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On New Types of Grill Sets and A Decomposition of Continuity

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In the present paper, we introduce the notions of $g^\alpha$-sets, $G^\alpha$-sets, $g^\alpha$-continuity and $G^\alpha$-continuity and investigate the relation between such sets (functions) with other grill sets (functions) and obtain a decomposition of continuity.

Keywords: Topological space; Grill; $\tau_G$-closed set; $G^\alpha$-open set and $G^\alpha$-continuous function

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1. Introduction

The idea of grill on a topological space was first introduced by Choquet [1] in 1947. It is observed from literature that the concept of grills is a powerful supporting tool, like nets and filters, in dealing with many a topological concept quite effectively. In [2], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. In [3], Ravi and Ganesan have defined and studied $G^\alpha$-open sets and $G^\alpha$-continuous functions in grill topological spaces. In this paper, we introduce the notions of $ga^\alpha$-sets, $Ga^\alpha$-sets, $ga^\alpha$-continuous functions and $Ga^\alpha$-continuous functions and investigate the relation between such sets (functions) with other grill sets (functions) and obtain a decomposition of continuity.

2. Preliminaries

Throughout this paper, $(X, \tau)$ (or $X$) represent a topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $X$, $Cl(A)$ and $Int(A)$ denote the closure of $A$ and the interior of $A$, respectively. The power set of $X$ will be denoted by $\wp(X)$. A collection $G$ of a nonempty subsets of a space $X$ is called a grill [1] on $X$ if

1. $A \in G$ and $A \subset B \Rightarrow B \in G$,
2. $A, B \subset X$ and $A \cup B \in G \Rightarrow A \in G$ or $B \in G$.

For any point $x$ of a topological space $(X, \tau)$, $\tau(x)$ denote the collection of all open neighborhoods of $x$.

We recall the following results which are useful in the sequel.

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Definition 2.1 [2] Let $(X, \tau)$ be a topological space and $G$ be a grill on $X$. The mapping $\Phi : \wp(X) \rightarrow \wp(X)$, denoted by $\Phi_G(A, \tau)$ for $A \in \wp(X)$ or simply $\Phi(A)$ called the operator associated with the grill $G$ and the topology $\tau$ and is defined by $\Phi_G(A) = \{ x \in X | A \cap U \in G, \forall U \in \tau(x) \}$.

Let $G$ be a grill on a space $X$. Then a map $\Psi : \wp(X) \rightarrow \wp(X)$ is defined by $\Psi(A) = A \cup \Phi(A)$, for all $A \in \wp(X)$. The map $\Psi$ satisfies Kuratowski closure axioms. Corresponding to a grill $G$ on a space $(X, \tau)$, there exists a unique topology $\tau_G$ on $X$ given by $\tau_G = \{ U \subset X | \Psi(X - U) = X - U \}$, where for any $A \subset X$, $\Psi(A) = A \cup \Phi(A) = \tau_G - Cl(A)$. For any grill $G$ on a topological space $(X, \tau)$, $\tau \subset \tau_G$. If $(X, \tau)$ is a topological space and $G$ is a grill on $X$, then we denote a grill topological space by $(X, \tau, G)$.

Theorem 2.2 [2]

1. If $G_1$ and $G_2$ are two grills on a space $X$ with $G_1 \subset G_2$, then $\tau_{G_1} \subset \tau_{G_2}$.
2. If $G$ is a grill on a space $X$ and $B \notin G$, then $B$ is closed in $(X, \tau, G)$.
3. For any subset $A$ of a space $X$ and any grill $G$ on $X$, $\Phi(A)$ is $\tau_G$-closed.

