RESEARCH ARTICLE

Regular Weakly $\mu$-Closed Sets, Almost Weakly $\mu$-Regular, Almost Weakly $\mu$-Normal and Mildly Weakly $\mu$-Normal in Topological Spaces

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In this paper, a new kind of sets called regular weakly $\mu$-closed (briefly, rw$\mu$-closed) sets are introduced and studied in topological spaces. Some of their properties are investigated. Finally, some characterizations of almost rw$\mu$-regular, almost rw$\mu$-normal and mildly rw$\mu$-normal spaces have been given.

Keywords: $w\mu$-closed, rw$\mu$-closed sets, rw$\mu$-$T_{\frac{1}{2}}$ space, almost $w\mu$-regular space, almost $w\mu$-normal space, mildly $w\mu$-normal space.

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1. Introduction

In 1970, the concept of generalized closed set in a topological space was introduced by N. Levine [1] in order to extend many of the important properties of closed sets to a larger family. After that, the concept of generalized closed set has been investigated by many mathematicians because the notion of generalized closed set is a natural generalization of closed set (see [1], [2], [3] for details). It is also well known that separation axioms are one of the basic subjects of study in general topology and in several branches of mathematics. In literature separation axioms have been studied by different mathematicians. In 1973, Singal et al. introduced the concept of almost regular [4], almost normal [5] and mildly normal [6] spaces. Recently, Ekici, Malghan, Navalagi, Noiri and Park [7–12] continued the study of several weaker forms of separation axioms, while different forms of continuity have been studied in [13]. The aim of this paper is to unify such types of existing spaces by using the concept of generalized topology introduced by A. Császár.

Throughout this paper $(X, \tau)$ always means a topological space on which no separation axioms are assumed unless explicitly stated. The closure and interior of a set $A \subseteq X$ is denoted by $Cl(A)$ and $Int(A)$ respectively. A subset $A$ is said to be regular open (resp., regular closed) if $A = Int(Cl(A))$ (resp., $A = Cl(Int(A))$). The collection of all regular open (regular closed) sets in a topological space $(X, \tau)$ is denoted by $RO(X)$ (resp., $RC(X)$). Let $(X, \tau)$ be a space and $A \subseteq X$. A point $x \in X$ is called a condensation point of $A$ if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called $\omega$-closed [14] if it contains all its condensation points. The complement of an $\omega$-closed set is called $\omega$-open. It is well known that the family of all $\omega$-open subsets of a space $(X, \tau)$, denoted by $\tau_\omega$, forms a topology on $X$ finer than $\tau$. A subset $A$ of a space $X$ is said to be preopen [15] (resp., semi-open [16], $\alpha$-open [17]) if $A \subseteq Int(Cl(A))$ (resp., $A \subseteq Cl(Int(A))$, $A \subseteq Int(Cl(Int(A)))$). The family of all preopen (resp., semi-open, $\alpha$-open) sets in a space $X$ is denoted by $PO(X)$ (resp., $SO(X)$, $\alpha O(X)$).

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We recall some notions defined in [18]. Let \( X \) be a non-empty set, \( \exp X \) denotes the power set of \( X \). We call a class \( \mu \subseteq \exp X \) a generalized topology [18], (briefly, GT) if \( \phi \in \mu \) and union of elements of \( \mu \) belongs to \( \mu \). A set \( X \), with a GT \( \mu \) on it is said to be a generalized topological space (briefly, GTS) and is denoted by \((X, \mu)\). For a GTS \((X, \mu)\), the elements of \( \mu \) are called \( \mu \)-open sets and the complements of \( \mu \)-open sets are called \( \mu \)-closed sets. For \( A \subseteq X \), we denote by \( c_\mu(A) \) the intersection of all \( \mu \)-closed sets containing \( A \), i.e., the smallest \( \mu \)-closed set containing \( A \); and by \( i_\mu(A) \) the union of all \( \mu \)-open sets contained in \( A \), i.e., the largest \( \mu \)-open set contained in \( A \) (see [18–20] for details).

Obviously in a topological space \((X, \tau)\), if one takes \( \tau \) as the GT, then \( c_\mu \) becomes equivalent to the usual closure operator. Similarly, \( c_\mu \) becomes preclosure, semiclosure, \( \alpha \)-closure (briefly, \( pcl, scl, cl_\alpha \)) if \( \mu \) stands for \( PO(X), SO(X), \alpha O(X) \) respectively.

