RESEARCH ARTICLE

Common Fixed Point Theorems in Generalized Fuzzy Metric Spaces with \(t\)-Norm of Hadžić Type

Sanjay Kumar * and Asha Rani †

* Department of Mathematics, D. C. R. University of Science and Technology Muṛthal (Sonepat), Haryana, India.
† B. M. Institute of Engineering and Technology, Sonepat-131001, Haryana, India.

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In this paper, we introduce the notion of generalized fuzzy metric spaces and prove a common fixed point theorem for weakly compatible maps using continuous \(t\)-norm of Hadžić-type (in short \(H\)-type) for a class of generalized Kannan type mappings in the newly defined spaces. At the end, we give example in support of our theorem.

Keywords: Generalized fuzzy metric spaces; Weakly compatible maps; \(n\)-th order \(t\)-norm; \(t\)-norm of Hadžić-type.

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1. Introduction

Branciari [1] introduced the concept of generalized metric space by replacing triangular inequality with quadrangular inequality. The definition of generalized metric space is as follows:

Definition 1.1 [1] Let \(X\) be a nonempty set and \(\mathbb{R}^+\) be the set of all positive real numbers and let \(d : X \times X \to \mathbb{R}^+\) be a mapping such that for all \(x, y \in X\) and for all points \(\xi, \eta \in X\), each of them different from \(x\) and \(y\), satisfying:

1. \(d(x, y) = 0 \iff x = y\),
2. \(d(x, y) = d(y, x)\) and,
3. \(d(x, y) \leq d(x, \xi) + d(\xi, \eta) + d(\eta, y)\).

Branciari [1] showed that there exist generalized metric spaces which are not metric spaces and further proved Banach contraction mapping theorem in setting of generalized metric spaces.

Recently, Choudhury and Das [2] introduced the notion of generalized Menger spaces and proved fixed points theorem for generalized Kannan type mappings in this space.

Definition 1.2 [3, 4] Let \((X, d)\) be a metric space and \(f\) be a mapping on \(X\). The mapping \(f\) is called a Kannan type mapping if there exists \(0 \leq \alpha < \frac{1}{2}\) such that

\[d(fx, fy) \leq \alpha[d(x, fx) + d(y, fy)] \text{ for all } x, y \in X.\]

It has been shown that every contraction and Kannan type mapping on a complete metric space have unique fixed points. A difference between contraction mappings and Kannan type mappings

\[\text{Corresponding author}
Email: asha.dahiya27@gmail.com\]
is that contraction mappings are always continuous but Kannan type mappings are not necessarily
continuous. In [5] it is shown that every metric space \( X \) is that contraction mappings are always continuous but Kannan type mappings are not necessarily
continuous. In [5] it is shown that every metric space \( X \) is complete if and only if every Kannan type
mapping has a fixed point. One can refer to [6] for similarity between contraction and Kannan type
mappings.

Definition 1.3 \((n\)-th order \( t\)-norm [7]) A mapping \( T : \prod_{i=1}^{n} [0, 1] \to [0, 1] \) is called a \( n \)-th order \( t\)-norm if the following conditions are satisfied:

1. \( T(0, 0, \ldots, 0) = 0, T(a, 1, 1, \ldots, 1) = a \) for all \( a \in [0, 1] \),
2. \( T(a_1, a_2, a_3, \ldots, a_n) = T(a_2, a_1, a_3, \ldots, a_n) = T(a_2, a_3, a_1, \ldots, a_n) = \cdots = T(a_2, a_3, a_4, \ldots, a_n, a_1) \),
3. \( a_i \geq b_i, i = 1, 2, 3, \ldots, n \) implies \( T(a_1, a_2, a_3, \ldots, a_n) \geq T(b_1, b_2, b_3, \ldots, b_n) \),
4. For \( \lambda \) a \( t\)-norm and \( \Pi_{i=1}^{n} (0, 1) \) such that \( \lambda \in [0, 1] \) such that \( \lim_{n\to\infty} b_n = 1 \) and \( T(b_n, b_n) = b_n \) for all \( n \in \mathbb{N} \), then \( T \) is of Hadzic-type but converse need not be true (see [10]).

Definition 1.4 [8] A mapping \( * : [0, 1] \times [0, 1] \to [0, 1] \) is called a continuous \( t\)-norm if \(([0, 1], *)\) is an
closed topological monoid with unit 1 such that \( a * b \leq c * d \) for \( a \leq c \) and \( b \leq d \).