Theorem 2.3 [2] Let $(X, \tau)$ be a topological space and $G$ be any grill on $X$. Then

1. $A \subseteq B(\subseteq X) \Rightarrow \Phi(A) \subseteq \Phi(B)$;
2. $A \subseteq X$ and $A \notin G \Rightarrow \Phi(A) = \phi$;
3. $\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$, for any $A \subseteq X$;
4. $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$ for any $A, B \subseteq X$;
5. $A \subseteq \Phi(A) \Rightarrow Cl(A) = \tau_G - Cl(A) = Cl(\Phi(A)) = \Phi(A)$;
6. $U \in \tau$ and $\tau \setminus \{ \phi \} \subseteq G \Rightarrow U \subseteq \Phi(U)$;
7. If $U \in \tau$ then $U \cap \Phi(A) = U \cap \Phi(U) \cap A$, for any $A \subseteq X$.

Theorem 2.4 Let $(X, \tau)$ be a topological space and $G$ be any grill on $X$. Then, for any $A, B \subseteq X$,

1. $A \subseteq \Psi(A)$ [2];
2. $\Psi(\phi) = \phi$ [2];
3. $\Psi(A \cup B) = \Psi(A) \cup \Psi(B)$ [2];
4. $\Psi(\Psi(A)) = \Psi(A)$ [2];
5. $Int(\Psi(A)) \subset Int(\Psi(A))$;
6. $Int(\Psi(A \cap B)) \subset Int(\Psi(A))$;
7. $Int(\Psi(A \cap B)) \subset Int(\Psi(B))$;
8. $Int(\Psi(A)) \subset \Psi(A)$;
9. $A \subseteq B \Rightarrow \Psi(A) \subseteq \Psi(B)$.

Theorem 2.5 [3] Let $(X, \tau)$ be a topological space and $G$ be any grill on $X$. Then, for any $A, B \subseteq X$,

1. $\Phi(A) \subseteq \Psi(A) = \tau_G - Cl(A) \subseteq Cl(A)$;
2. $A \cup \Psi(\Psi(\Phi(A)) \subseteq Cl(A)$;
3. $A \subseteq \Phi(A)$ and $B \subseteq \Phi(B) \Rightarrow \Psi(A \cap B) \subseteq \Psi(A) \cap \Psi(B)$.

Definition 2.6 [4] A subset $A$ of a space $(X, \tau)$ is called an $\alpha^*$-set if $Int(Cl(\Phi(A))) = Int(\Phi(A))$.

Definition 2.7 [5] Let $(X, \tau)$ be a topological space and $G$ be a grill on $X$. A subset $A$ in $X$ is said to be:

1. $\Phi$-open set if $A \subseteq Int(\Phi(A))$;
2. $g$-set if $Int(\Psi(A)) = Int(A)$;
3. $g\Phi$-set if $Int(\Psi(A)) = Int(A)$.

Definition 2.8 [5] Let $(X, \tau)$ be a topological space and $G$ be a grill on $X$. A subset $A$ in $X$ is said to be $G$-preopen if $A \subseteq Int(\Psi(A))$.

Definition 2.9 [3] Let $(X, \tau)$ be a topological space and $G$ be a grill on $X$. A subset $A$ in $X$ is said to be $G$-$\alpha$-open set if $A \subseteq Int(\Psi(\Phi(A)))$. The complement of $G$-$\alpha$-open set is said to be $G$-$\alpha$-closed.
Remark 3.7

Definition 2.10 [5] Let \((X, \tau)\) be a topological space and \(G\) be a grill on \(X\). A subset \(A\) in \(X\) is said to be \(G\)-set (resp. \(G\Phi\)-set) if there is a \(M \in \tau\) and a \(g\)-set (resp. \(g\Phi\)-set) \(N\) in \((X, \tau, G)\) such that \(A = M \cap N\).

3. \(g\alpha^*\)-Sets and \(G\alpha^*\)-Sets

We introduce a new type of sets as follows:

Definition 3.1 Let \((X, \tau)\) be a topological space and \(G\) be a grill on \(X\). A subset \(A\) in \(X\) is said to be \(g\alpha^*\)-set if \(\text{Int}(\Phi(\text{Int}(A))) = \text{Int}(A)\).