It is easy to observe that \( i_\mu \) and \( c_\mu \) are idempotent and monotonic, where \( \gamma : \exp X \to \exp X \) is said to be idempotent iff \( \gamma(\gamma(A)) = \gamma(A) \) and monotonic iff \( A \subseteq B \subseteq X \) implies \( \gamma(A) \subseteq \gamma(B) \). It is also well known from [19, 21] that if \( \mu \) is a GT on \( X \) and \( A \subseteq X \), \( x \in X \), then \( x \in c_\mu(A) \) iff \( x \in M \in \mu \Rightarrow M \cap A \neq \phi \) and \( c_\mu(X\setminus A) = X\setminus i_\mu(A) \).

We recall the following definitions to be used in sequel.

Definition 1.1 A subset \( A \) of a space \((X, \tau)\) is called:

1. generalized closed (briefly, \( g \)-closed) [1] if \( Cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \in \tau \).
2. regular generalized closed (briefly, \( r_g \)-closed) [3] if \( Cl(A) \subseteq U \) whenever \( A \subseteq U \in RO(X) \).
3. generalized preregular closed [22] (briefly, \( gpr \)-closed), or preregular generalized closed [23] if \( pCl(A) \subseteq U \) whenever \( A \subseteq U \in RO(X) \).
4. regular semi-open [24] if there is a regular open set \( U \) such that \( U \subseteq A \subseteq Cl(U) \). The family of all regular semi-open sets of \( X \) is denoted by \( RSO(X) \).
5. regular weakly generalized closed set (briefly, \( rwg \)-closed) [25] if \( Cl(\text{Int}(A)) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular open in \( X \).
6. regular \( w \)-closed set (briefly, \( rw \)-closed set) [26] if \( Cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a regular semi-open in \((X, \tau)\).

2. Properties of \( rw_\mu \)-Closed Sets

Definition 2.1 Let \( \mu \) be a GT on a topological space \((X, \tau)\). Then \( A \subseteq X \) is called a regular weakly \( \mu \)-closed set or simply an \( rw_\mu \)-closed set (resp., \( w_\mu \)-closed set) if \( c_\mu(A) \subseteq U \) whenever \( A \subseteq U \in RSO(X) \) (resp., \( A \subseteq U \in SO(X) \)). The complement of an \( rw_\mu \)-closed set (resp., \( w_\mu \)-closed) is called an \( rw_\mu \)-open set (resp., \( w_\mu \)-open).

Remark 2.2 Let \( \mu \) be a GT on a topological space \((X, \tau)\). Then we have the following relation between \( rw_\mu \)-closed sets and other known sets:

\[
\mu \text{-closed set} \rightarrow w_\mu \text{-closed set} \rightarrow rw_\mu \text{-closed set}.
\]

The next example shows that none of the implications is reversible.

Example 2.3 Consider the topological space \((X, \tau)\), where \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{b\}, \{c\}, \{b, c\}\} \). Let \( \mu = \{X, \phi, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\} \) be a GT on the space \( X \). Then it is easy to check that \( \{b, c\} \) is \( rw_\mu \)-closed in \((X, \tau)\) but neither a \( w_\mu \)-closed set nor a \( \mu \)-closed set.

Remark 2.4 Obviously if on a space \((X, \tau)\) one takes the GT \( \mu = \tau \), then \( rw_\mu \)-closed sets become equivalent to \( rw \)-closed sets [26]. Similarly, \( rw_\mu \)-closed sets becomes \( gpr \)-closed sets [22], \( rwg \)-closed sets [25], if the role of \( \mu \) is taken to stand for \( PO(X) \), \( RO(X) \) respectively.

The next two examples show that union (intersection) of two \( rw_\mu \)-closed sets is not in general an \( rw_\mu \)-closed set.
Example 2.5 (1) Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$. Let $\mu = \{X, \phi, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ be a GT on the space $X$. The $rw\mu$-closed sets are $X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}$. Then $\{a\}$ and $\{c\}$ are two $rw\mu$-closed sets but their union $\{a, c\}$ is not $rw\mu$-closed set.

(2) Let $X = \{a, b, c, d\}$ and $\tau = \mu = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. The $rw\mu$ closed sets are $X, \phi, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, d\}$. Then $\{a, b\}$ and $\{a, c\}$ are two $rw\mu$-closed sets on $X$ but their intersection $\{a, c\}$ is not an $rw\mu$-closed set in $X$.

Theorem 2.6 Let $\mu$ be a GT on a topological space $(X, \tau)$. Let $A \subseteq X$ be an $rw\mu$-closed subset of $X$. Then $c_\mu(A)\setminus A$ does not contain any non-empty regular closed set.