Basic examples of \( t\)-norm are the Lukasiewicz \( t\)-norm \( T_L, T_L(a, b) = \max(a + b - 1, 0) \), \( t\)-norm \( T_P, T_P(a, b) = ab \) and \( t\)-norm \( T_N, T_N(a, b) = \min(a, b) \).

Definition 1.5 [9] A special class of \( t\)-norms (called a Hadzic-type \( t\)-norm) is introduced as follows:

Let \( T \) be a \( t\)-norm and let \( T_n : [0, 1] \to [0, 1] \) \( n \in \mathbb{N} \) be defined in the following way:

\[
T_1(x) = T(x, x), \quad T_{n+1}(x) = T(T_n(x), x) \quad (n \in \mathbb{N}, \quad x \in [0, 1]).
\]

We say that the \( t\)-norm \( T \) is of \( H\)-type if \( T \) is continuous and the family \( \{T_n(x), \quad n \in \mathbb{N}\} \) is equi-
continuous at \( x = 1 \).

The family \( \{T_n(x), \quad n \in \mathbb{N}\} \) is equicontinuous at \( x = 1 \) if for every \( \lambda \in (0, 1) \) there exists \( \delta(\lambda) \in (0, 1) \) such that the following implication holds:

\[
x > 1 - \delta(\lambda) \implies T_n(x) > 1 - \lambda \quad \text{for all} \quad n \in \mathbb{N}.
\]

A trivial example of \( t\)-norm of \( H\)-type is \( T = T_M \).

Remark 1.6 Every \( t\)-norm \( T_M \) is of Hadzic-type but converse need not be true (see [10]).

There is a nice characterization of continuous \( t\)-norm [8].

1. If there exists a strictly increasing sequence \( \{b_n\}_{n \in \mathbb{N}} \in [0, 1] \) such that \( \lim_{n\to\infty} b_n = 1 \) and
2. \( T(b_n, b_n) = b_n \) for all \( n \in \mathbb{N} \), then \( T \) is of Hadzic-type.

Definition 1.7 [10] If \( T \) is a \( t\)-norm and \( (x_1, x_2, \ldots, x_n) \in [0, 1]^n \) \( n \in \mathbb{N} \), then \( T_{i=1}^{n} x_i \) is defined
recurrently by 1, if \( n = 0 \) and \( T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n) \) for all \( n = 1 \). If \( \{x_i\}_{i \in \mathbb{N}} \) is a sequence of
numbers from \( [0, 1] \), then \( T_{i=1}^{\infty} x_i \) is defined as \( \lim_{n\to\infty} T_{i=1}^{n} x_i \) (this limit always exists) and \( T_{i=n}^{\infty} x_i \) as
A sequence \( \{x_n\} \subset [0,1] \) such that \( \lim_{n \to \infty} x_n = 1 \) and \( \lim_{n \to \infty} T_{n=1}^{\infty} x_{n+i} = 1 \).

It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [11] which laid the foundation of fuzzy mathematics. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. Using the idea of fuzzy sets Kramosil and Michalek [12] introduced the concept of fuzzy metric spaces. Later on, George and Veeramani [13] modified the idea and defined Hausdorff topology on fuzzy metric space and proved some interesting results. This proved a milestone in fixed point theory of fuzzy metric space and afterwards a flood of papers appeared for fixed point theorems in fuzzy metric space.

**Definition 1.8** [11] Let \( X \) be any non empty set. A fuzzy set \( M \) in \( X \) is a function with domain \( X \) and values in \([0,1] \).

**Definition 1.9** [12] The 3-tuple \((X, M, *)\) is called a fuzzy metric space in the sense of Kramosil and Michalek if \( X \) is an arbitrary set, \(*\) is a continuous \( t\)-norm and \( M \) is a fuzzy set in \( X^2 \times [0, \infty) \) satisfying the following conditions:

1. \( M(x,y,t) > 0 \),
2. \( M(x,y,t) = 1 \) for all \( t > 0 \) if and only if \( x = y \),
3. \( M(x,y,t) = M(y,x,t) \),
4. \( M(x,y,t) * M(y,z,s) \leq M(x,z,t+s) \),
5. \( M(x,y,:) : [0, \infty) \to [0,1] \) is left continuous function, for all \( x, y, z \in X \) and \( t, s > 0 \).

