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Some New Types of Open and Closed Functions

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The purpose of this paper is to introduce some new classes of functions by using \( b \)-open sets. We investigate some of their fundamental properties and the connections between these functions and other known existing topological functions are studied.

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1. Introduction and Preliminaries

Functions and of course open functions stand among the most important notions in the whole of mathematical science. Many different forms of open functions have been introduced over the years. Various interesting problems arise when one considers openness. Its importance is significant in various areas of mathematics and related sciences. Since 1996, when Andrijevic [1] has introduced a weak form of open sets called \( b \)-open sets. In the same year, this notion was also called \( sp \)-open sets in the sense of Dontchev and Przemski [2] but one year later are called \( \gamma \)-open sets due to El-Atik [3].

The purpose of the present paper is to continue the study of related functions by involving \( b \)-open sets. In the second section, we obtain some characterizations of contra pre-\( b \)-open and contra pre-\( b \)-closed functions. In the third section, we investigate the group structure of the family \( brh(X, \tau) \cup contbrh(X, \tau) \), where \( brh(X, \tau) \) is the collection of all \( br \)-homeomorphism and \( contbrh(X, \tau) \) is the collection of all contra-\( br \)-homeomorphism.

Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned and \( f : (X, \tau) \rightarrow (Y, \sigma) \) denotes a function \( f \) of a space \((X, \tau)\) into a space \((Y, \sigma)\). Let \( A \) be a subset of a space \( X \). The closure and the interior of \( A \) are denoted by \( Cl(A) \) and \( Int(A) \), respectively.

Definition 1.1 A subset \( A \) of a space \((X, \tau)\) is called \( b \)-open [1] (= \( sp \)-open [2], \( \gamma \)-open [3]) (resp. regular open [4], \( \alpha \)-open [5], preopen [6]) if \( A \subset Cl(Int(A)) \cup Int(Cl(A)) \) (resp. \( A = Int(Cl(A)) \), \( A \subset Int(Cl(Int(A))) \), \( A \subset Int(Cl(A)) \)). The complement of \( b \)-open (resp. \( \alpha \)-open) set is called \( b \)-closed (resp. \( \alpha \)-closed) set.

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The union (resp. intersection) of all \( b- \)open (resp. \( b- \)closed) sets, each contained in (resp. containing) a set \( A \) in a space \( X \) is called the \( b- \)interior (resp. \( b- \)closure) of \( A \) and is denoted by \( b\text{Int}(A) \) (resp. \( b\text{Cl}(A) \)) [1].

**Definition 1.2** A function \( f : (X, \tau) \to (Y, \sigma) \) is called:

1. \( b- \)irresolute [3] (\( b- \)continuous [3]) if \( f^{-1}(V) \) is \( b- \)closed in \( X \) for every \( b- \)closed (resp. \( b- \)closed) subset \( V \) of \( Y \);
2. \( b- \)open [3] (resp. \( b- \)closed [3]) if \( f(V) \) is \( b- \)open (resp. \( b- \)closed) in \( Y \) for every open (resp. closed) subset of \( X \);
3. \( b- \)strongly continuous [7] if the inverse image of every subset of \( Y \) is a clopen subset of \( X \).

**Definition 1.3** A function \( f : (X, \tau) \to (Y, \sigma) \) is called:

1. \( b- \)weakly open [8] if \( f(U) \subseteq \text{Int}(f(\text{Cl}(U))) \) for each open set \( U \) of \( X \);
2. \( b- \)almost open [9] provided that \( f(U) \) is open in \( f(\text{Cl}(U)) \) for every open subset \( U \) of \( X \);
3. \( b- \)preopen [6] if \( f(U) \) is a preopen set of \( Y \) for each open set \( U \) in \( X \).

2. **Contra Pre-\( b- \)Open Functions**

**Definition 2.1** A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be:

1. contra pre-\( b- \)open if \( f(U) \in BC(Y) \) for each \( U \in BO(X) \).
2. contra pre-\( b- \)closed if \( f(U) \in BO(Y) \) for each \( U \in BC(X) \).