Proof Let $F$ be a regular closed subset of $(X, \tau)$ such that $F \subseteq c_\mu(A)\setminus A$. Then $F \subseteq X\setminus A$ and hence $A \subseteq X\setminus F \in SO(X)$. Since $A$ is $rw\mu$-closed, $c_\mu(A) \subseteq X\setminus F$ and hence $F \subseteq X\setminus c_\mu(A)$. So $F \subseteq c_\mu(A)\cap (X\setminus c_\mu(A)) = \emptyset$.

That the converse of the above theorem is false as shown by the next example.

Example 2.7 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$. Then $(X, \tau)$ is a topological space. Consider the GT $\mu = \{X, \phi, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ on $X$. Then $RO(X) = \{X, \phi, \{b\}, \{c\}\}$. Consider $A = \{a, c\}$. Then $c_\mu(A)\setminus A = X\setminus \{a, c\} = \{b\}$ does not contain any non-empty regular closed set. But $A$ is not $rw\mu$-closed.

Theorem 2.8 Let $\mu$ be a GT on a topological space $(X, \tau)$. Then a subset $A$ is $rw\mu$-open iff $F \subseteq i_\mu(A)$ whenever $F$ is a regular semi closed subset such that $F \subseteq A$.

Proof Let $A$ be an $rw\mu$-open subset of $X$ and $F$ be a regular semi closed subset of $X$ such that $F \subseteq A$. Then $X\setminus A$ is an $rw\mu$-closed set and $X\setminus A \subseteq X\setminus F \in RSO(X)$. So $c_\mu(X\setminus A) = X\setminus i_\mu(A) \subseteq X\setminus F$. Thus $F \subseteq i_\mu(A)$.

Conversely, let $F \subseteq i_\mu(A)$ whenever $F$ is a regular semi closed such that $F \subseteq A$. Let $X\setminus A \subseteq U$ where $U \in RSO(X)$. Then $X\setminus U \subseteq A$ and $X\setminus U$ is regular semi closed. By the assumption, $X\setminus U \subseteq i_\mu(A)$ and hence $c_\mu(X\setminus A) = X\setminus i_\mu(A) \subseteq U$. Hence $X\setminus A$ is $rw\mu$-closed and hence $A$ is $rw\mu$-open.

Theorem 2.9 Let $\mu$ be a GT on a topological space $(X, \tau)$ and $A$ be an $rw\mu$-closed subset of $X$. If $B \subseteq X$ be such that $A \subseteq B \subseteq c_\mu(A)$, then $B$ is also an $rw\mu$-closed set.

Proof Let $A$ be an $rw\mu$-closed set and $B \subseteq U \in RSO(X)$. Then $A \subseteq U \in RSO(X)$, and hence $c_\mu(A) \subseteq U$. Thus $B \subseteq c_\mu(A)$ and $B \subseteq U$, showing $B$ to be $rw\mu$-closed.

Theorem 2.10 Let $(X, \tau)$ be a topological space and $\mu$ be a GT on $X$. If $A$ is an $rw\mu$-closed subset of $X$, then $c_\mu(A)\setminus A$ is $rw\mu$-open.

Proof Let $A$ be an $rw\mu$-closed subset of $(X, \tau)$ and $F$ be a regular closed subset such that $F \subseteq c_\mu(A)\setminus A$, so by Theorem 2.6 $F = \emptyset$ and thus $F \subseteq i_\mu(c_\mu(A)\setminus A)$. So by Theorem 2.8, $c_\mu(A)\setminus A$ is $rw\mu$-open.

Example 2.11 Consider Example 2.7 once again. If we take $A = \{a, c\}$ then $c_\mu(A)\setminus A = \{b\}$ is $rw\mu$-closed but $\{a, c\}$ is not $rw\mu$-closed.

Definition 2.12 Let $\mu$ be a GT on a topological space $(X, \tau)$. Then $(X, \tau)$ is said to be $rw\mu$-$T_\frac{3}{4}$ if every $rw\mu$-closed set in $(X, \tau)$ is $\mu$-closed.

Theorem 2.13 Let $\mu$ be a GT on a topological space $(X, \tau)$. Then the following are equivalent:

1. $(X, \tau)$ is $rw\mu$-$T_\frac{3}{4}$.
2. Every singleton is either regular closed or $\mu$-open.

Proof (1) $\Rightarrow$ (2) : Suppose $\{x\}$ is not regular closed for some $x \in X$. Then $X\setminus \{x\}$ is not regular open and hence $X$ is the only regular open set containing $X\setminus \{x\}$. Thus $X\setminus \{x\}$ is $rw\mu$-closed. Hence
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$X \setminus \{x\}$ is $\mu$-closed (by (1)). Thus $\{x\}$ is $\mu$-open.