Note that \( M(x,y,t) \) can be thought as degree of nearness between \( x \) and \( y \) with respect to \( t \). It is known that \( M(x,y,:) \) is nondecreasing for all \( x, y \in X \) ([13]).

A sequence \( \{x_n\} \) in \( X \) converges to \( x \) if and only if for each \( t > 0 \) there exists \( n_0 \in N \), such that \( M(x_n, x, t) = 1 \), for all \( n \geq n_0 \).

The sequence \( \{x_n\}_{n \in N} \) is called Cauchy sequence if \( \lim_{n \to \infty} M(x_n, x_{n+p}, t) = 1 \), for all \( t > 0 \) and \( p \in N \). A fuzzy metric space \( X \) is called complete if every Cauchy sequence is convergent in \( X \).

**Lemma 1.10** [13] Let \((X, M, *)\) be a fuzzy metric space. Then \( M \) is a continuous function on \( X^2 \times (0, \infty) \).

Now we introduce the notion of generalized fuzzy metric spaces akin to the notion of generalized menger spaces [2].

**Definition 1.11** (Generalized Fuzzy metric space) Let \( X \) be a non-empty set and \( M \) is a fuzzy set on \( X^2 \times [0, \infty) \), and \( T \) is a 3rd order \( t\)-norm (Definition 1.3). Then \((X, M, T)\) is said to be a generalized fuzzy metric space if for all \( x, y \in X \) and all distinct points \( z, w \in X \) each of them different from \( x \) and \( y \), the following conditions are satisfied:

1. \( M(x,y,0) = 0 \),
2. \( M(x,y,t) = 1 \) for all \( t > 0 \) if and only if \( x = y \),
3. \( M(x,y,t) = M(y,x,t) \) for all \( t > 0 \) and for all \( x, y \in X \),
4. \( M(x,y,t) \geq T(M(x,z,t_1), M(z,w,t_2), M(w,y,t_3)) \), where \( t_1 + t_2 + t_3 = t \) and \( T \) is a 3rd order \( t\)-norm.

Topology of generalized fuzzy metric spaces is similar to the topology of generalized Menger spaces [2].

**Remark 1.12** Notice that Definition 1.1 is special case of Definition 1.11. Let \((X, d)\) be a generalized metric space. This space can be treated as a generalized fuzzy metric space if we put

\[
M(x,y,t) = \frac{t}{t + d(x,y)}
\]
and the $t$-norm $T$ is taken as $T_M$ which is defined as $T_M(\alpha, \beta, \gamma) = \min\{\alpha, \beta, \gamma\}$ that is $T_M$ is the $3^{rd}$ order minimum $t$-norm.

Conditions (1) and (2) of Definition 1.1 trivially follow from conditions (2) and (3) of Definition 1.11, respectively. We now show that condition (3) of Definition 1.1 follows from conditions (4) of Definition 1.11.

Let $x, y \in X$ and $z, w$ be two distinct points in $X$ different from $x$ and $y$. Suppose,

$$d(x, y) \leq d(x, z) + d(z, w) + d(w, y) \quad (1)$$

be true. For some $t_1, t_2, t_3 > 0$ and $t = t_1 + t_2 + t_3$, the inequality

$$M(x, y, t) \geq T_M(M(x, z, t_1), M(z, w, t_2), M(w, y, t_3)) \quad (2)$$

be false. Inequality (2) will be false only if $M(x, y, t) = 0$ and $M(x, z, t_1) = 1$, $M(z, w, t_2) = 1$, $M(w, y, t_3) = 1$, and all these implies that $x = z$ and $z = w$ and $w = y$, which is impossible.

Thus we conclude that condition (3) of Definition 1.1 follows from condition (4) of Definition 1.11. Hence generalized metric space is a special case of generalized fuzzy metric space.

Definition 1.13 Let $(X, M, \ast)$ be a generalized fuzzy metric space. A sequence $\{x_n\} \subset X$ is said to converge to some point $x \in X$ if given $\epsilon > 0$, $\lambda > 0$, we can find a positive integer $N_{\epsilon, \lambda}$ such that for all $n > N_{\epsilon, \lambda}$, $M(x_n, x, \epsilon) > 1 - \lambda$.