**Remark 2.2** The following example shows that contra pre-\( b- \)closedness and contra \( b- \)openness are independent.

**Example 2.3** Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\emptyset, \{a\}, X\} \). Let \( f : (X, \tau) \to (X, \tau) \) be given by

\[
f(a) = f(b) = f(c) = b
\]

and \( g : (X, \tau) \to (X, \tau) \) by

\[
g(a) = g(b) = g(c) = a.
\]

Obviously \( f \) is contra pre-\( b- \)open but not contra pre-\( b- \)closed and \( g \) is contra pre-\( b- \)closed but not contra pre-\( b- \)open.

**Remark 2.4** Contra pre-\( b- \)openness and contra pre-\( b- \)closedness are equivalent if the function is bijective.

**Theorem 2.5** For an injective function \( f : (X, \tau) \to (Y, \sigma) \) the following are equivalent:

1. \( f \) is contra pre-\( b- \)open;
2. For every subset \( B \) of \( Y \) and for every \( b- \)closed subset \( F \) of \( X \) with \( f^{-1}(B) \subset F \), there exists a \( b- \)open subset \( A \) of \( Y \) with \( B \subset A \) and \( f^{-1}(A) \subset F \);
3. For every \( y \in Y \) and for every \( b- \)closed subset \( F \) of \( X \) with \( f^{-1}(y) \subset F \), there exists a \( b- \)open subset \( A \) of \( Y \) with \( y \in A \) and \( f^{-1}(A) \subset F \).

**Proof**

(1)\( \Rightarrow \) (ii): Let \( B \) be a subset of \( Y \) and let \( F \) be a \( b- \)closed subset of \( X \) with \( f^{-1}(B) \subset F \). Put \( A = f(F^c)^c \). Since \( f \) is contra pre-\( b- \)open, then \( A \) is a \( b- \)open set of \( Y \) and since \( f^{-1}(B) \subset F \) we have \( f(F^c) \subset B^c \) and hence \( B \subset A \). Moreover \( f^{-1}(A) = f^{-1}(f(F^c)^c) \subset F \).

(2)\( \Rightarrow \) (3): It is sufficient put \( B = \{y\} \).
(3)⇒(1): Let $A$ be a $b$-open subset of $X$. Then let $y \in f(A)^c$ and let $F = A^c$. By (3), there exists a $b$-open subset $B_y$ of $Y$ with $y \in B_y$ and $f^{-1}(B_y) \subset F$. Then we see that $y \in B_y \subset f(A)^c$. Hence $f(A)^c = \bigcup \{B_y : y \in f(A)^c\}$ is $b$-open. Therefore, $f(A)$ is a $b$-closed subset of $Y$. ■

**Theorem 2.6** For an injective function $f : (X, \tau) \to (Y, \sigma)$ the following are equivalent:

1. $f$ is contra pre-$b$-closed;
2. For every subset $B$ of $Y$ and for every $b$-open subset $A$ of $X$ with $f^{-1}(B) \subset A$, there exist a $b$-closed subset $F$ of $Y$ with $B \subset F$ and $f^{-1}(F) \subset A$.

**Proof** (1)⇒(2): Let $B$ be a subset of $Y$ and let $A$ be a $b$-open subset of $X$ with $f^{-1}(B) \subset A$. Put $F = f(A)^c$. Since $f$ is contra pre-$b$-closed, then $F$ is $b$-closed in $Y$ and since $f^{-1}(B) \subset A$, we have $f(A)^c \subset B^c$ and hence $B \subset F$. Moreover, $f^{-1}(F) \subset A$.

(2)⇒(1): Let $E$ be a $b$-closed subset of $X$. Put $B = f(E)^c$ and let $A = E^c$. Hence $f^{-1}(B) = (f^{-1}(f(E)))^c \subseteq A$. By assumption there exists a $b$-closed set $F \subset Y$ for which $B \subset F$ and $f^{-1}(F) \subset A$. It follows that $B = F$. But if $y \in F$ and $y \notin B$, then $y \in f(E)$. Therefore, $y = f(x)$ for some $x \in E$ and we have that $x \in f^{-1}(F) \subset A = E^c$, which is a contradiction. Since $B = F$, $f(E)$ is $b$-open and hence $f$ is contra pre-$b$-closed. ■

As a result of the Theorem 2.6, we have the following corollary.

