RESEARCH ARTICLE

On Decomposition of \((1, 2)^g\)-Continuity

O.Ravi *, S. Jeyashri† and M. Krishnamoorthy‡

*† Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai Dt., Tamil Nadu, India.
† Department of Mathematics, R. V. S. Engineering College, Dindigul, Tamil Nadu, India.
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In this paper, we introduce the concepts of generalized continuity called \((1, 2)^g\)-\(\hat{g}\)-continuity, \((1, 2)^g\)-closed sets and \((1, 2)^g\)-neighborhood in bitopological spaces. Also we investigate their properties and obtain a decomposition of continuity.

Keywords: \((1, 2)^g\)-closed set; \((1, 2)^g\)-continuous map; \((1, 2)^g\)-neighborhood and \((1, 2)^g\)-\(slc\)-continuous map.

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1. Introduction

Different types of generalizations of continuous maps were introduced and studied by various authors in the recent development of topology. The decomposition of continuity is one of the many problems in general topology. Tong [1] introduced the notions of \(A\)-sets and \(A\)-continuity and established a decomposition of continuity. Also Tong [2] introduced the notions of \(B\)-sets and \(B\)-continuity and used them to obtain another decomposition of continuity and Ganster and Reilly [3] have improved Tong’s decomposition result. Przemski [4] obtained some decompositions of continuity. Recently, Ravi et al. [5–7] obtained decompositions of continuity in bitopological spaces. In this paper, we obtain a decomposition of continuity in bitopological spaces using \((1, 2)^g\)-\(\hat{g}\)-continuity. We also obtain characterizations of \((1, 2)^g\)-\(\hat{g}\)-continuous maps.

2. Preliminaries

Through out this paper \((X, \tau_1,\tau_2), (Y, \sigma_1, \sigma_2)\) and \((Z, \eta_1, \eta_2)\) (or simply \(X, Y\) and \(Z\)) denote bitopological spaces.

Definition 2.1 [8] Let \(S\) be a subset of \(X\). Then \(S\) is said to be \(\tau_{1,2}\)-open if \(S = A \cup B\) where \(A \in \tau_1\) and \(B \in \tau_2\). The complement of \(\tau_{1,2}\)-open set is called \(\tau_{1,2}\)-closed.

Remark 2.2 [8] Notice that \(\tau_{1,2}\)-open subsets of \(X\) need not necessarily form a topology.

Definition 2.3 [8] Let \(S\) be a subset of \(X\). Then:

1. the \(\tau_{1,2}\)-closure of \(S\), denoted by \(\tau_{1,2}\)-\(Cl(S)\), is defined as

\[
\bigcap\{F : S \subseteq F \text{ and } F \text{ is } \tau_{1,2}\text{-closed}\},
\]
(2) the $\tau_{1,2}$-interior of $S$, denoted by $\tau_{1,2-}Int(S)$, is defined as
\[ \bigcup \{ F : F \subseteq S \text{ and } F \text{ is } \tau_{1,2}\text{-open} \}. \]

We recall the following definitions and results which we used in this paper.

**Definition 2.4** A subset $A$ of $X$ is called:

1. a $(1,2)^*$-generalized closed set (briefly $(1,2)^*g$-closed) [9] if $\tau_{1,2}\text{-}Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_{1,2}$-open in $X$,
2. a $(1,2)^*g$-open set [9] if its complement, $A^C$ is $(1,2)^*g$-closed,
3. a $(1,2)^*g$-semi-open set [8] if $A \subseteq \tau_{1,2}\text{-}Cl(\tau_{1,2} - Int(A))$.

**Definition 2.5** [10] A set $A \subseteq X$ is called $(1,2)^*\hat{g}$-closed if $\tau_{1,2}\text{-}Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $(1,2)^*g$-semi-open. The complement of $(1,2)^*\hat{g}$-closed set is called $(1,2)^*\hat{g}$-open.

**Proposition 2.6** Every $\tau_{1,2}$-closed set is $(1,2)^*\hat{g}$-closed but not conversely.

**Proof** Let $A$ be any $\tau_{1,2}$-closed set and $U$ be any $(1,2)^*g$-semi-open set such that $A \subseteq U$. Then $\tau_{1,2}\text{-}Cl(A) \subseteq U$, since $\tau_{1,2}\text{-}Cl(A) = A$ and hence $A$ is $(1,2)^*\hat{g}$-closed.

**Example 2.7** Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a, b\}\}$ and $\tau_2 = \{\emptyset, X, \{b, c\}\}$. Then the sets in $\{\emptyset, X, \{a, b\}, \{b, c\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{a\}, \{c\}\}$ are called $\tau_{1,2}$-closed. Here $A = \{a, c\}$ is $(1,2)^*\hat{g}$-closed set but not $\tau_{1,2}$-closed.