Definition 3.2 Let \((X, \tau)\) be a topological space and \(G\) be a grill on \(X\). A subset \(A\) in \(X\) is said to be \(G\alpha^*\)-set if there is a \(M \in \tau\) and a \(g\alpha^*\)-set \(N\) in \((X, \tau, G)\) such that \(A = M \cap N\).

Example 3.3 Let \(X = \{a, b, c\}\) and \(\tau = \{\phi, X, \{a\}, \{b, c\}\}\). If \(G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}\), then \(G\) is a grill on \(X\) such that \(\tau - \{\phi\} \subset G\).

Example 3.4 Let \(X = \{a, b, c\}\) and \(\tau = \{\phi, X, \{a, b\}\}\). If \(G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}\), then \(G\) is a grill on \(X\) such that \(\tau - \{\phi\} \subset G\).

Example 3.5 Let \(X = \{a, b, c\}\) and \(\tau = \{\phi, X, \{a\}\}\). If \(G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}\), then \(G\) is a grill on \(X\) such that \(\tau - \{\phi\} \subset G\).

Example 3.6 Let \(X = \{a, b, c\}\) and \(\tau = \{\phi, X, \{a, b\}\}\). If \(G = \{X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}\), then \(G\) is a grill on \(X\) such that \(\tau - \{\phi\} \subset G\).

Remark 3.7

(1) Every open set is \(G\alpha^*\)-set but not conversely.
(2) Every \(g\alpha^*\)-set is \(G\alpha^*\)-set but not conversely.

In Example 3.4,

(1) Here \(\{a, c\}\) is \(G\alpha^*\)-set but not open set.
(2) Here \(\{a, b\}\) is \(G\alpha^*\)-set but not \(g\alpha^*\)-set.

Remark 3.8 [5]

(1) Every \(g\Phi\)-set is \(g\)-set but not conversely.
(2) Every \(g\Phi\)-set is \(G\Phi\)-set but not conversely.
(3) Every \(g\)-set is \(G\)-set but not conversely.
(4) Every \(G\)-set is \(G\Phi\)-set but not conversely.
(5) Every open set is \(G\)-set but not conversely.
(6) Every open set is \(G\Phi\)-set but not conversely.

Proposition 3.9 [5] Every open set is \(G\)-preopen set.

Remark 3.10 The converse of Proposition 3.9 is not true.

In Example 3.3, \(\{a, b\}\) is \(G\)-preopen set but not open set.

Proposition 3.11 [3] Every open set is \(G\alpha\)-open set but not conversely.

Proposition 3.12 Every closed set is a \(g\)-set.

Proof Let \(A\) be a closed set. Then \(\Psi(A) = A \cup \Phi(A) \subseteq A \cup \text{Cl}(A)\) (by Theorem 2.3 (3)) and \(\Psi(A) \subseteq \text{Cl}(A) = A\). We have \(\Psi(A) \subseteq A\). But \(A \subseteq \Psi(A)\). Therefore \(\Psi(A) = A\) implies \(\text{Int}(\Psi(A)) = \text{Int}(A)\). Thus \(A\) is a \(g\)-set.

Remark 3.13 The converse of Proposition 3.12 is not true.

In Example 3.3, \(\{a, c\}\) is \(g\)-set but not closed.
Remark 3.14 $G$-$\alpha$-open sets and $g\alpha^*$-sets are independent of each other.

Example 3.15 In Example 3.5, the set $\{b,c\}$ is an $g\alpha^*$-set but it is not a $G$-$\alpha$-open. Since every open set is an $G$-$\alpha$-open set, $\{a\}$ is $G$-$\alpha$-open set but it is not a $g\alpha^*$-set.

Proposition 3.16 Any $\tau_G$-closed set is an $g\alpha^*$-set($G\alpha^*$-set).