**Corollary 2.1** If $f : (X, \tau) \to (Y, \sigma)$ is contra pre-$b$-closed, then for every $y \in Y$ and every $b$-open subset $A$ of $X$ with $f^{-1}(y) \subset A$, there exist a $b$-closed subset $F$ of $Y$ with $y \in F$ and $f^{-1}(F) \subset A$.

**Theorem 2.7** If a function $f : (X, \tau) \to (Y, \sigma)$ is weakly $b$-open, $BC(Y)$ is closed under unions and if for each $b$-closed subset $F$ of $X$ and each fiber $f^{-1}(y) \subset X \setminus F$ there exists an open subset $U$ of $X$ for which $F \subset U$ and $f^{-1}(y) \cap Cl(U) = \emptyset$, then $f$ is contra pre-$b$-closed.

**Proof** Let $F$ be a $b$-closed subset of $X$ and $y \in Y \setminus f(F)$. Thus $f^{-1}(y) \subset X \setminus F$ and hence there exists an open subset $U$ of $X$ for which $F \subset U$ and $f^{-1}(y) \cap Cl(U) = \emptyset$. Therefore, $y \in Y \setminus f(Cl(U)) \subset Y \setminus f(F)$. Since $f$ is weakly $b$-open, $f(U) \subseteq bInt(f(Cl(U)))$. Therefore, $y \in bCl(Y \setminus f(Cl(U))) \subset Y \setminus f(F)$. Put $B_y = bCl(Y \setminus f(Cl(U)))$. Then $B_y$ is a $b$-closed subset of $Y$ containing $y$. Hence $Y \setminus f(F) = \bigcup \{B_y : y \in Y \setminus f(F)\}$ is $b$-closed and hence $f(F)$ is $b$-open. ■

**Definition 2.8** A space $(X, \tau)$ is called $b$-$T_1$ [11] if for any pair of distinct points $x$ and $y$ in $X$, there exist a $b$-open set $U$ in $X$ containing $x$ but not $y$ and a $b$-open set $V$ in $X$ containing $y$ but not $x$.

**Remark 2.9** The converse of corollary 2.1 does not hold since for any function with $b$-$T_1$ codomain we may take $F$ to be $\{y\}$.

**Theorem 2.10** Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then,

1. If $f$ is contra pre-$b$-open, then $bCl(f(A)) \subset f(bCl(A))$ for every $b$-open subset $A$ of $X$.
2. If $f$ is contra pre-$b$-closed, then $f(A) \subset bInt(f(bCl(A)))$ for every subset $A$ of $X$.

**Proof** (1) Since $f$ is contra pre-$b$-open, then $bCl(f(A)) = f(A) \subset f(bCl(A))$ for every $A \in BO(X)$.

(2) Since $f$ is contra pre-$b$-closed and since $bCl(A)$ is $b$-closed, then $f(A) \subset f(bCl(A)) = bInt(f(bCl(A)))$ for every subset $A$ of $X$. ■

**Definition 2.11** A function $f : (X, \tau) \to (Y, \sigma)$ is called:

1. pre-$b$-closed if for every $b$-closed set $B$ of $X$, $f(B)$ is $b$-closed in $Y$;
2. pre-$b$-open if for every $b$-open set $B$ of $X$, $f(B)$ is $b$-open in $Y$;
3. $b$-preclosed if $bCl(bInt(f(B))) \subset f(B)$ for every $b$-closed set $B$ of $X$;
4. $b$-preopen if $f(B) \subset bInt(bCl(f(B)))$ for every $b$-open set $B$ of $X$.

**Theorem 2.12** For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties hold:

1. $f$ is pre-$b$-closed, whenever $f$ is contra pre-$b$-closed and $b$-preclosed;
2. $f$ is pre-$b$-open, whenever $f$ is contra pre-$b$-open and $b$-preopen.
Proof (1) Let \( F \) be a \( b \)-closed subset of \( X \). Since \( f \) is \( b \)-preclosed \( \text{bCl}(\text{bInt}(f(F))) \subset f(F) \). Since \( f(F) \) is \( b \)-open, \( \text{bCl}(\text{bInt}(f(F))) = \text{bCl}(f(F)) \subset f(F) \). Hence \( f(F) \) is \( b \)-closed.