**Proposition 2.8** [10] Every $(1,2)^*\hat{g}$-closed set is $(1,2)^*g$-closed but not conversely.

**Proposition 2.9** [10] Every $\tau_{1,2}$-open set is $(1,2)^*g$-open but not conversely.

### 3. Some Properties of $(1,2)^*\hat{g}$-Continuous Maps

**Definition 3.1** A subset $A$ of $X$ is called $(1,2)^*g$-closed set if $\tau_{1,2}\text{-}Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $(1,2)^*g$-open in $X$.

**Definition 3.2** A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called:

1. $(1,2)^*g$-continuous [10] if $f^{-1}(V)$ is a $(1,2)^*g$-closed set of $X$ for every $\sigma_{1,2}$-closed set $V$ of $Y$,
2. $(1,2)^*g$-continuous if $f^{-1}(V)$ is a $(1,2)^*g$-closed set of $X$ for every $\sigma_{1,2}$-closed set $V$ of $Y$,
3. $(1,2)^*g$-continuous [8] if $f^{-1}(V)$ is a $\tau_{1,2}$-closed set of $X$ for every $\sigma_{1,2}$-closed set $V$ of $Y$.

**Definition 3.3** A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called $(1,2)^*\hat{g}$-continuous if $f^{-1}(F)$ is $(1,2)^*\hat{g}$-closed in $X$ for every $\sigma_{1,2}$-closed set $F$ in $Y$.

**Example 3.4** Let $X = Y = \{a, b, c, d\}$ with $\tau_1 = \{\emptyset, X, \{d\}, \{b, c, d\}\}$ and $\tau_2 = \{\emptyset, X, \{a, b, c\}\}$. Then the sets in $\{\emptyset, X, \{d\}, \{b, c, d\}, \{a, b, c\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{a\}, \{d\}, \{a, b, c\}\}$ are called $\tau_{1,2}$-closed. Let $f_1 = \emptyset, Y, \{a, b\}$ and $f_2 = \emptyset, Y, \{b, c, d\}$. Then the sets in $\{\emptyset, Y, \{a, b\}, \{b, c, d\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\emptyset, Y, \{a\}, \{d\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is $(1,2)^*g$-continuous, because every subset of $X$ is $(1,2)^*\hat{g}$-closed.

**Proposition 3.5** Every $(1,2)^*g$-continuous map is $(1,2)^*\hat{g}$-continuous but not conversely.

**Proof** The proof follows from Proposition 2.6.

**Example 3.6** Let $X = Y = \{a, b, c, d\}$ with $\tau_1 = \{\emptyset, X, \{d\}, \{b, c, d\}\}$ and $\tau_2 = \{\emptyset, X, \{a, b, c\}\}$. Then the sets in $\{\emptyset, X, \{d\}, \{b, c, d\}, \{a, b, c\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{a\}, \{d\}, \{a, b, c\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \emptyset, Y, \{b, d\}$ and $\sigma_2 = \emptyset, Y, \{a, b, c\}$. Then the sets in $\{\emptyset, Y, \{b, d\}, \{a, b, c\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\emptyset, Y, \{d\}, \{a, c\}\}$ are called $\sigma_{1,2}$-closed.
Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is $(1, 2)^{-\ast}\hat{g}$-continuous but not $(1, 2)^{\ast}$-continuous, because for the set $A = \{a, c\}$ in $Y$, $f^{-1}(A) = \{a, c\}$, which is not $\tau_{1,2}$-closed in $X$.

Proposition 3.7 Every $(1, 2)^{\ast}\hat{g}$-continuous map is $(1, 2)^{\ast}-g$-continuous but not conversely.

**Proof** The proof follows from Proposition 2.8. ■

Example 3.8 Let $X = Y = \{a, b, c, d\}$ with $\tau_1 = \{\emptyset, X, \{a\}, \{a, c\}\}$ and $\tau_2 = \{\emptyset, X, \{b, c, d\}\}$. Then the sets in $\{\emptyset, X, \{a\}, \{a, c\}, \{b, c, d\}\}$ are $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{a\}, \{b, d\}, \{b, c, d\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\emptyset, Y, \{a, b\}\}$ and $\sigma_2 = \{\emptyset, Y, \{b, c, d\}\}$. Then the sets in $\{\emptyset, Y, \{a, b\}, \{b, c, d\}\}$ are $\sigma_{1,2}$-open and the sets in $\{\emptyset, Y, \{a\}, \{d\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is $(1, 2)^{\ast}\hat{g}$-continuous but not $(1, 2)^{\ast}$-continuous, because for the set $A = \{d\}$ in $Y$, $f^{-1}(A) = \{d\}$, which is not $(1, 2)^{\ast}\hat{g}$-closed in $X$.