(2) $\Rightarrow$ (1) : Let $A$ be any $rw\mu$-closed subset of $(X, \tau)$ and $x \in c_\mu(A)$. We have to show that $x \in A$. If $\{x\}$ is regular closed and $x \notin A$, then $x \in c_\mu(A) \setminus A$. Thus $c_\mu(A) \setminus A$ contains a non-empty regular closed set $\{x\}$, a contradiction to Theorem 2.6. So $x \in A$. Again if $\{x\}$ is $\mu$-open, then since $x \in c_\mu(A)$, it follows that $x \in A$. So in both the cases $x \in A$. Thus $A$ is $\mu$-closed. ■

Theorem 2.14 Let $\mu$ be a GT on a topological space $(X, \tau)$. Then the following are equivalent:

(1) Every regular semiopen set of $X$ is $\mu$-closed.
(2) Every subset of $X$ is $rw\mu$-closed.

Proof (1) $\Rightarrow$ (2) : Let $A \subseteq U \in RSO(X)$. Then by (1) $U$ is $\mu$-closed and so $c_\mu(A) \subseteq c_\mu(U) = U$. Thus $A$ is $rw\mu$-closed.

(2) $\Rightarrow$ (1) Let $U \in RSO(X)$. Then by (2), $U$ is $rw\mu$-closed and hence $c_\mu(U) \subseteq U$, showing $U$ to be a $\mu$-closed set. ■

Theorem 2.15 Let $\mu$ be a GT on a topological space $(X, \tau)$. If $A$ is $rw\mu$-open then $U = X$ whenever $U$ is regular semi-open and $i_\mu(A) \cup (X \setminus A) \subseteq U$.

Proof Let $U \in RSO(X)$ and $i_\mu(A) \cup (X \setminus A) \subseteq U$ for an $rw\mu$-open set $A$. Then $X \setminus U \subseteq [X \setminus i_\mu(A)] \cap A$, i.e., $X \setminus U \subseteq c_\mu(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is $rw\mu$-closed by Theorem 2.6, $X \setminus U = \phi$ and hence $U = X$. ■

The converse of the above theorem is not always true as shown by the following example.

Example 2.16 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\}$. Then $(X, \tau)$ is a topological space. Consider the GT $\mu = \tau$. Let $A = \{b, c, d\}$. Then $X$ is the only regular open set containing $i_\mu(A) \cup (X \setminus A)$ but $A$ is not $rw\mu$-open in $X$.

3. Almost $w\mu$-Regular, Almost $w\mu$-Normal and Mildly $w\mu$-Normal spaces

Definition 3.1 Let $(X, \tau)$ be a topological space and $\mu$ be a GT on $X$. Then $(X, \tau)$ is said to be almost $w\mu$-regular if for each regular closed set $F$ of $X$ and each $x \notin F$ there exist disjoint $w\mu$-open sets $U$ and $V$ such that $x \in U$, $F \subseteq V$.

Example 3.2 Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{b\}, \{a, c\}\}$. Then $(X, \tau)$ is a topological space. Consider the GT $\mu = \tau$. The possible regular closed non-empty, proper subsets of $X$ are $\{a, c\}$ and $\{b\}$. Now, if consider the regular closed set $\{a, c\}$ and $\{b\}$, then $\{a, c\}$ and $\{b\}$ such that $\{a, c\} \subseteq \{a, c\}, b \in \{b\}$ and $\{a, c\} \cap \{b\} = \phi$. Similarly we can prove regular closed set $\{b\}$. Hence $(X, \tau)$ is a almost $w\mu$-regular.

Remark 3.3 Let $\mu$ be a GT on a space $(X, \tau)$. Then every almost $w\mu$-regular space reduces to an almost regular [4] (resp., almost $p$-regular [8]) space if one takes $\mu$ to be $\tau$. (resp., $PO(X)$).

Theorem 3.4 Let $\mu$ be a GT on a topological space $(X, \tau)$. Then the following statements are equivalent:

(1) $X$ is almost $w\mu$-regular.
(2) For each $x \in X$ and each $U \in RO(X)$ with $x \in U$, there exists $V \in w\mu$ such that $x \in V \subseteq c_\mu(V) \subseteq U$.
(3) For each regular closed set $F$ of $X$, $\cap \{c_\mu(V) : F \subseteq V \in w\mu\} = F$.
(4) For each $A \subseteq X$ and each $U \in RO(X)$ with $A \cap U \neq \phi$, there exists $V \in w\mu$ such that $A \cap V \neq \phi$ and $c_\mu(V) \subseteq U$.
(5) For each non-empty subset $A$ of $X$ and each regular closed subset $F$ of $X$ with $A \cap F = \phi$, there exist $V, W \in w\mu$ such that $A \cap V \neq \phi$, $F \subseteq W$ and $W \cap V = \phi$. 
(6) For each regular closed set $F$ and $x \notin F$, there exist $U \in w\mu$ and an $rwm\mu$-open set $V$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.