(2) Let \( A \) be a \( b \)-open subset of \( X \). Since \( f \) is \( b \)-preopen, \( f(A) \subset \text{bInt}(\text{bCl}(f(A))) \). Since \( f(A) \) is \( b \)-closed, \( \text{bInt}(\text{bCl}(f(A))) = \text{bInt}(f(A)) \) and hence \( f(A) \subset \text{bInt}(f(A)) \), that is, \( f(A) = \text{bInt}(f(A)) \). Thus, \( f(A) \) is \( b \)-open.

Definition 2.13 A subset \( A \) of a space \((X, \tau)\) is said to be generalized \( b \)-closed \([12]\) (briefly, \( gb \)-closed) if \( \text{bCl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( b \)-open in \( X \).

Theorem 2.14 If a function \( f : (X, \tau) \to (Y, \sigma) \) is \( b \)-irresolute and contra \( b \)-closed, then the inverse image of every subset of \( Y \) is \( gb \)-closed in \( X \).

Proof Let \( A \) be subset of \( Y \). Suppose that \( f^{-1}(A) \subset U \) where \( U \in \text{BO}(X) \). Taking complements we obtain \( f(U^c) \subset A^c \). Since \( f \) is contra \( b \)-closed, \( f(U^c) \) is \( b \)-open. Then \( f(U^c) = \text{bInt}(f(U^c)) \subset \text{bInt}(A^c) = (\text{bCl}(A))^c \). Hence \( f^{-1}(\text{bCl}(A)) \subset U \). Since \( f \) is \( b \)-irresolute, \( f^{-1}(\text{bCl}(A)) \) is \( b \)-closed. Thus, we have \( \text{bCl}(f^{-1}(A)) \subset \text{bCl}(f^{-1}(\text{bCl}(A))) = f^{-1}(\text{bCl}(A)) \subset U \). This implies that \( f^{-1}(A) \) is \( gb \)-closed in \( X \).

Regarding the restriction \( f|_A \) of a function \( f : (X, \tau) \to (Y, \sigma) \) to a subset \( A \) of \( X \) we have the following.

Theorem 2.15 For a function \( f : (X, \tau) \to (Y, \sigma) \), the following statements are true

1. If \( f \) is contra \( b \)-closed and \( A \) is \( b \)-closed set in \( X \), then the function \( f|_A : (A, \tau_A) \to (Y, \sigma) \) is contra \( b \)-closed.
2. If \( f \) is contra \( b \)-open and \( A \) is \( b \)-open set of \( X \), then the function \( f|_A : (A, \tau_A) \to (Y, \sigma) \) is contra \( b \)-open.

Proof (1) Suppose \( B \) be an arbitrary \( b \)-closed subset of \( A \). Since \( A \) is \( b \)-closed subset of \( X \), then using Proposition 2.7 in [13], \( B \) is \( b \)-closed in \( X \). Then \( f|_A(B) = f(B) \) is \( b \)-open in \( Y \). Thus \( f|_A \) is contra \( b \)-closed.

(2): Similar to (1).

The proof of the following two theorems are straightforward and hence omitted.

Theorem 2.16 Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two functions such that \( g \circ f : (X, \tau) \to (Z, \eta) \).

1. \( g \circ f \) is contra \( b \)-open, if \( f \) is \( b \)-open and \( g \) is contra \( b \)-open;
2. \( g \circ f \) is contra \( b \)-open, if \( f \) is contra \( b \)-open and \( g \) is \( b \)-open.

Theorem 2.17 Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two functions such that \( g \circ f : (X, \tau) \to (Z, \eta) \).

1. \( g \circ f \) is contra \( b \)-closed, if \( f \) is \( b \)-closed and \( g \) is contra \( b \)-closed;
2. \( g \circ f \) is contra \( b \)-closed, if \( f \) is contra \( b \)-closed and \( g \) is \( b \)-open.