Remark 3.9 The following examples show that $(1, 2)^{\ast}\hat{g}$-continuity and $(1, 2)^{\ast}-g^{\ast}$-continuity are independent.

Example 3.10 Let $X = Y = \{a, b, c\}$ with $\tau_1 = \{\emptyset, X, \{c\}\}$ and $\tau_2 = \{\emptyset, X, \{a, b\}\}$. Then the sets in $\{\emptyset, X, \{c\}, \{a\}\}$ are $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{a\}, \{b\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\emptyset, Y, \{a\}\}$ and $\sigma_2 = \{\emptyset, Y, \{b\}\}$. Then the sets in $\{\emptyset, Y, \{a\}, \{b\}\}$ are $\sigma_{1,2}$-open and the sets in $\{\emptyset, Y, \{c\}, \{a, c\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is $(1, 2)^{\ast}\hat{g}$-continuous but not $(1, 2)^{\ast}-g^{\ast}$-continuous, because for the set $A = \{b, c\}$ in $Y$, $f^{-1}(A) = \{d\}$, which is not $g^{\ast}$-closed in $X$.

Example 3.11 Let $X = Y = \{a, b, c\}$ with $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{a, c\}\}$. Then the sets in $\{\emptyset, X, \{a\}, \{a, c\}\}$ are $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{b\}, \{b, c\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\emptyset, Y, \{a\}\}$ and $\sigma_2 = \{\emptyset, Y\}$. Then the sets in $\{\emptyset, Y, \{a\}, \{a, c\}\}$ are $\sigma_{1,2}$-open and the sets in $\{\emptyset, Y, \{c\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is $(1, 2)^{\ast}-g^{\ast}$-continuous but not $(1, 2)^{\ast}\hat{g}$-continuous, because for the set $A = \{a, b\}$ is $\tau_{1,2}$-open in $Y$, $f^{-1}(A) = \{a, b\}$, which is not $(1, 2)^{\ast}\hat{g}$-closed in $X$.

Remark 3.12 Every $(1, 2)^{\ast}$-continuous map is $(1, 2)^{\ast}-g^{\ast}$-continuous but not conversely.

**Proof** It follows from the fact that every $\tau_{1,2}$-closed set is $(1, 2)^{\ast}-g^{\ast}$-closed. ■

The following example supports that the converse of the above theorem is not true in general.

Example 3.13 In Example 3.6, $f$ is not $(1, 2)^{\ast}$-continuous. However $f$ is $(1, 2)^{\ast}-g^{\ast}$-continuous.

Remark 3.14 Every $(1, 2)^{\ast}-g^{\ast}$-continuous map is $(1, 2)^{\ast}$-continuous but not conversely.

**Proof** Using Proposition 2.9, Definition 2.4 and 3.1, we obtain this remark. ■

Example 3.15 Let $X = Y = \{a, b, c\}$ with $\tau_1 = \{\emptyset, X, \{c\}\}$ and $\tau_2 = \{\emptyset, X, \{a, b\}\}$. Then the sets in $\{\emptyset, X, \{c\}, \{a\}\}$ are $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{c\}, \{a\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_1 = \{\emptyset, Y, \{a, c\}\}$ and $\sigma_2 = \{\emptyset, Y\}$. Then the sets in $\{\emptyset, Y, \{a, c\}\}$ are $\sigma_{1,2}$-open and the sets in $\{\emptyset, Y, \{b\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be an identity map. Then $f$ is $(1, 2)^{\ast}-g^{\ast}$-continuous but not $(1, 2)^{\ast}\hat{g}$-continuous, because for the set $A = \{b\}$ in $Y$, $f^{-1}(A) = \{b\}$, which is not $(1, 2)^{\ast}-g^{\ast}$-closed in $X$. 
where $A \rightarrow B$ (resp. $A \nrightarrow B$) represents $A$ implies $B$ but not conversely (resp. $A$ and $B$ are independent).

Remark 3.17 The composition of two $(1,2)^*\hat{g}$-continuous maps need not be $(1,2)^*\hat{g}$-continuous and this is shown by the following example.