(7) For each $A \subseteq X$ and each regular closed set $F$ with $A \cap F = \phi$, there exist $U \in w\mu$ and an $rwm\mu$-open set $V$ such that $A \cap U \neq \phi$, $F \subseteq V$ and $U \cap V = \phi$.

**Proof** (1) $\Rightarrow$ (2): Let $U \in RO(X)$ with $x \in U$. Then $x \notin X \setminus U \in RC(X)$. Thus by (1), there exist disjoint $G, V \in w\mu$ such that $x \in V$, $X \setminus U \subseteq G$. So, $x \in V \subseteq c_{\mu}(V) \subseteq c_{\mu}(X \setminus G) = X \setminus G \subseteq U$.

(2) $\Rightarrow$ (3): Let $X \setminus F \in RO(X)$ and $x \in X \setminus F$. Then by (2), there exists $U \in w\mu$ such that $x \in U \subseteq c_{\mu}(U) \subseteq X \setminus F$. So $F \subseteq X \setminus c_{\mu}(U) = V$ (say) $\in w\mu$ and $U \cap V = \phi$. Then $x \notin c_{\mu}(V)$. Thus $F \ni \{c_{\mu}(V): F \ni V \in w\mu\}$.

(3) $\Rightarrow$ (4): Let $A$ be a subset of $X$ and $U \in RO(X)$ be such that $A \cap U \neq \phi$. Let $x \in A \cap U$. Then $x \notin X \cap U$. Hence by (3), there exists $W \in w\mu$ such that $X \setminus U \subseteq W$ and $x \notin c_{\mu}(W)$. Put $V = X \setminus c_{\mu}(W)$. Then $V \in w\mu$ contains $x$ and hence $A \cap V \neq \phi$. Now $V \subseteq X \setminus W$, so $c_{\mu}(V) \subseteq X \setminus W \subseteq U$.

(4) $\Rightarrow$ (5): Let $F$ be a set as in the hypothesis of (5). Then $X \setminus F \in RO(X)$ with $A \cap (X \setminus F) \neq \phi$ and hence by (4), there exists $V \ni w\mu$ such that $A \cap V \neq \phi$ and $c_{\mu}(V) \subseteq X \setminus F$. If we put $W = X \setminus c_{\mu}(V)$, then $W \in w\mu$, $F \subseteq W$ and $W \cap V = \phi$.

(5) $\Rightarrow$ (1): Let $F$ be a regular closed set such that $x \notin F$. Then $F \cap \{x\} = \phi$. Thus by (5), there exist $U, V \ni w\mu$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.

(1) $\Rightarrow$ (6): Trivial in view of Remark 2.2.

(6) $\Rightarrow$ (7): Let $A \subseteq X$ and $F$ be a regular closed set with $A \cap F = \phi$. Then for $a \in A$, $a \notin F$ and hence by (6), there exist $U \in w\mu$ and an $rwm\mu$-open set $V$ such that $a \in U$, $F \subseteq V$ and $U \cap V = \phi$. So $A \cap U \neq \phi$, $F \subseteq V$ and $U \cap V = \phi$.

(7) $\Rightarrow$ (1): Let $x \notin F$ where $F$ is regular closed in $X$. Since $\{x\} \cap F = \phi$, by (7) there exist $U \in w\mu$ and an $rwm\mu$-open set $W$ such that $x \in U$, $F \subseteq W$ and $U \cap W = \phi$. Then $F \subseteq c_{\mu}(W) = V \in w\mu$ (by Theorem 2.8) and hence $V \cap U = \phi$.

**Note:** If in a topological space $(X, \tau)$ we take $\mu = \alpha O(X)$, then an almost $w\mu$-regular space reduces to an almost regular space [10].

**Definition 3.5** Let $\mu$ be a GT on a topological space $(X, \tau)$. Then $(X, \tau)$ is said to be almost $w\mu$-normal if for each closed set $A$ and each regular closed set $B$ of $X$ with $A \cap B = \phi$, there exist two disjoint $w\mu$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Example 3.6** Consider the topological space $(X, \tau)$ are defined in Example 3.2 and the GT $\mu = \tau$. Clearly $(X, \tau)$ is almost $w\mu$-normal.