Lemma 2.18 \([1]\) Let \((X, \tau)\) be a topological space and let \( A \) be a subset of \( X \). Then \( x \in \text{bCl}(A) \) if and only if for every \( b \)-open set \( U \) of \( X \) containing \( x \), \( U \cap A \neq \emptyset \).

Definition 2.19 A subset \( A \) of \( X \) is said to be \( b \)-dense \([3]\) in \( X \) if and only if \( b \text{Cl}(A) \cap X \).

Theorem 2.20 For a function \( f : (X, \tau) \to (Y, \sigma) \), the following properties holds:

1. If \( f \) is contra \( b \)-open and \( B \subset Y \) has the property that \( B \) is not contained in proper \( b \)-open sets then \( f^{-1}(B) \) is \( b \)-dense \( X \);
2. If \( f \) is contra \( b \)-closed and \( A \) is a \( b \)-dense subset of \( Y \), then \( f^{-1}(A) \) is not contained in a proper \( b \)-open set.
Proof (1) Let \( x \in X \) and let \( A \in BO(X, x) \). Then \( f(A) \) is \( b \)-closed and \((f(A))^c \) is a proper \( b \)-open subset of \( Y \). Thus, \( B \not\subseteq (f(A))^c \) and hence there exists \( y \in B \) such that \( y \in f(A) \). Let \( z \in A \) for which \( y = f(z) \). Then \( z \in A \cap f^{-1}(B) \). Hence \( A \cap f^{-1}(B) \neq \emptyset \) and thus by Lemma 2.18, \( x \in bCl(f^{-1}(B)) \).
Then by definition, \( f^{-1}(B) \) is \( b \)-dense in \( X \).

(2) Assume that \( f^{-1}(A) \subseteq B \) where \( B \) is a proper \( b \)-open subset of \( X \). Then we have that \( f(B^c) \) is a nonempty \( b \)-open set of \( Y \) such that \( f(B^c) \cap A = \emptyset \), which by Lemma 2.18, contradicts the fact that \( A \) is \( b \)-dense.

3. Contra \( b \)-Irresolute Functions

Definition 3.1 A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be contra \( b \)-irresolute if for every \( V \in BO(Y, \sigma), f^{-1}(V) \in BC(X, \tau) \).

Remark 3.2 It is clear that the notions contra pre-\( b \)-openness (resp. contra pre-\( b \)-closedness) and contra \( b \)-irresoluteness are independent.

Theorem 3.3 Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be two functions; \( 1_X : (X, \tau) \to (X, \tau) \) be the identity function on \( (X, \tau) \). Then, for the composite \( g \circ f : (X, \tau) \to (Z, \eta) \) we have the following properties.

1. If \( f \) and \( g \) are \( b \)-irresolute, then \( g \circ f \) is \( b \)-irresolute.
2. The identity function \( 1_X : (X, \tau) \to (X, \tau) \) is \( b \)-irresolute.
3. If \( f \) and \( g \) are contra-\( b \)-irresolute, then \( g \circ f \) is \( b \)-irresolute.
4. If \( f \) is contra-\( b \)-irresolute and \( g \) is \( b \)-irresolute then \( g \circ f \) is \( b \)-irresolute.
5. If \( f \) is \( b \)-irresolute and \( g \) is contra-\( b \)-irresolute, then \( g \circ f \) is contra-\( b \)-irresolute.

Proof The proofs are obvious.

We shall construct some families of functions from a topological space onto itself; we construct a new group of functions.

Definition 3.4 A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be:

1. \( br \)-homeomorphism [12] if \( f \) is bijective \( b \)-irresolute and \( f^{-1} \) is also \( b \)-irresolute
2. contra \( br \)-homeomorphism if \( f \) is bijective contra-\( b \)-irresolute and \( f^{-1} \) is also contra-\( b \)-irresolute
3. \( b \)-homeomorphism [12] if \( f \) is bijective \( b \)-continuous and \( f^{-1} \) is also \( b \)-continuous.

