Pre-(\(\omega\))separation axioms

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Abstract

In this paper we use the notion of (\(\omega\))preopen sets to introduce and study pre-separation axioms in an (\(\omega\))topological space.

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1. Introduction

The notion of an (\(\omega\))topological space is introduced in [2]. These spaces could have possible applications in digital topology, infinite neural networks and other infinite networks as elucidated in [3]. While the notions of connectedness and hyperconnectedness would be useful in the study of infinite neural networks and other infinite networks, from [4] and [6] it is established that weak separation axioms have some applications in digital topology. In [3], [8] and [9] we have studied the notions of connectedness and hyperconnectedness in an (\(\omega\))topological space.

So for the purpose of establishing (\(\omega\))topological spaces as a tool to study digital topologies we propose to define various types of weak separation axioms on an (\(\omega\))topological space. We begin this process by using the notion of semi-(\(\omega\))open sets [3] to introduce semi-(\(\omega\))separation axioms in [10]. In [7] we introduce and study (\(\omega\))preopen sets analogous to the notion of preopen sets introduced in [1]. In [5] the authors have introduced and studied three new separation axioms called pre-\(T_0\), pre-\(T_1\), pre-\(T_2\) using preopen sets. The aim of this communication is to introduce and study these pre-separation axioms in an (\(\omega\))topological space analogous the work done in [3]. We provide some characterizations of pre-(\(\omega\))\(T_0\), pre-(\(\omega\))\(T_1\) and pre-(\(\omega\))\(T_2\) spaces and establish the interrelations between the separation axioms (\(\omega\))\(T_0\), (\(\omega\))\(T_1\) and (\(\omega\))\(T_2\) and the new pre-(\(\omega\))pre-separation axioms.

2. Preliminaries

A set \(X\) equipped with a countable number of topologies \(\mathcal{J}_n\) satisfying \(\mathcal{J}_n \subset \mathcal{J}_{n+1}\), \(n \in \mathbb{N}\) is called an (\(\omega\))topological space and is denoted by \((X, \{\mathcal{J}_n\})\), or simply by \(X\). A set \(G(\subseteq X)\) is said to be (\(\omega\))open if
\( G \in \bigcup_n \mathcal{F}_n \), for some \( n \). A set \( F \) is \((\omega)\)closed if its complement \( X - F \) is \((\omega)\)open. In the sequel, \( N \) denotes the set of natural numbers and \( n \) denotes a natural number. We require the following notions introduced in [3].

**Definition 2.1.** \( X \) is said to be an \((\omega)T_0\)-space if for every pair of distinct points \( x \) and \( y \) of \( X \), there exists an \((\omega)\)open set \( G \) such that \( x \in G \) and \( y \notin G \).

**Definition 2.2.** \( X \) is said to be an \((\omega)T_1\)-space if for every pair of distinct points \( x \) and \( y \) of \( X \), there exists a \( n \in N \) such that for some \( U, V \in \mathcal{F}_n \), we have \( x \in U \), \( y \in V \), \( y \notin U \) and \( x \notin V \).

The following definition is from [2].

**Definition 2.3.** \( X \) is said to be \((\omega)T_2\)-space if for any two distinct points \( x, y \) of \( X \), there exists an \( n \) such that for some \( U, V \in \mathcal{F}_n \), we have \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \).

Finally we require the following definitions and results from [7].

**Definition 2.4.** A set \( A \subset X \) is said to be \((\omega)\)preopen if there exists an \( n \) such that for some \( U \in \mathcal{F}_n \), we have \( A \subset U \subset (\omega)clA \). In particular we write \( A \) is \((\mathcal{F}_n - \omega)\)preopen, since \( U \in \mathcal{F}_n \). The set of all \((\omega)\)preopen subsets of \( X \) is denoted by \( PO(X) \).

**Definition 2.5.** The complement of a \((\omega)\)preopen set is said to be \((\omega)\)preclosed. A set whose complement is \((\mathcal{F}_n - \omega)\)preopen is said to be \((\mathcal{F}_n - \omega)\)preclosed. The set of all \((\omega)\)preclosed subsets of \( X \) is denoted by \( PC(X) \).

**Definition 2.6.** For any \( n \in N \), the smallest \((\mathcal{F}_n - \omega)\)preclosed set containing a set \( A \) is called the \((\mathcal{F}_n - \omega)\)preclosure of \( A \) and is denoted by \((\mathcal{F}_n) pcl A \). \((\omega) pcl A \) denotes the intersection of all \((\omega)\)preclosed sets containing \( A \). It is called the \((\omega)\)preclosure of \( A \).

\((\omega) pcl A \) need not be \((\omega)\)preclosed.

**Lemma 2.7.** Let \( A \subset Y \subset X \) and \( Y \) be \((\omega)\)preopen in \( X \), then \( A \in PO(X) \) if and only if \( A \in PO(Y) \).

**Definition 2.8.** A set \( N(\subset X) \) is said to be a \((\omega)\)preneighbourhood a point \( x \in X \) if there exists an \((\omega)\)preopen set \( G \) such that \( x \in G \subset N \).