**Remark 3.7** Let $\mu$ be a GT on a space $(X, \tau)$. Then an almost $w\mu$-normal space reduces to an almost normal [5] (resp., almost $p$-normal [9, 11]) space if one takes $\mu$ to be $\tau$. (resp., $PO(X)$).

**Theorem 3.8** Let $\mu$ be a GT on a topological space $(X, \tau)$. Then the following statements are equivalent:

1. $X$ is almost $w\mu$-normal.
2. For any closed set $A$ and any regular closed set $B$ of $X$ with $A \cap B = \phi$, there exist disjoint $rwm\mu$-open sets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$.
3. For each closed set $A$ and each regular open set $B$ containing $A$, there exists an $w\mu$-open set $V$ of $X$ such that $A \subseteq V \subseteq c_{\mu}(V)$.
4. For each regular closed set $A$ and each regular open set $B$ containing $A$, there exists an $rwm\mu$-open set $V$ of $X$ such that $Cl(A) \subseteq V \subseteq c_{\mu}(V)$.
5. For each $rg$-closed set $A$ and each regular open set $B$ containing $A$, there exists an $w\mu$-open set $V$ of $X$ such that $Cl(A) \subseteq V \subseteq c_{\mu}(V)$.
6. For each $q$-closed set $A$ and each regular open set $B$ containing $A$, there exists a $w\mu$-open set $V$ such that $Cl(A) \subseteq V \subseteq c_{\mu}(V)$. 


Proof (1) ⇒ (2): Obvious by Remark 2.2.

(2) ⇒ (3): Let \( A \) be a closed set and \( B \) be a regular open set containing \( A \). Then \( A \cap (X \setminus B) = \emptyset \), where \( A \) is closed and \( X \setminus B \) is regular closed. So by (2) there exist disjoint \( w_\mu \)-open sets \( V \) and \( W \) such that \( A \subseteq V \) and \( X \setminus B \subseteq W \). Thus by Remark 2.2 and Theorem 2.8, \( X \setminus B \subseteq i_\mu(W) \) and \( V \cap i_\mu(W) = \emptyset \). Hence \( c_\mu(V) \cap i_\mu(W) = \emptyset \) and hence \( A \subseteq V \subseteq c_\mu(V) \subseteq X \setminus i_\mu(W) \subseteq B \).

(3) ⇒ (4): Let \( A \) be \( rg \)-closed and \( B \) be a regular open set containing \( A \). Then \( Cl(A) \subseteq B \). The rest follows from (3).

(4) ⇒ (5): This follows from (4) and the fact that a subset \( A \) is \( w_\mu \)-open iff \( F \subseteq i_\mu(A) \) whenever \( F \subseteq A \) and \( F \) is closed.

(5) ⇒ (6): Follows from (5) and the fact that every \( g \)-closed set is an \( rg \)-closed set.

(6) ⇒ (1): Let \( A \) be any closed set and \( B \) be a regular closed set such that \( A \cap B = \emptyset \). Then \( X \setminus B \) is a regular open set containing \( A \) where \( A \) is \( g \)-closed (as every closed set is \( g \)-closed [7]). So there exists a \( w_\mu \)-open set \( G \) of \( X \) such that \( Cl(A) \subseteq G \subseteq c_\mu(G) \subseteq X \setminus B \). Put \( U = i_\mu(G) \) and \( V = X \setminus c_\mu(G) \). Then \( U \) and \( V \) are two disjoint \( w_\mu \)-open subsets of \( X \) such that \( Cl(A) \subseteq U \) (as \( G \) is \( rw_\mu \)-open [27]), i.e., \( A \subseteq U \) and \( B \subseteq V \). Hence \( X \) is almost \( w_\mu \)-normal.

Note: If in a topological space \( (X, \tau) \), if we take \( \mu = \alpha O(X) \), then an almost \( w_\mu \)-normal space reduces to an almost normal space.

Definition 3.9 Let \( \mu \) be a GT on a topological space \( (X, \tau) \). Then \( (X, \tau) \) is said to be mildly \( w_\mu \)-normal if for any two disjoint regular closed sets \( A \) and \( B \) there exist two disjoint \( w_\mu \)-open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \).

Example 3.10 Consider the topological space \( (X, \tau) \) are defined in Example 3.2 and the GT \( \mu = \tau \). Clearly \( (X, \tau) \) is mildly \( w_\mu \)-normal.

Remark 3.11 Let \( \mu \) be a GT on a space \( (X, \tau) \). Then a mildly \( \mu \)-normal space reduces to a mildly normal [6] (resp., mildly \( p \)-normal [9, 11] if one takes \( \mu \) to be \( \tau \). (resp., \( PO(X) \)).