We use the following notation on families of two functions above.

\[
brh(X, \tau) = \{ f | f : (X, \tau) \to (X, \tau) \text{ is a } br \text{-homeomorphism} \},
\]

\[
conbrh(X, \tau) = \{ f | f : (X, \tau) \to (X, \tau) \text{ is a contra-}br \text{-homeomorphism} \}.
\]

For the homeomorphism group \( h(X, \tau) \), we have \( h(X, \tau) \subseteq brh(X, \tau) \). Indeed, we recall

\[
h(X, \tau) = \{ a | a : (X, \tau) \to (X, \tau) \text{ is a homeomorphism} \}.
\]

for every homeomorphism \( f : (X, \tau) \to (Y, \sigma) \), every subset \( F \in BC(X, \tau) \) and for every subset \( V \in PC(X, \tau) \), it is shown that \( f^{-1}(F) \in BC(X, \tau) \) and \( f(V) \in PC(Y, \sigma) \) hold; \( f \) and \( f^{-1} \) are \( b \)-irresolute. Thus, we have \( f \in brh(X, \tau) \) holds for every \( f \in h(X, \tau) \) and hence \( h(X, \tau) \subseteq brh(X, \tau) \).

Let \( (X, \tau) \) be a topological space where \( X = \{ a, b, c \} \) and \( \tau = \{ \emptyset, \{ a \}, \{ b \}, \{ a, b \}, \{ a, c \}, X \} \). Then it is shown that \( h(X, \tau) = \{ 1_X \} = brh(X, \tau) \) and \( conbrh(X, \tau) = \{ h_b \} \), where \( 1_X : (X, \tau) \to (X, \tau) \) is the identity function and \( h_b : (X, \tau) \to (X, \tau) \) is a function defined by \( h_b(b) = \{ b \} \), \( h_b(a) = c \), \( h_b(c) = a \).
These properties show that \( brh(X, \tau) \) and \( brh(X, \tau) \cup contbrh(X, \tau) = \{1_X, h_b\} \) form groups under the composition of functions.

**Theorem 3.5** Let \((X, \tau)\) be a topological space.

1. The union of two families \( brh(X, \tau) \) and \( contbrh(X, \tau) \), that is \( brh(X, \tau) \cup contbrh(X, \tau) \) forms a group under the composition of functions.
2. The family \( brh(X, \tau) \) forms a subgroup of \( brh(X, \tau) \cup contbrh(X, \tau) \).
3. The group \( h(X, \tau) \) is a subgroup of \( brh(X, \tau) \) and \( h(X, \tau) \) is also a subgroup of \( brh(X, \tau) \cup contbrh(X, \tau) \).

**Proof** Set \( \mathcal{H}_X = brh(X, \tau) \cup contbrh(X, \tau) \) for a topological space \((X, \tau)\) throughout this proof.

(1) A binary operation \( \omega : \mathcal{H}_X \times \mathcal{H}_X \to \mathcal{H}_X \) is defined by \( \omega(a, b) = b \circ a \), where \( a, b \in \mathcal{H}_X \) and \( b \circ a \) denotes the composite of two functions \( a \) and \( b \) defined by \((b \circ a)(x) = b(a(x))\) for any \( x \in X \). By Theorem 3.3 (1) and (2), it is shown that \( \omega(a, b) = b \circ a \in \mathcal{H}_X \) for any \( a, b \in \mathcal{H}_X \). We observe that the axioms of group are satisfied. The identity function \( 1_X : (X, \tau) \to (X, \tau) \) is the identity element of the group \( \mathcal{H}_X \) by Theorem 3.3.

(2) Let \( a \in brh(X, \tau) \) and \( b \in brh(X, \tau) \). Then, it is shown that \( brh(X, \tau) \neq \emptyset \) because \( 1_X \in brh(X, \tau) \) and \( \omega(a, b^{-1}) = b^{-1} \circ a \in brh(X, \tau) \) by Theorem 3.3 (1) and definition 3.4(1), where \( \omega : \mathcal{H}_X \times \mathcal{H}_X \to \mathcal{H}_X \). Thus, \( brh(X, \tau) \) is a subgroup of \( \mathcal{H}_X = brh(X, \tau) \cup contbrh(X, \tau) \) under the binary operation \( \omega_X = \omega((brh(X, \tau) \times brh(X, \tau)) \).