Example 3.18 Let $X = Y = Z = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{a, b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{b, c, d\}\}$. Then the sets in $\{\emptyset, X, \{a\}, \{a, b, c\}, \{b, c, d\}\}$ are called $\tau_1$-open and the sets in $\{\emptyset, X, \{a\}, \{a, b, c\}\}$ are called $\tau_1$-closed. Let $\sigma_1 = \{\emptyset, Y, \{a, c\}\}$ and $\sigma_2 = \{\emptyset, Y, \{b, c\}\}$. Then the sets in $\{\emptyset, Y, \{a\}, \{a, b, c\}\}$ are called $\sigma_1$-open and the sets in $\{\emptyset, Y, \{b\}, \{a, b, c\}\}$ are closed. Let $\eta_1 = \{\emptyset, Z, \{b\}\}$ and $\eta_2 = \{\emptyset, Z, \{a, b, c\}\}$. Then the sets in $\{\emptyset, Z, \{a\}, \{a, b, c\}\}$ are called $\eta_1$-open and the sets in $\{\emptyset, Z, \{d\}, \{b, c, d\}\}$ are called $\eta_1$-closed. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ be defined as $f(a) = g(a) = b$, $f(b) = g(b) = a$ and $f(c) = g(c) = c$, $f(d) = g(d) = d$. Then $f$ and $g$ are $(1,2)^*\hat{g}$-continuous but their composition $g \circ f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is not $(1,2)^*\hat{g}$-continuous, because $A = \{b, c, d\}$ is $\eta_1$-closed in $Z$ but $(g \circ f)^{-1}(A) \neq f^{-1}(g^{-1}(A)) = f^{-1}(g^{-1}(\{b, c, d\})) = \{b, c, d\}$, which is not $(1,2)^*\hat{g}$-closed in $X$.

Definition 3.19 Let $x$ be a point of $X$ and $W$ a subset of $X$. Then $W$ is called a $(1,2)^*\hat{g}$-neighborhood of $x$ in $X$ if there exists an $\tau_1$-open set $U$ of $X$ such that $x \in U \subseteq W$.

Definition 3.20 Let $x$ be a point of $X$ and $W$ a subset of $X$. Then $W$ is called an $(1,2)^*\hat{g}$-neighborhood of $x$ in $X$ if there exists an $(1,2)^*\hat{g}$-open set $U$ of $X$ such that $x \in U \subseteq W$.

Definition 3.21 For every set $E \subseteq X$, we define the $(1,2)^*\hat{g}$-closure of $E$ to be the intersection of all $(1,2)^*\hat{g}$-closed sets containing $E$. In symbols, $(1,2)^*\hat{g}$-$Cl(E) = \cap \{A : E \subseteq A \subseteq l\}$, where $l$ is the class of $(1,2)^*\hat{g}$-closed sets in $X$.

Result 3.22

1. For any subset $A$ of $X$, $A \subseteq (1,2)^*\hat{g}$-$Cl(A)$.
2. If $A$ is $(1,2)^*\hat{g}$-closed set, then $A = (1,2)^*\hat{g}$-$Cl(A)$.
3. If $A \subseteq B$, then $(1,2)^*\hat{g}$-$Cl(A) \subseteq (1,2)^*\hat{g}$-$Cl(B)$.

Proposition 3.23 Let $A$ be a subset of $X$. Then $x \in (1,2)^*\hat{g}$-$Cl(A)$ if and only if for any $(1,2)^*\hat{g}$-neighborhood $W_x$ of $x$ in $X$, $A \cap W_x \neq \emptyset$.

Proof (Necessity): Assume $x \in (1,2)^*\hat{g}$-$Cl(A)$. Suppose that there is a $(1,2)^*\hat{g}$-neighborhood $W$ of the point $x$ in $X$ such that $W \cap A = \emptyset$. Since $W$ is a $(1,2)^*\hat{g}$-neighborhood of $x$ in $X$, by Definition 3.20, there exists a $(1,2)^*\hat{g}$-open set $U_x$ such that $x \in U_x \subseteq W$. Therefore, we have $U_x \cap A = \emptyset$ and so $A \subseteq (U_x)^C$. Since $(U_x)^C$ is a $(1,2)^*\hat{g}$-closed set containing $A$, we have by Definition 3.21, $(1,2)^*\hat{g}$-$Cl(A) \subseteq (U_x)^C$ and therefore $x \in (1,2)^*\hat{g}$-$Cl(A)$, which is a contradiction.

(Sufficiency): Assume for each $(1,2)^*\hat{g}$-neighborhood $W_x$ of $x$ in $X$, $A \cap W_x \neq \emptyset$. Suppose that $x \notin (1,2)^*\hat{g}$-$Cl(A)$. Then by Definition 3.21, there exists a $(1,2)^*\hat{g}$-closed set $F$ of $X$ such that $A \subseteq F$ and $x \notin F$. Thus $x \notin F^C$ and $F^C$ is a $(1,2)^*\hat{g}$-open in $X$ and hence $F^C$ is a $(1,2)^*\hat{g}$-neighborhood of $x$ in $X$. But $A \cap F^C = \emptyset$, which is a contradiction. ■
Proposition 3.24 A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $(1, 2)^*\hat{g}$-continuous if and only if $f^{-1}(U)$ is $(1, 2)^*\hat{g}$-open in $X$ for every $\sigma_1, \sigma_2$-open set $U$ in $Y$.