Proof Let $A$ be a subset in $(X,\tau,G)$. Then $\Phi(A)$ is $\tau_G$-closed. Now $\text{Int}(\Phi(A)) \subseteq \Phi(A)$ implies $\Psi(\text{Int}(\Phi(A))) \subseteq \Psi(\Phi(A))$. Then $\text{Int}(\Psi(\text{Int}(\Phi(A)))) \subseteq \text{Int}(\Phi(A)) = \text{Int}(\Phi(A))$ (since every $\tau_G$-closed is equivalent to $g$-set [5]). On other hand, $\text{Int}(\Phi(A)) \subseteq \Psi(\text{Int}(\Phi(A)))$ implies $\text{Int}(\Phi(A)) \subseteq \text{Int}(\Psi(\Phi(A)))$. Therefore $\text{Int}(\Psi(\Phi(A))) = \text{Int}(\Phi(A))$. Thus $\Phi(A)$ is $g\alpha^*$-set. Also, by Remark 3.7, $\Phi(A)$ is $G\alpha^*$-set.

Remark 3.17 The converse of Proposition 3.16 is not true.

In Example 3.5, $\{a\}$ is $G\alpha^*$-set but not $\tau_G$-closed.

In Example 3.3, $\{b\}$ is $g\alpha^*$-set but not $\tau_G$-closed.

Proposition 3.18 Every $g\Phi$-set is $g\alpha^*$-set.

Proof Since $\Phi(\text{Int}(A)) \subseteq \Phi(A)$, $\text{Int}(\Phi(\text{Int}(A))) \subseteq \text{Int}(\Phi(A)) = \text{Int}(A)$. We have $\text{Int}(\Phi(\text{Int}(A))) \subseteq \text{Int}(A)$. Now $\text{Int}(\Psi(\Phi(\text{Int}(A)))) = \text{Int}(\text{Int}(A) \cup \Phi(\text{Int}(A))) \supseteq \text{Int}(\text{Int}(A)) \cup \text{Int}(\Phi(\text{Int}(A))) = \text{Int}(A)$. Hence $A$ is $g\alpha^*$-set.

Remark 3.19 The converse of Proposition 3.18 is not true.

In Example 3.3, $\{a,b\}$ is $g\alpha^*$-set but not $g\Phi$-set.

Remark 3.20 $G\Phi$-sets and $g\alpha^*$-sets are independent of each other.

In Example 3.5, $\{a\}$ is $G\Phi$-set but not $g\alpha^*$-set.

In Example 3.4, $\{a\}$ is $g\alpha^*$-set but not $G\Phi$-set.

Proposition 3.21 Every $\alpha^*$-set is a $g\alpha^*$-set.

Proof $\Psi(\text{Int}(A)) = \text{Int}(A) \cup \Phi(\text{Int}(A)) \subset \text{Int}(A) \cup \text{Cl}(\text{Int}(A)) = \text{Cl}(\text{Int}(A))$. Then $\text{Int}(\Psi(\text{Int}(A))) \subseteq \text{Int}(\text{Cl}(\text{Int}(A))) = \text{Int}(A)$. On other hand, $\text{Int}(A) \subseteq \Psi(\text{Int}(A))$ implies $\text{Int}(A) \subseteq \text{Int}(\Psi(\text{Int}(A)))$. Therefore $\text{Int}(\Psi(\Phi(\text{Int}(A)))) = \text{Int}(A)$. Thus $A$ is $g\alpha^*$-set.

Proposition 3.22 If $A, B$ are two $g\alpha^*$-sets then $A \cap B$ is a $g\alpha^*$-set.

Proof $\text{Int}(A \cap B) \subset \text{Int}(\Psi(\text{Int}(A \cap B))) = \text{Int}(\Psi(\text{Int}(A) \cap \text{Int}(B))) = \text{Int}(\Psi(\text{Int}(A) \cap \text{Int}(B))) \cap \Psi(\text{Int}(A) \cap \text{Int}(B)) = \text{Int}(\Psi(\text{Int}(A) \cap \text{Int}(B))) \cap \text{Int}(\Psi(\text{Int}(A) \cap \text{Int}(B))) \subset \text{Int}(\Psi(\text{Int}(A))) \cap \text{Int}(\Psi(\text{Int}(B))) = \text{Int}(A) \cap \text{Int}(B) = \text{Int}(A \cap B)$. Then $A \cap B$ is $g\alpha^*$-set.