Definition 1.14 A sequence $\{x_n\}$ is said to be a Cauchy sequence in $X$ if $\epsilon > 0$, $\lambda > 0$, there exists a positive integer $N_{\epsilon, \lambda}$ such that $M(x_n, x_m, \epsilon) > 1 - \lambda$ for all $m, n > N_{\epsilon, \lambda}$.

Definition 1.15 A generalized fuzzy metric space $(X, M, \ast)$ is said to be complete if every Cauchy sequence is convergent in it.

Definition 1.16 Two self-maps $f$ and $g$ of a generalized fuzzy metric space $(X, M, \ast)$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $fp = gp$ for some $p \in X$ then $f gp = g fp$.

A new class of fixed point problems in metric spaces was addressed by Khan, Swaleh and Sessa in [14]. They introduced a control function called altering distance function which alters the distance between any two points in a given metric space. They proved fixed point theorems for mappings satisfying certain inequalities involving this altering distance function. Afterwards a number of works have appeared in which altering distance functions and their generalizations have been used in metric spaces for obtaining fixed point results. We note some of these in references [15], [16] and [17].

This idea of control function in fixed point theory has opened the possibility of proving new fixed point results. Some recent results on fixed point and coincidence point have been obtained in works like [18], [19], [20] and [17].

Definition 1.17 ($\Psi$-function) A function $\psi : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a $\Psi$-function if it satisfies the following conditions:

1. $\psi$ is monotone increasing and continuous,
2. $\psi(x, x) \geq x$ for all $0 < x < 1$,
3. $\psi(1, a) \geq a$, $\psi(0, 0) = 0$,
4. $\psi(1, 1) = 1$.

An example of $\Psi$-function:

$$\psi(x, y) = \sqrt{xy}.$$
The purpose of this paper is to prove a fixed point result for a class of Kannan type mappings in generalized fuzzy metric spaces using \( \Psi \)-functions as defined in Definition 1.17. We shall support our result by constructing an example.

Proposition 1.18 [10] Let \((x_n, n \in \mathbb{N})\) be a sequence of numbers in \([0, 1]\) such that \(\lim_{n \to \infty} x_n = 1\) and the \(t\)-norm \(T\) is of \(H\)-type, then

\[
\lim_{n \to \infty} T_{i=n}^\infty x_i = \lim_{n \to \infty} T_{i=1}^\infty x_{n+i} = 1.
\]

2. Main Result

Theorem 2.1 Let \((X, M, T)\) be a complete generalized fuzzy metric space, where \(T\) is the 3\(rd\) order \(t\)-norm of Hadžić-type and the mappings \(f\) and \(g : X \to X\) be mappings which satisfy the following conditions:

1. \(f(X) \subseteq g(X)\),
2. any one of \(f(X)\) and \(g(X)\) is complete,
3. \(M(fx, fy, t) = \psi(M(fx, gx, \frac{t_1}{a}), M(fy, gy, \frac{t_2}{b}))\) for all \(x, y \in X\) and for \(t_1, t_2, t > 0\) with \(t = t_1 + t_2, a, b > 0\) and \(0 < a + b < 1\) where \(\psi\) is a \(\Psi\)-function.

For any \(x_0 \in X\), the sequence \(\{y_n\}\) in \(X\) be constructed as follows: \(y_n = fx_n = gx_{n+1}, n = 0, 1, 2, 3, \ldots\), such that for \(\mu \in (c, 1)\) and \(c = a + b\), the following condition holds:

\[
\lim_{n \to \infty} T_{i=n}^\infty M(y_0, y_1, \frac{1}{\mu^t}) = 1,
\]

Then \(f\) and \(g\) have a unique common fixed point provided \(f\) and \(g\) are weakly compatible on \(X\).