Proof Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be $(1, 2)^*\hat{g}$-continuous and $U$ be a $\sigma_1, \sigma_2$-open set in $Y$. Then $U^C$ is $\sigma_1, \sigma_2$-closed in $Y$ and since $f$ is $(1, 2)^*\hat{g}$-continuous, $f^{-1}(U^C)$ is $(1, 2)^*\hat{g}$-closed in $X$. But $f^{-1}(U^C) = (f^{-1}(U))^C$ and so $f^{-1}(U)$ is $(1, 2)^*\hat{g}$-open in $X$.

Conversely, assume that $f^{-1}(U)$ is $(1, 2)^*\hat{g}$-open in $X$ for each $\sigma_1, \sigma_2$-open set $U$ in $Y$. Let $F$ be a $\sigma_1, \sigma_2$-closed set in $Y$. Then $F^C$ is $\sigma_1, \sigma_2$-open in $Y$ and by assumption, $f^{-1}(F^C)$ is $(1, 2)^*\hat{g}$-open in $X$. Since $f^{-1}(F^C) = (f^{-1}(F))^C$, we have $f^{-1}(F)$ is $(1, 2)^*\hat{g}$-closed in $X$ and so $f$ is $(1, 2)^*\hat{g}$-continuous.

\[\Box\]

Remark 3.25 The following example shows that:

1. the union of two $(1, 2)^*\hat{g}$-open sets need not be $(1, 2)^*\hat{g}$-open.
2. the intersection of two $(1, 2)^*\hat{g}$-closed sets need not be $(1, 2)^*\hat{g}$-closed.

Example 3.26 Let $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{b\}, \{a, b\}\}$ and $\tau_2 = \{\emptyset, X, \{d\}, \{c, d\}, \{a, c, d\}\}$. Then the sets in $\{\emptyset, X, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ are called $\tau_1, \tau_2$-open and the sets in $\{\emptyset, X, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$ are called $\tau_1, \tau_2$-closed:

1. Here $A = \{a, b\}$ and $B = \{b, c\}$ are $(1, 2)^*\hat{g}$-open sets. But $A \cup B = \{a, b, c\}$ is not $\hat{g}$-open set.
2. Here $A = \{a, d\}$ and $B = \{c, d\}$ are $(1, 2)^*\hat{g}$-closed sets. But $A \cap B = \{d\}$ is not $\hat{g}$-closed set.

Definition 3.27 A bitopological space $(X, \tau_1, \tau_2)$ equipped with the family of all $\tau_1, \tau_2$-open sets will be called DRT-space if $Int_{\tau_1}(S) = Int_{\tau_2}(S)$ for each $\tau_1, \tau_2$-closed subset $S$ of $X$.

Example 3.28 Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{a, b\}\}$. Then $(X, \tau_1, \tau_2)$ is DRT space.

Example 3.29 Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{b, c\}\}$. Then $(X, \tau_1, \tau_2)$ is not DRT space, since $Int_{\tau_1}(\{a\}) = \{a\} \neq \emptyset = Int_{\tau_2}(\{a\})$.

Remark 3.30 In a DRT space, union of arbitrary $(1, 2)^*\hat{g}$-open sets is $(1, 2)^*\hat{g}$-open.

Theorem 3.31 For a DRT space $X$, let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function. Then the following statements are equivalent:

1. The function $f$ is $(1, 2)^*\hat{g}$-continuous.
2. The inverse of each $\sigma_1, \sigma_2$-open set is $(1, 2)^*\hat{g}$-open.
3. For each point $x$ in $X$ and each $\sigma_1, \sigma_2$-open set $V$ in $Y$ with $f(x) \in V$, there is an $(1, 2)^*\hat{g}$-open set $U$ in $X$ such that $x \in U$, $f(U) \subseteq V$.
4. The inverse of each $\sigma_1, \sigma_2$-closed set is $(1, 2)^*\hat{g}$-closed.
5. For each $x$ in $X$, the inverse of every $(1, 2)^*\hat{g}$-neighborhood of $f(x)$ is $(1, 2)^*\hat{g}$-neighborhood of $x$.
6. For each $x$ in $X$ and each $(1, 2)^*\hat{g}$-neighborhood $N$ of $f(x)$, there is $(1, 2)^*\hat{g}$-neighborhood $W$ of $x$ such that $f(W) \subseteq N$.
7. For each subset $A$ of $X$, $f((1, 2)^*\hat{g}\text{-Cl}(A)) \subseteq \sigma_1, \sigma_2 - Cl(f(A))$.
8. For each subset $B$ of $Y$, $(1, 2)^*\hat{g}\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1, \sigma_2 - Cl(B))$.