3. Pre-(\(\omega\))separation axioms

In this section, we use the notion of \((\omega)\)preopeness to define the notions of pre-(\(\omega\))\(T_0\)-spaces followed by pre-(\(\omega\))\(T_1\)-spaces and pre-(\(\omega\))\(T_2\)-spaces. We then go on to study the characterizations of these notions and their interrelations with \((\omega)T_0\)-, \((\omega)T_1\)- and \((\omega)T_2\)-spaces.

**Definition 3.1.** \( X \) is a pre-(\(\omega\))\(T_0\)-space if for any every pair of distinct points \( x, y \in X \), there exists an \((\omega)\)preopen set \( G \) such that \( x \in G \) and \( y \notin G \).

Obviously an \((\omega)T_0\)-space is pre-(\(\omega\))\(T_0\), but the converse is not true as seen below.

**Example 3.2.** Consider the \((\omega)\)topological space \((X, \mathcal{F}_n)\) where \( X = N \) and \( \mathcal{F}_n \) is defined as follows:

\[
\mathcal{F}_1 = \emptyset, X \bigcup P(3) \\
\mathcal{F}_2 = \emptyset, X \bigcup P(3, 4) \text{ and in general,} \\
\mathcal{F}_n = \emptyset, X \bigcup P(3, 4, \ldots, n) \text{ for all } n.
\]

Then \( X \) is not an \((\omega)\)\(T_0\)-space but it is pre-(\(\omega\))\(T_0\).

**Theorem 3.3.** The \((\omega)\)precloseress of distinct points are distinct in \( X \).
Proof. Let \( x, y \) be two distinct points in \( X \), and let \( A = X - \{ x \} \). Then it is easy to see that \((\omega)\text{cl}A = A\) or \(X\). If \((\omega)\text{cl}A = A\), then \(A\) is an \((\omega)\)-closed and hence \((\omega)\)-preclosed set which contains \(y\), so \(x \not\in (\omega)\text{pcl}[y]\) and so \((\omega)\text{pcl}[x] \neq (\omega)\text{pcl}[y]\). Alternatively if \((\omega)\text{cl}A = X\) then \(A\) is \((\omega)\)-preopen, then \([x]\) is \((\omega)\)-preclosed so \((\omega)\text{pcl}[x] = [x]\), hence \((\omega)\text{pcl}[x] \neq (\omega)\text{pcl}[y]\).

**Theorem 3.4.** If in an \((\omega)\)-topological space \(X\), \((\omega)\)-preclosures of distinct points are distinct, then \(X\) is pre-(\(\omega\))-\(T_0\).

Proof. Suppose that \(x, y \in X\) are two distinct points such that \((\omega)\text{pcl}[x] \neq (\omega)\text{pcl}[y]\). Then without loss of generality we may assume that there exists at least one point \(z\) such that \(z \in (\omega)\text{pcl}[x]\) and \(z \not\in (\omega)\text{pcl}[y]\). Then we claim that \(x \not\in (\omega)\text{pcl}[y]\), since otherwise we have \(z \in (\omega)\text{pcl}[y]\). Then there exists an \((\omega)\)-preclosed set \(A\) such that \(x \in A\) and \(y \not\in A\). So \(x \in X - A\) and \(y \not\in X - A\), where \(X - A\) is \((\omega)\)-preopen. Thus it follows that \(X\) is pre-(\(\omega\))-\(T_0\).

**Theorem 3.5.** Every \((\omega)\)-topological space is pre-(\(\omega\))-\(T_0\).

Proof. Follows from Theorem 3.3 and 3.4.

**Definition 3.6.** \(X\) is a pre-(\(\omega\))-\(T_1\)-space if for each pair of distinct points \(x, y \in X\), there exist two \((\omega)\)-preopen sets \(U\) and \(V\) such that \(x \in U, y \not\in V\) and \(y \in V, x \not\in U\).

Obviously a pre-(\(\omega\))-\(T_1\)-space is pre-(\(\omega\))-\(T_0\), but the converse is not true as seen in the following example.

**Example 3.7.** The \((\omega)\)-topological space in Example 3.2 is pre-(\(\omega\))-\(T_0\) but not pre-(\(\omega\))-\(T_1\).

Since every topological space is also an \((\omega)\)-topological space, the following example establishes that an pre-(\(\omega\))-\(T_1\)-space need not be \((\omega)\)-\(T_1\). The converse is obviously always true as seen in the following example [5].

**Example 3.8.** Let \(X = [a, b]\) with the indiscrete topology \(\tau\). Then \(X\) is pre-(\(\omega\))-\(T_1\) but not \((\omega)\)-\(T_1\).

**Theorem 3.9.** \(X\) is pre-(\(\omega\))-\(T_1\) iff each singleton set is the intersection of \((\omega)\)-preclosed sets.