Proof Let \(x_0 \in X\), we now construct a sequence \(\{y_n\}\) by \(y_n = gx_{n+1} = fx_n, n \in \mathbb{N}\), where \(\mathbb{N}\) is the set of all positive integers. Now we have for \(t, t_1, t_2 > 0\), with \(t = t_1 + t_2\) such that

\[
M(y_{n+1}, y_n, t) \geq M(fx_{n+1}, fx_n, t)
\]

\[
\geq \psi\{M(fx_{n+1}, gx_{n+1}, \frac{t_1}{a}), M(fx_n, gx_n, \frac{t_2}{b})\}
\]

\[
= \psi\{M(y_{n+1}, y_n, \frac{t_1}{a}), M(y_n, y_{n-1}, \frac{t_2}{b})\}.
\]

Let \(t_1 = \frac{at}{a+b}, t_2 = \frac{bt}{a+b}\) and \(c = a + b\), then \(0 < c < 1\), we have

\[
M(y_{n+1}, y_n, t) \geq \psi\{M(y_{n+1}, y_n, \frac{t}{c}), M(y_n, y_{n-1}, \frac{t}{c})\}.
\]

We now claim that for all \(t > 0\),

\[
M(y_{n+1}, y_n, \frac{t}{c}) \geq M(y_n, y_{n-1}, \frac{t}{c}).
\]

If possible, let for some \(t > 0, M(y_{n+1}, y_n, \frac{t}{c}) < M(y_n, y_{n-1}, \frac{t}{c})\), then we have

\[
M(y_{n+1}, y_n, t) \geq \psi\{M(y_{n+1}, y_n, \frac{t}{c}), M(y_{n+1}, y_n, \frac{t}{c})\}
\]

\[
\geq M(y_{n+1}, y_n, \frac{t}{c})
\]

\[\geq M(y_{n+1}, y_n, t),\]
since $0 < c < 1$, which leads to a contradiction.

Therefore for all $t > 0$, $M(y_n+1, y_n, t) \geq M(y_n, y_{n-1}, t)$, which leads to the following inequality, i.e.,

$$M(y_{n+1}, y_n, t) \geq \psi\{M(y_{n+1}, y_n, \frac{t}{c}), M(y_n, y_{n-1}, \frac{t}{c})\}$$

$$\geq \psi\{M(y_n, y_{n-1}, \frac{t}{c}), M(y_n, y_{n-1}, \frac{t}{c})\}$$

$$\geq M(y_n, y_{n-1}, \frac{t}{c})$$

$$\geq M(y_{n-1}, y_{n-2}, \frac{t}{c^2})$$

$$\vdots$$

$$\geq M(y_1, y_0, \frac{t}{c^n})$$

that gives,

$$M(y_{n+1}, y_n, t) \geq M(y_1, y_0, \frac{t}{c^n}).$$

Therefore, letting $n \to \infty$, we have

$$\lim_{n \to \infty} M(y_{n+1}, y_n, t) = 1,$$

for all $t > 0$.

Next we show that sequence $\{y_n\}$ is a Cauchy sequence.

Let $\sigma = \frac{c}{\mu}$, where $\mu \in (c, 1)$ and $c \in (0, 1)$, then $0 < \sigma < 1$, therefore the series $\sum_{i=1}^{\infty} \sigma^i$ is convergent and there exists $m_0 \in \mathbb{N}$ such that $\sum_{i=m_0}^{\infty} \sigma^i < 1$. Now for every $m > m_0$ and for every $s \in \mathbb{N}$, we have $t > t \sum_{i=m_0}^{\infty} \sigma^i \geq t \sum_{i=m}^{m+s} \sigma^i$, which implies that

$$M(y_{m+s+1}, y_m, t) \geq M(y_{m+s+1}, y_m, t \sum_{i=m}^{m+s} \sigma^i)$$

$$\geq T(T \ldots T(M(y_{m+s+1}, y_m, t \sigma^{m+s}), M(y_{m+s}, y_{m+s-1}, t \sigma^{m+s-1}), \ldots, M(y_{m+1}, y_m t \sigma^m)))$$

$$\geq T(T \ldots T(M(y_0, y_1, t \sigma^{m+s} \sigma^m \ldots \sigma^m), \ldots, M(y_0, y_1, t \sigma^m \sigma^m \ldots \sigma^m)))$$

$$\geq T_{i=m}^{m+s} M(y_0, y_1, \frac{t}{\mu^s}) \geq T_{i=m}^{\infty} M(y_0, y_1, \frac{t}{\mu^s}).$$

It is obvious that $\lim_{n \to \infty} T_{i=n}^{\infty} M(y_0, y_1, \frac{1}{\mu^s}) = 1$ implies that $\lim_{n \to \infty} T_{i=n}^{\infty} M(y_0, y_1, (\frac{t}{\mu^s})) = 1$ for every $t > 0$.