(3) First recall that \( h(X, \tau) \subseteq brh(X, \tau) \) and \( h(X, \tau) \neq \emptyset \). Moreover, \( \omega_X (a, b^{-1}) = b^{-1} \circ a \in h(X, \tau) \) for any \( a, b \in h(X, \tau) \).

**Theorem 3.6** Let \((X, \tau), (Y, \sigma)\) and \((Z, \eta)\) be a topological spaces.

1. If \( f : (X, \tau) \to (Y, \sigma) \) is a \( br\)-homeomorphism (resp. contra-\( br\)-homeomorphism), then the function \( f \) induces an isomorphism \( f_* : brh(X, \tau) \cup contbrh(X, \tau) \to brh(Y, \sigma) \cup contbrh(Y, \sigma) \), where \( f_* \) is defined by \( f_*(a) = f \circ a \circ f^{-1} \) for any \( a \in brh(X, \tau) \cup contbrh(X, \tau) \).
2. \((g \circ f)_* = g_* \circ f_* : brh(X, \tau) \cup contbrh(X, \tau) \to brh(Y, \sigma) \cup contbrh(Y, \sigma) \) holds, where \( g : (Y, \sigma) \to (Z, \eta) \) is a \( bc \)-homeomorphism (respectively contra-\( bc \)-homeomorphism).
3. \((1_X)_* = 1 : brh(X, \tau) \cup contbrh(X, \tau) \to brh(X, \tau) \cup contbrh(X, \tau) \) is the identity isomorphism.
4. \( f_*(brh(X, \tau)) = brh(Y, \sigma), f_*(h(X, \tau)) \subseteq brh(Y, \sigma) \) and \( f_*(contbrh(X, \tau)) = contbrh(Y, \sigma) \) hold.
5. if a function \( f : (X, \tau) \to (Y, \sigma) \) is a homeomorphism and \( g : (Y, \sigma) \to (Z, \eta) \) is a homeomorphism, then the induced functions

\[ f_* : brh(X, \tau) \cup contbrh(X, \tau) \to brh(Y, \sigma) \cup contbrh(Y, \sigma) \]

and

\[ g_* : brh(Y, \sigma) \cup contbrh(Y, \sigma) \to brh(Z, \eta) \cup contbrh(Z, \eta) \]

are isomorphisms. Moreover, they have the same property of (2), (3) and (4) in above.

**Proof** Let \( \mathcal{H}_X = brh(X, \tau) \cup contbrh(X, \tau) \) for a topological space \((X, \tau)\) throughout this proof.

(1) By using Theorem 3.3, it is shown that the function \( f_* \) is well defined and \( f_* \mathcal{H}_X \to \mathcal{H}_Y \) is an isomorphism of groups.

(2) and (3) For an element \( a \in \mathcal{H}_X, (g \circ f)_*(a) = (g \circ f) \circ a \circ (g \circ f)^{-1} = g \circ (f \circ a \circ f^{-1}) \circ g^{-1} = g_* (f_* (a)) \) and \( f_*(1_X) = f \circ 1_X \circ f^{-1} = 1_X \) hold.