Proof. Suppose that \(X\) is pre-(\(\omega\))-\(T_1\) and let \(x\) be a point in \(X\). Then for any \(y \in X, y \neq x\), there exists an \((\omega)\)-preopen set \(G_y\) containing \(y\) but not \(x\). Let \(G = \cup\{G_y : y \in \{x\}^c\}\), then it is easy to see that \(\{x\}^c = G\) and so \(\{x\}^c\) is the union of \((\omega)\)-preopen sets and hence \([x]\) is the intersection of \((\omega)\)-preclosed.

Conversely, let \(x, y\) be two distinct points in \(X\), then \([x]\) and \([y]\) are both expressible as the intersection of be \((\omega)\)-preclosed sets. Then there exists two \((\omega)\)-preopen sets \(G\) and \(H\) such that \(x \in G\) and \(y \not\in G\) aslo \(y \in H\) and \(x \not\in H\). Then it follows that \(X\) is pre-(\(\omega\))-\(T_1\).

**Theorem 3.10.** Every \((\omega)\)-open subspace of a pre-(\(\omega\))-\(T_1\)-space is pre-(\(\omega\))-\(T_1\).

Proof. The line proof is similar to that of Theorem 3.16.

**Definition 3.11.** \(X\) is a pre-(\(\omega\))-\(T_2\)-space if for each pair of distinct points \(x, y \in X\), there exist two \((\omega)\)-preopen sets \(U\) and \(V\) such that \(x \in U, y \not\in V\) and \(U \cap V = \emptyset\).

It is evident that a \((\omega)\)-\(T_2\)-space is pre-(\(\omega\))-\(T_2\), but the converse is not true as seen below.

**Example 3.12.** The \((\omega)\)-topological space defined in Example 3.8 is seen to be pre-(\(\omega\))-\(T_2\), but not \((\omega)\)-\(T_2\).

The following example from- [5] shows that a pre-(\(\omega\))-\(T_1\)-space is not always pre-(\(\omega\))-\(T_2\), the converse is obviously true.

**Example 3.13.** Let \(X = E \cup \{a\} \cup \{b\}\), where \(E\) is an infinite set and \(a, b\) are two distinct points not in \(E\). Let \(\tau\) be the family of subsets of \(X\) containing \((i)\) \(A\) if \(A \subset E\) and \((ii)\) \(A\) if \(a \text{ or } b \in A\) but \(A^c\) contains only a finite number of points from \(E\). Then the topology \((X, \tau)\) is pre-(\(\omega\))-\(T_1\) but not pre-(\(\omega\))-\(T_2\).
The next two theorems provide characterizations of a pre-$(\omega)T_2$-space.

**Theorem 3.14.** An $(\omega)$topological space $X$ is pre-$(\omega)T_2$ iff for any $x \in X$, and each $y \neq x$ there is a $(\omega)$preneighbourhood $N$ of $x$ such that $y \notin (\omega)\text{pcl}N$.

Proof. Let $X$ be a pre-$(\omega)T_2$-space and $x \in X$. Then for any $y \in X$, with $x \neq y$ there exist two $(\omega)$preopen sets $G, H$ such that $x \in G, y \in H$ and $G \cap H = \emptyset$. Then $x \in G \subset X - H = N, (\text{say})$ so $N$ is an $(\omega)$preneighbourhood of $x$ which does not contain $y$.

Conversely suppose $x, y$ are two distinct points in $X$. Then there exists a $(\omega)$preneighbourhood $N$ of $x$ such that $y \notin (\omega)\text{pcl}N$. Then $x \notin X - (\omega)\text{pcl}N, y \in X - (\omega)\text{pcl}N$, then as $(\omega)\text{pcl}N$ is the intersection of all $(\omega)$preclosed sets containing $N$, $X - (\omega)\text{pcl}N$ is the union of $(\omega)$preopen sets. Then there exists an $(\omega)$preopen set say $V$ such that $y \in V \subset X - (\omega)\text{pcl}N$ also there exists an $(\omega)$preopen set $U$ such that $x \in U \subset N$ (since $N$ is $(\omega)$preneighbourhood of $x$). Hence it follows that $X$ is pre-$(\omega)T_2$.

**Theorem 3.15.** $X$ is pre-$(\omega)T_2$ iff for each $x \in X$, we have $\{x\} = \cap\{(\omega)\text{pcl}N : N$ is a $(\omega)$preneighbourhood of $x\}$.

Proof. Follows from Theorem 3.14.

**Theorem 3.16.** Every $(\omega)$open subspace of a pre-$(\omega)T_2$-space is pre-$(\omega)T_2$.

Proof. Let $A$ be a $(\omega)$open subset of $X$ and $x, y$ be two distinct points in $A$. Since $X$ is pre-$(\omega)T_2$, so there exist two disjoint $(\omega)$preopen sets $U, V$ such that $x \in U$ and $y \in V$. Now we see by Lemma 2.7 that $U_1 = A \cap U$ and $V_1 = A \cap V$ are $(\omega)$preopen in $A$, and $x \in U_1$, $y \in V_1$ with $U_1 \cap V_1 = \emptyset$.

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References