Remark 3.23 The union of two $g\alpha^*$-sets need not be a $g\alpha^*$-set.

Example 3.24 In Example 3.6, the set $\{a\}$ and the set $\{b\}$ are $g\alpha^*$-sets, but $\{a\} \cup \{b\} = \{a,b\}$ is not a $g\alpha^*$-set.

Proposition 3.25 Every $G\Phi$-set is $G\alpha^*$-set.

Proof It follows from Definitions and Proposition 3.18.

Remark 3.26 The converse of Proposition 3.25 is not true.

In Example 3.3, $\{b\}$ is $G\alpha^*$-set but not $G\Phi$-set.

Proposition 3.27 [5] A subset $A$ in a space $(X,\tau,G)$ is open if and only if it is a $G$-preopen and a $G$-set.

Proposition 3.28 Every $G$-set is $G\alpha^*$-set.

Proof It follows from Remark 3.8 (4) and Proposition 3.25.

Remark 3.29 The converse of Proposition 3.28 is not true.
In Example 3.3, \( \{a, b\} \) is \( G\alpha^* \)-set but not \( G\)-set.

Proposition 3.30 [3] Every \(G\)-\(\alpha\)-open set is \(G\)-preopen set but not conversely.

Remark 3.31 \( g\) sets and \(G\)-preopen sets are independent of each other.

In Example 3.5,
(1) \( \{a, b\} \) is \( G\)-preopen set but not \( g\)-set.
(2) \( \{c\} \) is \( g\)-set but not \( G\)-preopen set.

Remark 3.32 [5] \( G\)-sets and \( G\)-preopen sets are independent of each other.

Proposition 3.33 Every \( g\)-set is \( G\Phi\)-set.

Proof It follows from Remark 3.8 (3) and (4).

Remark 3.34 The converse of Proposition 3.33 is not true.

In Example 3.5, \( \{a\} \) is \( G\Phi\)-set but not \( g\)-set.

Remark 3.35 \( G\alpha^*\)-sets and \( G\)-\(\alpha\)-open are independent of each other.

In Example 3.5,
(1) \( \{a, b\} \) is \( G\)-\(\alpha\)-open but not \( G\alpha^*\)-set.
(2) \( \{b, c\} \) is \( G\alpha^*\)-set but not \( G\)-\(\alpha\)-open set.

Proposition 3.36 A subset \( S \) in a space \((X, \tau, G)\) is open if and only if it is an \(G\)-\(\alpha\)-open set and a \(G\alpha^*\)-set.

Proof Necessity: It follows from Remark 3.7 and Proposition 3.11.

Sufficiency: Since \( S \) is a \(G\alpha^*\)-set, then \( S = U \cap A \) where \( U \) is an open set and \( \text{Int}(\text{Int}(A)) = \text{Int}(A) \). Since \( S \) is also \(G\)-\(\alpha\)-open set, we have \( S \subset \text{Int}(\text{Int}(S)) = \text{Int}(\text{Int}(U \cap A)) = \text{Int}(\text{Int}(U) \cap \text{Int}(A)) \). Therefore \( S \) is an open set.

Remark 3.37 From the previous Propositions, Examples and Remarks, we obtain the following diagram, where \( A \rightarrow B \) (resp. \( A \not\rightarrow B \)) represents \( A \) implies \( B \) but not conversely (resp. \( A \) and \( B \) are independent of each other).

Diagram I
4. Decomposition of Continuity

Definition 4.1 [5] A function $f: (X, \tau, G) \to (Y, \sigma)$ is said to be $G$-continuous (resp. $G\Phi$-continuous, $G$-precontinuous) if for each open set in $Y$, $f^{-1}(V)$ is $G$-set (resp. $G\Phi$-set, $G$-preopen).