Now for every $t > 0$, there exist $m_1(t, \lambda)$ such that $M(y_{m+s+1}, y_m, t) > 1 - \lambda$ for every $m = m_1(t, \lambda)$ and every $s \in \mathbb{N}$.

This means that the sequence $\{y_n\}$ is Cauchy sequence and since $(X, M, T)$ is a complete generalized fuzzy metric space we have $\{y_n\}$ is convergent in $X$, therefore for some $w$ in $X$, $\lim_{n \to \infty} y_n = w = \lim_{n \to \infty} g x_n = \lim_{n \to \infty} f x_n$. 
Let $g(X)$ be complete, there exists a point $p \in X$ such that $gp = w$. We now show that $fp = w$. For some $t > 0$, since $0 < b < 1$, we can choose $\eta_1, \eta_2, t_1, t_2 > 0$ such that $t = \eta_1 + \eta_2 + t_1 + t_2$, and $\frac{t_2}{b} > t$. Then we have,

\[
M(w, fp, t) = M(w, fp, \eta_1 + \eta_2 + t_1 + t_2) \\
\geq T\{M(w, y_n, \eta_1), M(y_n, y_{n+1}, \eta_2), M(y_{n+1}, fp, t_1 + t_2)\} \\
\geq T\{M(w, y_n, \eta_1), M(y_n, y_{n+1}, \eta_2), \psi(M(fx_{n+1}, gx_{n+1}, \frac{t_1}{a}), M(fp, gp, \frac{t_2}{b}))\} \\
\geq T\{M(w, y_n, \eta_1), M(y_n, y_{n+1}, \eta_2), \psi(M(fx_{n+1}, gx_{n+1}, \frac{t_1}{a}), M(fp, gp, t))\}.
\]

Taking limit $n \to \infty$, we have

\[
M(w, fp, t) \geq T\{1, 1, \psi(1, M(w, fp, t))\} = \psi(1, M(w, fp, t)) \geq M(w, fp, t).
\]

This gives $M(w, fp, t) \geq M(w, fp, t)$, which is not possible. Therefore, $M(w, fp, t) = 1$ for all $t > 0$ and this imply that $w = fp = gp$. Since $f$ and $g$ are weakly compatible, it follows that $fgp = gfp$, that is, $fw = gw$.

Now, we show that $w$ is a fixed point of $f$ and $g$, from (3), we have

\[
M(fw, fx_n, t) \geq \psi(M(fw, gw, \frac{t_1}{a}), M(fx_n, gx_n, \frac{t_2}{b})), \quad \text{(where } t = t_1 + t_2)\]

i.e.,

\[
M(fw, y_n, t) \geq \psi(M(fw, gw, \frac{t_1}{a}), M(y_n, y_{n+1}, \frac{t_2}{b})).
\]

Taking $n \to \infty$, we have $M(fw, w, t) \geq \psi(1, 1) = 1$, gives us $fw = w = gw$. Thus $w$ is a fixed point of $f$ and $g$.

**Uniqueness.** Let $z$ be another fixed point of $f$ and $g$. Now for all $t > 0$, we have

\[
M(z, w, t) = M(fz, fw, t) \\
\geq \psi((M(fz, gz, \frac{t_1}{a}), M(fw, gw, \frac{t_2}{b})) \quad \text{(where } t_1, t_2 > 0 \text{ and } t = t_1 + t_2) \\
= \psi(M(z, z, \frac{t_1}{a}), M(w, w, \frac{t_2}{b})) \\
= \psi(1, 1) = 1,
\]

gives us $z = w$. This completes the proof of the theorem.

**Corollary 2.1** Let $(X, M, T)$ be a complete generalized fuzzy metric space, where $T$ is the 3\textsuperscript{rd} order $t$-norm of Hadžić-type and $f$ be self mapping on $X$ satisfy the following inequality:

\[
M(fx, fy, \phi(t)) = \psi(M(fx, x, \frac{t_1}{a}), M(fy, y, \frac{t_2}{b})),
\]

for all $x, y \in X$, and for $t_1, t_2, t > 0$ with $t = t_1 + t_2, a, b > 0$ with $0 < a + b < 1$ where $\psi$ is a $\Psi$-function.