Proof (1)$\Leftrightarrow$ (2): This follows from Proposition 3.24.

(1)$\Leftrightarrow$ (3): Suppose that (3) holds and let $V$ be a $\sigma_1, \sigma_2$-open set in $Y$ and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and thus there exists an $(1, 2)^*\hat{g}$-open set $U_x$ such that $x \in U_x$ and $f(U_x) \subseteq V$. Now, $x \in U_x \subseteq f^{-1}(V)$ and $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$. Then Remark 3.30, $f^{-1}(V)$ is $(1, 2)^*\hat{g}$-open in $(X, \tau)$ and therefore $f$ is $(1, 2)^*\hat{g}$-continuous.

Conversely, suppose that (1) holds and let $f(x) \in V$ where $V$ is $\sigma_1, \sigma_2$-open set in $Y$. Then $x \in f^{-1}(V)$ where $f^{-1}(V)$ is $(1, 2)^*\hat{g}$-open set in $X$, since $f$ is $(1, 2)^*\hat{g}$-continuous. Let $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subseteq V$. 

\[\Box\]
(2) \Leftrightarrow (4): This result follows from the fact that if \( A \) is a subset of \( Y \), then \( f^{-1}(A^C) = (f^{-1}(A))^C \).

(2) \Rightarrow (5): For \( x \) in \( X \), let \( N \) be a \((1, 2)^*\)-neighborhood of \( f(x) \). Then there exists an \( \sigma_{1,2} \)-open set \( U \) in \( Y \) such that \( f(x) \in U \subseteq N \). Consequently, \( f^{-1}(U) \) is an \((1, 2)^*\)-\( g \)-open set in \( X \) and \( x \in f^{-1}(U) \subseteq f^{-1}(N) \). Thus \( f^{-1}(N) \) is an \((1, 2)^*\)-\( g \)-neighborhood of \( x \).

(5) \Rightarrow (6): Let \( x \in X \) and let \( N \) be a \((1, 2)^*\)-neighborhood of \( f(x) \). Then by assumption, \( W = f^{-1}(N) \) is an \((1, 2)^*\)-\( g \)-neighborhood of \( x \) and \( f(W) = f(f^{-1}(N)) \subseteq N \).

(6) \Rightarrow (3): For \( x \) in \( X \), let \( V \) be a \( \tau_{1,2} \)-open set containing \( f(x) \). Then \( V \) is a \((1, 2)^*\)-neighborhood of \( f(x) \).

So by assumption, there exist \((1, 2)^*\)-\( g \)-neighborhood \( W \) of \( x \) such that \( f(W) \subseteq V \). Hence there exist \((1, 2)^*\)-\( g \)-open set \( U \) in \( X \) such that \( x \in U \subseteq W \) and so \( f(U) \subseteq f(W) \subseteq V \).

(7) \Leftrightarrow (4): Suppose that (4) holds and let \( A \) be a subset of \( X \). Since \( A \subseteq f^{-1}(f(A)) \), we have \( A \subseteq f^{-1}(\sigma_{1,2} - Cl(f(A))) \). Since \( \sigma_{1,2} - Cl(f(A)) \) is a \( \sigma_{1,2} \)-closed set in \( Y \), by assumption \( f^{-1}(\sigma_{1,2} - Cl(f(A))) \) is an \((1, 2)^*\)-\( g \)-closed set containing \( A \). Consequently, \((1, 2)^*\)-\( g \)-\( Cl(A) \subseteq f^{-1}(\sigma_{1,2} - Cl(f(A))) \). Thus \( f((1, 2)^*\)-\( g \)-\( Cl(A) \)) \subseteq f(f^{-1}(\sigma_{1,2} - Cl(f(A)))) \subseteq \sigma_{1,2} - Cl(f(A)) \).

Conversely, suppose that (7) holds for any subset \( A \) of \( X \). Let \( F \) be a \( \sigma_{1,2} \)-closed subset of \( Y \) and \( A = f^{-1}(F) \). Then by assumption, \( f((1, 2)^*\)-\( g \)-\( Cl(f^{-1}(F)) \)) \subseteq \sigma_{1,2} - Cl(f(f^{-1}(F))) \subseteq \sigma_{1,2} - Cl(F) \). i.e., \((1, 2)^*\)-\( g \)-\( Cl(f^{-1}(F)) \) \subseteq f^{-1}(F) \) and so \( f^{-1}(F) \) is \((1, 2)^*\)-\( g \)-closed.