We introduce a new class of mappings as follows:

Definition 4.2 A function $f: (X, \tau, G) \to (Y, \sigma)$ is said to be $g$-continuous (resp. $g\Phi$-continuous) if for each open set in $Y$, $f^{-1}(V)$ is $g$-set (resp. $g\Phi$-set).

Definition 4.3 A function $f: (X, \tau, G) \to (Y, \sigma)$ is said to be $Ga^*$-continuous (resp. $ga^*$-continuous) if for each open set in $Y$, $f^{-1}(V)$ is $Ga^*$-set (resp. $ga^*$-set).

Proposition 4.4 Every $ga^*$-continuous function is $Ga^*$-continuous but not conversely.

Proof This is an immediate consequence of Remark 3.7.

Example 4.5 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\emptyset\} \subset G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f : (X, \tau, G) \to (Y, \sigma)$ be defined as follows $f(a) = f(b) = a$ and $f(c) = b$. The inverse image of the open set $\{a\}$ is $\{a, b\}$ which is $Ga^*$-set but it is not a $ga^*$-set.

Proposition 4.6 Every $G$-continuous function is $Ga^*$-continuous but not conversely.

Proof This is an immediate consequence of Proposition 3.28.

Example 4.7 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\emptyset\} \subset G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f : (X, \tau, G) \to (Y, \sigma)$ be defined as follows $f(a) = f(c) = a$ and $f(b) = b$. The inverse image of the open set $\{b\}$ which is $Ga^*$-set but it is not a $G$-set.

Remark 4.8 $G\Phi$-continuity and $ga^*$-continuity are independent as per the following examples.

Example 4.9

1. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\emptyset\} \subset G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f : (X, \tau, G) \to (Y, \sigma)$ be defined as follows $f(a) = a, f(b) = f(c) = b$. The inverse image of the open set $\{a\}$ is $\{a\}$ which is $Ga^*$-set but it is not a $G\Phi$-set.

2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\emptyset\} \subset G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f : (X, \tau, G) \to (Y, \sigma)$ be defined as follows $f(a) = a, f(b) = f(c) = b$. The inverse image of the open set $\{a\}$ is $\{a\}$ which is $G\Phi$-set but it is not a $ga^*$-set.

Proposition 4.10 Every $g\Phi$-continuous function is $ga^*$-continuous but not conversely.

Proof This is an immediate consequence of Proposition 3.18.

Example 4.11 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\emptyset\} \subset G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\emptyset, Y, \{b\}\}$. Let $f : (X, \tau, G) \to (Y, \sigma)$ be defined as follows $f(a) = f(c) = a$ and $f(b) = b$. The inverse image of the open set $\{b\}$ is $\{b\}$ which is $Ga^*$-set but it is not a $G\Phi$-set.

Proposition 4.12 Every $G\Phi$-continuous function is $Ga^*$-continuous but not conversely.

Proof This is an immediate consequence of Proposition 3.25.

Example 4.13 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\emptyset\} \subset G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\emptyset, Y, \{b\}\}$. Let $f : (X, \tau, G) \to (Y, \sigma)$ be defined as follows $f(a) = b = f(c)$ and $f(b) = b$. The inverse image of the open set $\{b\}$ is $\{b\}$ which is $Ga^*$-set but it is not a $G\Phi$-set.

Remark 4.14 $G$-precontinuity and $G$-continuity are independent of each other.
Remark 4.15 [5] Every $G$-continuous function is $G\Phi$-continuous but not conversely.

Proposition 4.16 Every $g\Phi$-continuous function is $g$-continuous but not conversely.

Proof This is an immediate consequence of Remark 3.8 (1).

Example 4.17 Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. If $G = \{X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\phi\} \subseteq G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau, G) \rightarrow (Y, \sigma)$ be defined as follows $f(a) = a$, $f(b) = c$ and $f(c) = b$. The inverse image of the open set $\{a\}$ is $\{a\}$ which is $g$-set but it is not a $g\Phi$-set.

Proposition 4.18 Every $g\Phi$-continuous function is $G$-continuous but not conversely.