Theorem 3.12 Let \( \mu \) be a GT on a topological space \( (X, \tau) \). Then the following are equivalent:

1. \( X \) is mildly \( w_\mu \)-normal.
2. For \( H, K \in RC(X) \) with \( H \cap K = \emptyset \), there exist disjoint \( rw_\mu \)-open sets \( U \) and \( V \) such that \( H \subseteq U \) and \( K \subseteq V \).
3. For any \( H \in RC(X) \) and any \( V \in RO(X) \) with \( H \subseteq V \), there exists an \( rw_\mu \)-open set \( U \) of \( X \) such that \( H \subseteq U \subseteq c_\mu(U) \subseteq V \).
4. For any \( H \in RC(X) \) and any \( V \in RO(X) \) with \( H \subseteq V \), there exists a \( w_\mu \)-open set \( U \) of \( X \) such that \( H \subseteq U \subseteq c_\mu(U) \subseteq V \).

Proof (1) ⇒ (2): Follows from Remark 2.2.

(2) ⇒ (3): Let \( H \in RC(X) \) and \( V \in RO(X) \) be such that \( H \subseteq V \). Then by (3) there exist disjoint \( rw_\mu \)-open sets \( U \) and \( W \) such that \( H \subseteq U \) and \( X \setminus V \subseteq W \). Thus by Theorem 2.8, \( X \setminus V \subseteq i_\mu(W) \) and \( U \cap i_\mu(W) = \emptyset \). So \( c_\mu(U) \cap c_\mu(W) = \emptyset \) and hence \( H \subseteq U \subseteq c_\mu(U) \subseteq X \setminus i_\mu(W) \subseteq V \).

(3) ⇒ (4): Let \( H \in RC(X) \) and \( V \in RO(X) \) be such that \( H \subseteq V \). Then by (3) there exists an \( rw_\mu \)-open set \( G \) of \( X \) such that \( H \subseteq G \subseteq c_\mu(G) \subseteq V \). Since \( H \in RC(X) \), by Theorem 2.8, \( H \subseteq i_\mu(G) \subseteq U \) (say). Hence \( U \in \mu \) and \( H \subseteq U \subseteq c_\mu(U) \subseteq c_\mu(G) \subseteq V \).

(4) ⇒ (1): Let \( H, K \in RC(X) \) be such that \( H \cap K = \emptyset \). Then \( X \setminus K \cap \in RO(X) \) with \( H \subseteq X \setminus K \). Thus by (4) there exists a \( w_\mu \)-open set \( U \) of \( X \) such that \( H \subseteq U \subseteq c_\mu(U) \subseteq X \setminus K \). Put \( V = X \setminus c_\mu(U) \). Then \( U \) and \( V \) are disjoint \( w_\mu \)-open sets such that \( H \subseteq U \) and \( K \subseteq V \).

Note: If in a topological space \( (X, \tau) \) we take \( \mu = \alpha O(X) \), then a mildly \( w_\mu \)-normal space reduces to a mildly normal space [10].
4. Preservation Theorems

Definition 4.1 Let $\mu$ and $\lambda$ be two GT’s on two topological spaces $(X, \tau)$ and $(Y, \sigma)$ respectively. A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be:

1. an $R$ map \[28\] if $f^{-1}(V)$ is regular open in $X$ for every regular open set $V$ in $Y$.
2. rc-preserving \[29\] if $f(F)$ is regular closed in $Y$ for every regular closed $F$ in $X$.
3. $(\mu, \lambda)$-open \[27\] if $f(U)$ is $\lambda$-open in $Y$ for each $\mu$-open subset $U$ of $X$.
4. $\mu$-rw$\mu$-continuous if $f^{-1}(F)$ is $\omega\mu$-closed in $X$ for each $\lambda$-closed set $F$ in $Y$.
5. $\mu$-rw$\mu$-closed if $f(F)$ is $\omega\mu$-closed in $Y$ for each $\mu$-closed set $F$ of $X$.

Example 4.2 Let $X = Y = \{a, b, c\}$, $\tau = \sigma = \{X, \phi, \{b\}, \{c\}, \{a, b\}\}$, the GT $\mu = \lambda = \{x, \phi, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ and $f : (X, \tau) \to (Y, \sigma)$ be the identity map. The $\omega\mu$-closed sets are $X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}$. Then $f$ is $\mu$-rw$\mu$-continuous and $\mu$-rw$\mu$-closed.