(4) Let \( a \in phe(X, \tau), b \in h(X, \tau) \) and \( c \in contbrh(X, \tau) \). Then, \( f_*(a) = f \circ a \circ f^{-1} \in brh(Y, \sigma) \) by Theorem 3.3 and so \( f_*(brh(X, \tau)) \subseteq brh(Y, \sigma) \). Conversely, for each element \( h \in brh(Y, \sigma) \) we take an element \( f^{-1} \circ h \circ f \in brh(X, \tau) \). Thus we have that \( h = f_* (f^{-1} \circ h \circ f) \in f_* (brh(X, \tau)) \) and so \( brh(Y, \sigma) \subseteq \text{...} \).
of the group \( \sigma \). Namely, we have that \( \text{brh}(Y, \sigma) = f_*(\text{brh}(X, \tau)) \). For the element \( b \in h(X, \tau), f_*(b) = f \circ b \circ f^{-1} \in \text{brh}(Y, \sigma) \) and so \( f_*(h(X, \tau)) \subseteq \text{brh}(Y, \sigma) \). For the element \( c \in \text{contrbrh}(Y, \sigma), f_*(c) = f \circ c \circ f^{-1} \in \text{contrbrh}(Y, \sigma) \) and so \( f_*(\text{contrbrh}(X, \tau)) \subseteq \text{contrbrh}(Y, \sigma) \). Conversely, for any \( h \in \text{contrbrh}(Y, \sigma) \) we take an element \( f^{-1} \circ h \circ f \in \text{contrbrh}(X, \tau) \) by Theorem 3.3 \( h = f_*(f^{-1} \circ h \circ f) \in f_*(\text{contrbrh}(X, \tau)) \). Namely, we have that \( \text{contrbrh}(Y, \sigma) = f_*(\text{contrbrh}(X, \tau)) \).

(5) The proof is obtained by an argument similar to that in the proof of (1).

Definition 3.7 For a topological space \((X, \tau)\) and subset \(H\) of \(X\), we define the following families of functions:

1. \( \text{brh}(X, X/H, \tau) = \{a | a \in \text{brh}(X, \tau) \text{ and } a(X/H) = X/H\} \)
2. \( \text{brh}_0(X, X/H, \tau) = \{a | a \in \text{brh}(X, X/H, \tau) \text{ and } a(x) = x \text{ for every } x \in X/H\} \).

Theorem 3.8 Let \( H \) be a subset of a topological space \((X, \tau)\).

1. The family \( \text{brh}(X, X/H, \tau) \) forms a subgroup of \( \text{brh}(X, \tau) \) and \( \text{brh}(X, X/H, \tau) = \text{brh}(X, H, \tau) \) holds.
2. The family \( \text{brh}_0(X, X/H, \tau) \) forms a subgroup of \( \text{brh}(X, X/H, \tau) \) and hence \( \text{brh}_0(X, X/H, \tau) \) forms a subgroup of \( \text{brh}(X, H, \tau) \).

Proof (1) It is shown that \( \text{brh}(X, X/H, \tau) \) is a nonempty subset of \( \text{brh}(X, \tau) \), because \( 1_X \in \text{brh}(X, X/H, \tau) \). Moreover, we have that \( \omega_X(a, b^{-1}) = b^{-1} \circ a \in \text{brh}(X, X/H, \tau) \) for any elements \( a, b \in \text{brh}(X, X/H, \tau), \) where \( \omega_X = \omega_{X/H}(\text{brh}(X, X/H, \tau) \times \text{brh}(X, X/H, \tau)) \). Since \( \omega \) is the binary operation of the group \( \text{brh}(X, \tau) \). Thus, \( \text{brh}(X, X/H, \tau) \) is a subgroup of \( \text{brh}(X, \tau) \). Evidently, the identity function \( 1_X \) is the identity element of \( \text{brh}(X, X/H, \tau) \).

(2) It is shown that \( \text{brh}_0(X, X/H, \tau) \) is a nonempty subset of \( \text{brh}(X, X/H, \tau) \), because \( 1_X \in \text{brh}_0(X, X/H, \tau) \). We have that \( \omega_X(a, b^{-1}) = b^{-1} \circ a \in \text{brh}_0(X, X/H, \tau) \) for any elements \( a, b \in \text{brh}_0(X, X/H, \tau), \) where \( \omega_X = \omega_{X/H}(\text{brh}_0(X, X/H, \tau) \times \text{brh}_0(X, X/H, \tau)) \) since \( \omega_X \) is the binary operation of the group \( \text{brh}(X, X/H, \tau) \). Thus, \( \text{brh}_0(X, X/H, \tau) \) is a subgroup of \( \text{brh}(X, X/H, \tau) \) and the identity function \( 1_X \) is the identity element of \( \text{brh}_0(X, X/H, \tau) \). By using (1) above, \( \text{brh}_0(X, X/H, \tau) \) is a subgroup of \( \text{brh}(X, \tau) \). ■

References