Proof This is an immediate consequence of Remark 3.8 (2).

Example 4.19 Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a, b\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\phi\} \subseteq G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau, G) \rightarrow (Y, \sigma)$ be defined as follows $f(a) = f(b) = a$ and $f(c) = b$. The inverse image of the open set $\{a\}$ is $\{a, b\}$ which is $G\Phi$-set but it is not a $g\Phi$-set.

Proposition 4.20 Every $g$-continuous function is $G$-continuous but not conversely.

Proof This is an immediate consequence of Remark 3.8 (3).

Example 4.21 Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\phi\} \subseteq G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau, G) \rightarrow (Y, \sigma)$ be defined as follows $f(a) = f(b) = a$ and $f(c) = b$. The inverse image of the open set $\{a\}$ is $\{a\}$ which is $G$-set but it is not a $g$-set.

Proposition 4.22 Every continuous function is $G$-continuous but not conversely.

Proof This is an immediate consequence of Remark 3.8 (5).

Example 4.23 Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\phi\} \subseteq G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau, G) \rightarrow (Y, \sigma)$ be defined as follows $f(a) = f(c) = b$ and $f(b) = a$. The inverse image of the open set $\{a\}$ is $\{c\}$ which is $G$-set but it is not an open set.

Proposition 4.24 Every continuous function is $G$-precontinuous but not conversely.

Proof This is an immediate consequence of Proposition 3.9.

Example 4.25 Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\phi\} \subseteq G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau, G) \rightarrow (Y, \sigma)$ be defined as follows $f(a) = f(c) = b$ and $f(b) = a$. The inverse image of the open set $\{a\}$ is $\{b\}$ which is $G$-preopen set but it is not an open set.

Proposition 4.26 Every $g$-continuous function is $G\Phi$-continuous but not conversely.

Proof This is an immediate consequence of Proposition 3.33.

Example 4.27 Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ then $G$ is a grill on $X$ such that $\tau - \{\phi\} \subseteq G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau, G) \rightarrow (Y, \sigma)$ be defined as follows $f(a) = a$, $f(b) = c$ and $f(c) = b$. The inverse image of the open set $\{a\}$ is $\{a\}$ which is $G\Phi$-set but it is not a $g\Phi$-set.

Remark 4.28 $g$-continuity and $G$-precontinuity are independent as per the following examples.

Example 4.29 Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}\}$. If $G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, then $G$ is a grill on $X$ such that $\tau - \{\phi\} \subseteq G$. Let $Y = \{a, b\}$ with topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau, G) \rightarrow (Y, \sigma)$ be defined as follows $f(a) = a = f(b)$ and $f(c) = b$. The inverse image of the open set $\{a\}$ is $\{a, b\}$ which is $G$-preopen set but it is not a $g$-set. Moreover, $f : (X, \tau, G) \rightarrow (Y, \sigma)$ be
defined as follows \( f(a) = b = f(b) \) and \( f(c) = a \). The inverse image of the open set \( \{a\} \) is \( \{c\} \) which is \( g \)-set but it is not a \( G \)-preopen set.

**Remark 4.30** \( G\alpha^* \)-continuity and \( G\alpha \)-continuity are independent as per the following examples.

**Example 4.31** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a\}\} \). If \( G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} \), then \( G \) is a grill on \( X \) such that \( \tau - \{\phi\} \subset G \). Let \( Y = \{a, b\} \) with topology \( \sigma = \{\phi, Y, \{a\}\} \). Let \( f : (X, \tau, G) \to (Y, \sigma) \) be defined as follows \( f(a) = a = f(b) \) and \( f(c) = b \). The inverse image of the open set \( \{a\} \) is \( \{a, b\} \) which is \( G\alpha \)-open set but it is not a \( G\alpha^* \)-set.

