> M(z, Sz, t),$$

is a contradiction. Therefore $Sz = z$. Hence $Az = Tz = Sz = z$, that is, $z$ is a common fixed of $A, S$ and $T$.

Uniqueness, let $z'$ be another common fixed point of $A, S$ and $T$. Then $Az' = Sz' = Tz' = z'$ and by (1), we have

$$M(z', z, t) = M(Sz', Tz, t) \geq \psi(M(Az', Sz', t), M(Az, Tz, t), M(Az', Tz', t))$$

$$= \psi(1, 1, 1) = 1.$$  

Therefore, $z = z'$ that is $z$ is the unique common fixed point of self-maps $A, S$ and $T$.

**Corollary 2.1** Let $A, T$ and $S$ be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying:

1. $M(Sx, Ty, t) \geq a(t)M(Ax, Sx, t) + b(t)M(Ay, Ty, t) + c(t)M(Ax, Tx, t)$, for every $x, y \in X$, where $a, b, c : \mathbb{R}^+ \to [0, 1]$, are function such that $a(t) + b(t) + c(t) = 1$ for every $t > 0$,
2. let $A, T$ and $S$ have property (E.A),
3. the pair $(A, S)$ and $(A, T)$ are weakly commuting and $A$ is continuous, then $A, S$ and $T$ have a unique common fixed point in $X$.

**Proof** By Theorem 2.3, it is enough set $\psi(x, y, z) = a(t)x + b(t)y + c(t)z$.

**Example 2.4** Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 1]$, with t-norm defined $a * b = \min\{a, b\}$, for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t + |x - y|}$ for all $t > 0$ and $x, y \in X$. Define self-maps $A$ and $S$ on $X$ as follows:

$$Tx = 0, \quad Ax = \frac{x}{8}, \quad Sx = \frac{x}{x + 16}$$

for any $x \in X$.

$$M(ASx, SAx, t) = M(\frac{x}{x + 128}, \frac{x}{8x + 128}, t) = \frac{t}{t + \frac{|x - x|}{x + 16}} = M(Ax, Sx, t), \quad \text{for all } t > 0.$$  

Thus $(A, S)$ is weakly commuting. Also, $M(ATAx, TAx, t) = 1 \geq M(Ax, Tx, t)$, i.e., $(A, T)$ is weakly commuting. If define sequence $x_n = \frac{1}{n}$, then it is easy to see that

$$\lim_{n \to \infty} M(AXn, TXn, t) = \lim_{m \to \infty} M(Ax_m, Sx_m, t) = 1.$$
Thus by \(a = c = 0\) and \(b = 1\), it follows that the all conditions of Corollary 2.1 are holds, and \(x_0 = 0\) is a unique fixed point of \(A, S\) and \(T\).

**Corollary 2.2** Let \(S_i, T_j\) and \(A\) be self-mappings of a complete fuzzy metric space \((X, M, *)\) satisfying:

1. \(M(S_ix, T_jy, t) \geq a(t)M(Ax, S_ix, t) + b(t)M(Ay, T_jy, t) + c(t)M(Ax, T_jx, t)\), for every \(x, y \in X\) and \(i, j = 1, 2, \ldots\), where \(a, b, c: \mathbb{R}^+ \rightarrow [0, 1]\), are function such that \(a(t) + b(t) + c(t) = 1\) for every \(t > 0\),
2. there exists \(i_0, j_0 \in \mathbb{N}\) such that \(A, T_{i_0}\) and \(S_{j_0}\) have property (E.A),
3. the pair \((A, S_{i_0})\) and \((A, T_{j_0})\) are weakly commuting and \(A\) is continuous, then \(A, S_i\) and \(T_j\) have a unique common fixed point in \(X\) for \(i, j = 1, 2, \ldots\).

**Proof** By Corollary 2.1, \(A\) and \(S_{i_0}\) and \(T_{j_0}\) for some \(i_0, j_0 \in \mathbb{N}\), have a unique common fixed point in \(X\). That is, there exists a unique \(z \in X\) such that

\[
Az = S_{i_0}(z) = T_{j_0}(z) = z.
\]

Suppose there exists \(i \in \mathbb{N}\) such that \(i \neq i_0\). Then we have

\[
M(S_ix, z, t) = M(S_i z, T_{j_0}z, t) \geq a(t)M(Az, S_i z, t) + b(t)M(Az, T_{j_0}z, t) + c(t)M(Az, T_{j_0}z, t),
\]

Hence we get

\[
M(S_i z, z, t) \geq a(t)M(z, S_i z, t) + b(t)M(z, z, t) + c(t)M(z, z, t) > M(z, z, t),
\]

is a contradiction. Hence for every \(i \in \mathbb{N}\), it follows that \(S_i(z) = z\). Similarly for every \(j \in \mathbb{N}\), we get \(T_j z = z\). Therefore for every \(i, j \in \mathbb{N}\), we have

\[
S_i z = T_j z = Az = z.
\]

**Corollary 2.3** Let \(A\) be a self-mapping of a complete fuzzy metric space \((X, M, *)\) satisfying:

1. \(M(x, y, t) \geq a(t)M(Ax, x, t) + b(t)M(Ay, y, t) + c(t)M(Ax, x, t)\), for every \(x, y \in X\), where \(a, b, c: \mathbb{R}^+ \rightarrow [0, 1]\), are functions such that \(a(t) + b(t) + c(t) = 1\) for every \(t > 0\).

If there exists a sequence \(\{x_n\}\), such that \(\lim_{n \to \infty} M(Ax_n, x_n, t) = 1\), then \(A\) have a unique common fixed point in \(X\).

**Proof** It is enough set \(S = T = I\), identity map in Corollary 2.1.

### 3. Open Problem

How can obtain some results of this paper for intuitionistic fuzzy metric spaces?

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### References

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