Theorem 4.3 Let $\mu$ and $\lambda$ be two GT’s on two topological spaces $(X, \tau)$ and $(Y, \sigma)$ respectively. A surjective mapping $f : (X, \tau) \to (Y, \sigma)$ is $\mu$-rw$\mu$-closed if for each subset $B$ of $Y$ and each $U \in \mu$ containing $f^{-1}(B)$ there exists an $\omega\mu$-open set $V$ of $Y$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof Assume $f$ is $\mu$-rw$\mu$-closed, $\mu$ is a subset of $Y$ and $U(\in \mu)$ contains $f^{-1}(B)$. Put $V = Y \setminus f(X \setminus B)$. Then $V$ is an $\omega\mu$-open set of $Y$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Conversely, let $F$ be a $\mu$-closed set of $X$. Then $f^{-1}(Y \setminus f(F)) \subseteq X \setminus F \in \mu$. Thus there exists an $\omega\mu$-open set $V$ of $Y$ such that $Y \setminus f(F) \subseteq V$ and $f^{-1}(V) \subseteq \setminus X$. Therefore, we have $f(F) \subseteq Y \setminus V$ and $F \subseteq f^{-1}(Y \setminus V)$. Hence we obtain that $f(F)$ is $\omega\mu$-closed in $Y$. Thus $f$ is $\mu$-rw$\mu$-closed.

Theorem 4.4 Let $\mu$ and $\lambda$ be two GT’s on two topological spaces $(X, \tau)$ and $(Y, \sigma)$ respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective $(\mu, \lambda)$-open, $\mu$-rw$\mu$-closed $R$ map. If $X$ is almost $\mu\lambda$-regular, then $Y$ is almost $\omega\mu\lambda$-regular.

Proof Let $f \in RC(Y)$ and $y \in \setminus Y$. Then $f^{-1}(y)$ and $f^{-1}(F)$ are disjoint. Since $f$ is an $R$ map, $f^{-1}(F)$ is regular closed in $X$. For each $x \in f^{-1}(y)$, there exist disjoint $\mu\lambda$-open sets $U$ and $V$ of $X$ such that $x \subseteq U$ and $f^{-1}(F) \subseteq V$. Since $f$ is $(\mu, \lambda)$-open, we have $y = f(x) \in f(U) \subseteq \lambda$. Since $f$ is $\mu$-rw$\mu$-closed, by Theorem 2.6, there exists an $\omega\mu$-open set $W$ of $Y$ such that $F \subseteq W$ and $f^{-1}(W) \subseteq V$. Since $f(U)$ and $W$ are disjoint, by Theorem 3.4, we obtain $Y$ is almost $\omega\mu\lambda$-regular.

Theorem 4.5 Let $\mu$ and $\lambda$ be two GT’s on two topological spaces $(X, \tau)$ and $(Y, \sigma)$ respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be a surjective $\mu$-rw$\mu$-closed $R$ map. If $X$ is a mildly $\mu\lambda$-normal space, then $Y$ is also a mildly $\omega\mu\lambda$-normal space.

Proof Let $A$ and $B$ be two disjoint regular closed closed sets in $X$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint. Since $X$ is mildly $\mu\lambda$-normal, there exist disjoint $\mu\lambda$-open sets $U$ and $V$ of $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Then by Theorem 4.3, there exist $\omega\mu$-open sets $K$ and $L$ of $Y$ such that $A \subseteq K$, $B \subseteq L$, $f^{-1}(K) \subseteq U$ and $f^{-1}(L) \subseteq V$. Since $U$ and $V$ are disjoint, so are $K$ and $L$. By Theorem 2.8, then it follows that $A \subseteq i_{\omega\mu\lambda}(K)$, $B \subseteq i_{\omega\mu\lambda}(H)$ and $i_{\omega\mu\lambda}(K) \cap i_{\omega\mu\lambda}(H) = \phi$. This shows that $Y$ is mildly $\omega\mu\lambda$-normal.

Theorem 4.6 Let $\mu$ and $\lambda$ be two GT’s on two topological spaces $(X, \tau)$ and $(Y, \sigma)$ respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\mu$-rw$\mu$-continuous rc-preserving injection. If $Y$ is a mildly $\mu\lambda$-normal space, then $X$ is mildly $\omega\mu$-normal.

Proof Let $A$ and $B$ be any disjoint regular closed closed sets of $X$. Since $f$ is an rc-preserving injection, $f(A)$ and $f(B)$ are disjoint regular closed closed sets of $Y$. By mildly normality of $Y$, there exist disjoint $\omega\mu$-open sets $U$ and $V$ of $Y$ such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Since $f$ is $\mu$-rw$\mu$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $\omega\mu$-open sets containing $A$ and $B$ respectively. Hence by Theorem 3.8, $X$ is mildly $\omega\mu$-normal.
References