**Example 4.32** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a\}\} \). If \( G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} \), then \( G \) is a grill on \( X \) such that \( \tau - \{\phi\} \subset G \). Let \( Y = \{a, b\} \) with topology \( \sigma = \{\phi, Y, \{a\}\} \). Let \( f : (X, \tau, G) \to (Y, \sigma) \) be defined as follows \( f(a) = b, f(b) = f(c) = a \). The inverse image of the open set \( \{a, b\} \) which is \( G\alpha^* \)-set but it is not a \( G\alpha \)-open set.

**Proposition 4.33** Every \( G\alpha \)-continuous function is \( G \)-precontinuous but not conversely.

**Proof** This is an immediate consequence of Proposition 3.30.

**Example 4.34** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a\}, \{b, c\}\} \). If \( G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} \) then \( G \) is a grill on \( X \) such that \( \tau - \{\phi\} \subset G \). Let \( Y = \{a, b\} \) with topology \( \sigma = \{\phi, Y, \{a\}\} \). Let \( f : (X, \tau, G) \to (Y, \sigma) \) be defined as follows \( f(a) = f(b) = b, f(c) = a \). The inverse image of the open set \( \{a, b\} \) which is \( G\alpha^* \)-set but it is not a \( G\alpha \)-open set.

**Proposition 4.35** Every continuous function is \( G\Phi \)-continuous but not conversely.

**Proof** This is an immediate consequence of Remark 3.8 (4).

**Example 4.36** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a\}, \{b, c\}\} \). If \( G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} \) then \( G \) is a grill on \( X \) such that \( \tau - \{\phi\} \subset G \). Let \( Y = \{a, b\} \) with topology \( \sigma = \{\phi, Y, \{a\}\} \). Let \( f : (X, \tau, G) \to (Y, \sigma) \) be defined as follows \( f(a) = f(c) = a \) and \( f(b) = b \). Then inverse image of the open set \( \{a\} \) is \( \{a, c\} \) which is \( G\Phi \)-set but it is not an open set.

**Proposition 4.37** Every continuous function is \( G\alpha \)-continuous but not conversely.

**Proof** This is an immediate consequence of Proposition 3.11.

**Example 4.38** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a\}\} \). If \( G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} \) then \( G \) is a grill on \( X \) such that \( \tau - \{\phi\} \subset G \). Let \( Y = \{a, b\} \) with topology \( \sigma = \{\phi, Y, \{a\}\} \). Let \( f : (X, \tau, G) \to (Y, \sigma) \) be defined as follows \( f(a) = f(b) = a \) and \( f(c) = b \). Then inverse image of the open set \( \{a\} \) is \( \{a, b\} \) which is \( G\alpha \)-open set but it is not an open set.

**Proposition 4.39** Every continuous function is \( G\alpha^* \)-continuous but not conversely.

**Proof** This is an immediate consequence of Remark 3.7 (1).

**Example 4.40** Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a\}, \{b, c\}\} \). If \( G = \{X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} \) then \( G \) is a grill on \( X \) such that \( \tau - \{\phi\} \subset G \). Let \( Y = \{a, b\} \) with topology \( \sigma = \{\phi, Y, \{a\}\} \). Let \( f : (X, \tau, G) \to (Y, \sigma) \) be defined as follows \( f(a) = f(b) = a \) and \( f(c) = b \). Then inverse image of the open set \( \{a\} \) is \( \{a, b\} \) which is \( G\alpha^* \)-set but it is not an open set.

**Remark 4.41** The following diagram shows the relationships established between continuity and other maps where \( A \to B \) (resp. \( A \not\to B \)) represents \( A \) implies \( B \) but not conversely (resp. \( A \) and \( B \) are independent of each other).
We have the following decomposition of continuity inspired by Proposition 3.36.

Theorem 4.42 A function \( f : (X, \tau, G) \rightarrow (Y, \sigma) \) is continuous if and only if it is a \( G-\alpha \)-continuous and a \( G\alpha^* \)-continuous.

Proof This is an immediate consequence of Proposition 3.36. \qed

References

3. O. Ravi and S. Ganesan. On \( g-\alpha \)-open sets and \( g-\alpha \)-continuous functions. Submitted.