For any $x_0 \in X$, the sequence $\{x_n\}$ in $X$ be constructed as follows: $x_n = fx_{n-1}, n = 0, 1, 2, 3, \ldots,$
such that for $\mu \in (c, 1)$ and $c = a + b$, the following condition holds:

$$\lim_{n \to \infty} T_{i=n}^\infty M(x_0, x_1, \frac{1}{\mu^i}) = 1.$$ 

Then $f$ has a unique common fixed point.

**Proof** By putting $g(x) = I$ in Theorem 2.1, we get the required result. ■

**Corollary 2.2** Let $(X, M, T_M)$ be a complete generalized fuzzy metric space, where $T_M$ is the 3rd order minimum $t$-norm given by $T_M(\alpha, \beta, \gamma) = \min\{\alpha, \beta, \gamma\}$ and the mapping $f : X \to X$ be a self mapping which satisfy the following inequality for all $x, y \in X$,

$$M(fx, fy, t) = \psi(M(fx, x, \frac{t_1}{a}), M(fy, y, \frac{t_2}{b})), \quad (4)$$

where $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $a, b > 0$ with $0 < a + b < 1$ where $\psi$ is a $\Psi$-function. Then $f$ has a unique common fixed point.

**Proof** Since every $t$-norm $T_M$ is of Hadžič-type. Therefore proof follows from Corollary 2.1. ■

**Example 2.2** Let $X = \{1, 2, 3, 4\}$, $T_M(\alpha, \beta, \gamma) = \min\{\alpha, \beta, \gamma\}$, i.e., $T_M$ is the 3rd order minimum $t$-norm and $M(x, y, t)$ is defined as:

$$M(1, 2, t) = M(2, 1, t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.60, & \text{if } 0 < t < 3, \\ 1, & \text{if } t \geq 3. \end{cases}$$

$$M(1, 3, t) = M(3, 1, t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.80, & \text{if } 0 < t \leq 1.5, \\ 1, & \text{if } t > 1.5. \end{cases}$$

$$M(1, 4, t) = M(4, 1, t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.70, & \text{if } 0 < t \leq 2, \\ 1, & \text{if } t > 2. \end{cases}$$

$$M(2, 3, t) = M(3, 2, t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.85, & \text{if } 0 < t \leq 1.5, \\ 1, & \text{if } t > 1.5, \end{cases}$$

$$M(2, 4, t) = M(4, 2, t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.70, & \text{if } 0 < t \leq 2, \\ 1, & \text{if } t > 2. \end{cases}$$

$$M(3, 4, t) = M(4, 3, t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 0.60, & \text{if } 0 < t < 3, \\ 1, & \text{if } t \geq 3. \end{cases}$$

Then $(X, M, T_M)$ is a complete generalized fuzzy metric space.
Let $f : X \to X$ be given by $f(1) = f(2) = f(3) = f(4) = 1$, and $g(2) = g(3) = g(4) = 3$, $g(1) = 1$. Also $f(X) \subseteq g(X)$ and if we take $\psi(x, y) = \sqrt{xy}$ and $a = 0.2$, $b = 0.75$, then $f$ and $g$ satisfy all the conditions of Theorem 2.1 and 1 is the unique fixed point of $f$ and $g$ and we note that $(X, M, T_M)$ is not a fuzzy metric space as can be seen from the fact that

$$M(3, 4, 2) \neq T_M(M(3, 2, 1), M(2, 4, 1)).$$

This shows that generalized fuzzy metric spaces are effective generalization of generalized metric spaces.

Example 2.3 Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, $T_M(\alpha, \beta, \gamma) = \min\{\alpha, \beta, \gamma\}$, i.e., $T_M$ is the $3^rd$ order minimum $t$-norm and $M(x, y, t)$ be defined as $M(x, y, t) = \frac{t}{t + d(x, y)}$ and $d(x, y) = |x - y|$ with $t > 0$ and $x, y \in X$, as a fuzzy set. Then $(X, M, T_M)$ is a complete generalized fuzzy metric space.

Let

$$f(x) = 1, \quad g(x) = \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}, \end{cases}$$

on $X$. Also $f(X) \subseteq g(X)$ and if we take $\psi(x, y) = \sqrt{xy}$ and $a = 0.2$, $b = 0.75$, then $f$ and $g$ satisfy all the conditions of Theorem 2.1 and 1 is the unique fixed point of $f$ and $g$.

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References