(7) \Leftrightarrow (8): Suppose that (7) holds and \( B \) be any subset of \( Y \). Then replacing \( A \) by \( f^{-1}(B) \) in (7), we obtain \( f((1, 2)^*\)-\( g \)-\( Cl(f^{-1}(B)) \)) \subseteq \sigma_{1,2} - Cl(f(f^{-1}(B))) \subseteq \sigma_{1,2} - Cl(B) \). i.e., \((1, 2)^*\)-\( g \)-\( Cl(f^{-1}(B)) \) \subseteq f^{-1}(\sigma_{1,2} - Cl(f(A))) \). This completes the proof of the theorem.

### 4. Decomposition of \((1, 2)^*\)-Continuity.

In this section by using \((1, 2)^*\)-\( g \)-continuity we obtain a decomposition of continuity in bitopological spaces.

To obtain our decomposition of \((1, 2)^*\)-continuity, we first introduce the notions of \((1, 2)^*\)-\( slc \)-sets and \((1, 2)^*\)-\( slc \)-continuous mappings in bitopological spaces and we prove that a map is \((1, 2)^*\)-continuous if and only if it is both \((1, 2)^*\)-\( g \)-continuous and \((1, 2)^*\)-\( slc \)-continuous.

**Definition 4.1** A subset \( A \) of \( X \) is called \((1, 2)^*\)-semi-locally closed (briefly \((1, 2)^*\)-\( slc \)) if \( A = U \cap F \), where \( U \) is \((1, 2)^*\)-semi-open and \( F \) is \( \tau_{1,2} \)-semi-closed in \( X \).

**Definition 4.2** A subset \( A \) of \( X \) is called \((1, 2)^*\)-\( slc \)-set if \( A = U \cap F \) where \( U \) is \((1, 2)^*\)-semi-open and \( F \) is \( \tau_{1,2} \)-closed in \( X \).

**Example 4.3** Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{\emptyset, X, \{b, c, d\}\} \) and \( \tau_2 = \{\emptyset, X, \{a, c\}\} \). Then the sets in \( \{\emptyset, X, \{a, c\}\} \) are \( \tau_{1,2} \)-open and the sets in \( \{\emptyset, X, \{a\}, \{b, d\}\} \) are \( \tau_{1,2} \)-closed. Then the set \( A = \{a\} \) is a \( slc \)-set in \( X \).

**Remark 4.4** Every \( \tau_{1,2} \)-closed set is \((1, 2)^*\)-\( slc \)-set but not conversely. In Example 4.3, the set \( A = \{a, c\} \) is \((1, 2)^*\)-\( slc \)-set but it is not \( \tau_{1,2} \)-closed.

**Remark 4.5** \((1, 2)^*\)-\( g \)-closed sets and \((1, 2)^*\)-\( slc \)-sets are independent. In Example 4.3, the set \( A = \{a, b, d\} \) is \((1, 2)^*\)-\( g \)-closed set but it is not \((1, 2)^*\)-\( slc \)-set. The set \( A = \{a, b, c\} \) is \((1, 2)^*\)-\( slc \)-set but it is not \((1, 2)^*\)-\( g \)-closed.

**Proposition 4.6** Let \( X \) be a bitopological space. Then a subset \( A \) of \( X \) is \( \tau_{1,2} \)-closed if and only if it is both \((1, 2)^*\)-\( g \)-closed and \((1, 2)^*\)-\( slc \)-set.
Proof Necessity is trivial. To prove the sufficiency, assume that $A$ is both $(1,2)^{∗g}$-closed and $(1,2)^{∗\text{slc}^{∗}}$-set. Then $A = U \cap F$ where $U$ is $(1,2)^{∗}$-semi-open and $F$ is $\tau_{1,2}$-closed in $X$. Therefore, $A \subseteq U$ and $A \subseteq F$ and so by hypothesis, $\tau_{1,2}$-$\text{Cl}(A) \subseteq U$ and $\tau_{1,2}$-$\text{Cl}(A) \subseteq F$. Thus $\tau_{1,2}$-$\text{Cl}(A) \subseteq U \cap F = A$ and hence $\tau_{1,2}$-$\text{Cl}(A) = A$ i.e., $A$ is $\tau_{1,2}$-closed in $X$.

Definition 4.7 A mapping $f : (X, \tau_{1,2}) \rightarrow (Y, \sigma_{1,2})$ is said to be $(1,2)^{∗}$-$\text{slc}^{∗}$-continuous if for each $\sigma_{1,2}$ -closed set $F$ in $Y$, $f^{-1}(F)$ is a $(1,2)^{∗}$-$\text{slc}^{∗}$-set in $X$.

Example 4.8 Let $X = Y = \{a, b, c, d\}$ and $\tau_{1} = \{\emptyset, X, \{a, c\}\}$ and $\tau_{2} = \{\emptyset, X, \{b, c\}\}$. Then the sets in $\{\emptyset, X, \{a, c\}, \{b, c\}, \{a, d\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{a, b\}, \{b, d\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_{1} = \{\emptyset, Y, \{b\}\}$ and $\sigma_{2} = \{\emptyset, Y, \{d\}\}$. Then the sets in $\{\emptyset, Y, \{b\}, \{d\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\emptyset, Y, \{a, b\}, \{a, c\}, \{a, d\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_{1,2}) \rightarrow (Y, \sigma_{1,2})$ be an identity map. Then $f$ is $(1,2)^{∗}$-$\text{slc}^{∗}$-continuous but not $(1,2)^{∗g^{∗}}$-continuous.

Remark 4.9 From the Remark 4.4 it is clear that every $(1,2)^{∗}$-continuous map is $(1,2)^{∗}$-$\text{slc}^{∗}$-continuous. But the converse is not true. The map $f$ in Example 4.8. is $(1,2)^{∗}$-$\text{slc}^{∗}$-continuous but not $(1,2)^{∗}$-continuous, since for the $\sigma_{1,2}$-closed set $\{a, b, c\}$ in $Y$, $f^{-1}(\{a, b, c\}) = \{a, b, c\}$ which is not $\tau_{1,2}$-closed in $X$.

Remark 4.10 $(1,2)^{∗g}$-continuity and $(1,2)^{∗}$-$\text{slc}^{∗}$-continuity are independent.

The function $f$ in Example 4.8 is $(1,2)^{∗}$-$\text{slc}^{∗}$-continuous but not $(1,2)^{∗g}$-continuous. Let $X = Y = \{a, b, c, d\}$ and $\tau_{1} = \{\emptyset, X, \{a, c\}\}$ and $\tau_{2} = \{\emptyset, X, \{b, c\}\}$. Then the sets in $\{\emptyset, X, \{a, c\}, \{b, c\}, \{a, d\}\}$ are called $\tau_{1,2}$-open and the sets in $\{\emptyset, X, \{a, b\}, \{b, d\}\}$ are called $\tau_{1,2}$-closed. Let $\sigma_{1} = \{\emptyset, Y, \{c\}\}$ and $\sigma_{2} = \{\emptyset, Y\}$. Then the sets in $\{\emptyset, Y, \{c\}\}$ are called $\sigma_{1,2}$-open and the sets in $\{\emptyset, Y, \{a, b\}\}$ are called $\sigma_{1,2}$-closed. Let $f : (X, \tau_{1,2}) \rightarrow (Y, \sigma_{1,2})$ be the identity map. Then $f$ is not $(1,2)^{∗}$-$\text{slc}^{∗}$-continuous but it is $(1,2)^{∗g}$-continuous, because for the set $A = \{a, b, d\}$ in $X$, $f^{-1}(A) = \{a, b, d\}$, which is not $(1,2)^{∗}$-$\text{slc}^{∗}$-set in $X$.

We have the following decomposition for $(1,2)^{∗}$-continuity.

Theorem 4.11 A function $f : (X, \tau_{1,2}) \rightarrow (Y, \sigma_{1,2})$ is $(1,2)^{∗}$-continuous if and only if it is both $(1,2)^{∗g}$-continuous and $(1,2)^{∗}$-$\text{slc}^{∗}$-continuous.

Proof Assume that $f$ is $(1,2)^{∗}$-continuous. Then by Proposition 3.5 and Remark 4.9, $f$ is both $(1,2)^{∗g}$-continuous and $(1,2)^{∗}$-$\text{slc}^{∗}$-continuous. Conversely, assume that $f$ is both $(1,2)^{∗g}$-continuous and $(1,2)^{∗}$-$\text{slc}^{∗}$-continuous. Let $F$ be a $\sigma_{1,2}$ -closed subset of $Y$. Then $f^{-1}(F)$ is both $(1,2)^{∗g}$-closed and $(1,2)^{∗}$-$\text{slc}^{∗}$-set. As in Proposition 4.6, we prove that $f^{-1}(F)$ is a $\tau_{1,2}$-closed set in $X$ and so $f$ is $(1,2)^{∗}$-continuous.

References