A study on fuzzy tangent bundle of fuzzy Banach manifold

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Abstract

In this paper, we study some of fuzzy topological and analytical properties of fuzzy tangent bundle and fuzzy cotangent bundle of fuzzy Banach manifold.

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1. Introduction

The concept of fuzzy sets was first introduced by Zadeh [21] in 1965. Further, many authors developed the theory in different context. Some of the important developments are Fuzzy Banach Spaces and Fuzzy Metric Spaces which are introduced by many authors namely Bag et al. [1, 16] and Saadati and Vaezpour [18] in different perception. In 1993, Ferraro and Foster have introduced the concept of $C^1$ fuzzy manifold [6], and in 2003, Guner introduced tangent bundle on $C^1$ fuzzy manifold [7], with the similar approach in our previous papers [8, 9, 10] we have introduced the concept of fuzzy Banach manifold and studied some of its fuzzy topological properties.

In this paper, we study tangent bundle and some of its fuzzy topological properties on fuzzy Banach manifold. Further, we study fuzzy version of Hahn-Banach theorem on tangent bundle by introducing fuzzy norm on $TM$ and cotangent bundle $T^*M$, where the members of $T(M)$ are tangent vectors and members of $T^*(M)$ are fuzzy differential 1-forms so, in this study we extend bounded linear fuzzy differential 1-forms from submanifold $T(N)$ to whole of $T(M)$ in norm preserving manner.

2. Preliminaries

In this section we include some definitions and theorems which are used in this paper.

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Definition 2.1. [21] Let $X$ be a set. A fuzzy subset $A$ of $X$ is defined to be a function $\mu_A : X \rightarrow [0, 1]$. Thus we have the following:

$$A = \{(x, \mu_A) : \forall x \in X\} = \mu_A.$$

Definition 2.2. [15] A fuzzy subset in $X$ is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one say, $x \in X$. If its value at $x$ is $\lambda$ ($0 < \lambda \leq 1$) we denote this fuzzy point by $x_\lambda$, where the point $x$ is called its support.

Definition 2.3. [3] A fuzzy topology on a set $X$ is a family $\tau$ of fuzzy sets in $X$ which satisfies the following conditions:

1. $k_0, k_1 \in \tau$,
2. If $A, B \in \tau$, then $A \cap B \in \tau$,
3. If $A_j \in \tau$, $\forall j \in J$ (where $J$ is index set), then $\bigcup_{j \in J} A_j \in \tau$.

The pair $(X, \tau)$ is called a fuzzy topological space and the members of $\tau$ are called open fuzzy sets.

Definition 2.4. [15] Let $X$, $Y$ be fuzzy topological spaces. A bijection $f$ of $X$ onto $Y$ is said to be a fuzzy continuous map if for each open fuzzy subset $A$ in $Y$ the inverse image $f^{-1}(A)$ is in $X$.

Definition 2.5. [18] The fuzzy normed space $(X, N, \ast)$ is said to be a fuzzy Banach space whenever $X$ is complete with respect to the fuzzy metric induced by fuzzy norm.

Definition 2.6. [8] Let $M$ be a fuzzy topological space, $U$ be a fuzzy subset of $M$ such that $\sup \{\mu_U(x)\} = 1$, $\forall x \in M$ and $\phi$ is a fuzzy homeomorphism defined on the support of $U = \{x \in M : \mu_A(x) > 0\}$, which maps $U$ onto an open fuzzy set $\phi(U)$ in some fuzzy Banach space $E_i$. Then the pair $(U, \phi)$ is called as fuzzy Banach chart.

Definition 2.7. [8] A fuzzy Banach atlas $A$ of class $C^k$ on $M$ is a collection of pairs $(U_i, \phi_i)$ $(i$ ranging over index set $I)$ subject to the following conditions:

1. $\bigcup_{i \in I} U_i = M$ that is the domains of fuzzy Banach charts in $A$ cover $M$.
2. Each fuzzy homeomorphism $\phi_i$, defined on the support of $U_i = \{x \in X : \mu_{U_i}(x) > 0\}$ which maps $U_i$ onto an open fuzzy subset $\phi_i(U_i)$ in some fuzzy Banach space $E_i$ and for each $i, j$ in the index set $I$, $\phi_i(U_i \cap U_j)$ and $\phi_j(U_i \cap U_j)$ are open fuzzy sets in $E_i$.
3. The mapping $\phi_i \circ \phi_j^{-1}$ which maps $\phi_j(U_i \cap U_j)$ onto $\phi_i(U_i \cap U_j)$ is fuzzy diffeomorphism of class $C^k$ $(k \geq 1)$ for each pair of indices $i, j$.

The maps $\phi_i \circ \phi_j^{-1}$, $i, j \in I$ are called fuzzy transition maps. A fuzzy topological space $(M, \tau)$ modeled on fuzzy Banach space $(X, N, \ast)$, covered by fuzzy Banach atlases is called fuzzy Banach Manifold.

Definition 2.8. [11] A mapping $f : X \times Y \rightarrow [0, 1]$ is a fuzzy function if $\forall x \in X$ there exists $y \in Y \ni f(x, y) > 0$.

Let $X$, $Y$ be two fuzzy topological vector spaces and let $\phi$ be a mapping from $X$ into $Y$. Let $o(t)$ denote any function of real variable $t$ such that $\lim_{t \rightarrow 0} o(t)/t = 0$.

Definition 2.9. [7] The mapping $\phi$ is said to be tangent at 0 if given a neighborhood $W$ of 0, $0 < \delta \leq 1$, in $Y$ there exists a neighborhood $V$ of 0, for every $\lambda$, $0 < \lambda < \delta$, in $X$ such that

$$\phi(tV) \subset o(t)W,$$

for some function $o(t)$.
Definition 2.10. [5] Let \( X, Y \) be two fuzzy topological vector spaces. Let \( f : X \rightarrow Y \) be a fuzzy continuous mapping. Then \( f \) is said to be fuzzy differentiable at a point \( x \in X \) if there exists a linear fuzzy continuous mapping \( u \) of \( X \) into \( Y \) such that
\[
f(x + y) = f(x) + u(y) + \phi(y), \quad y \in X,
\]
where \( \phi \) is tangent to 0. The mapping \( u \) is called the fuzzy derivative of \( f \) at \( x \). The fuzzy derivative of \( f \) at \( x \) is denoted by \( f'(x) \); it is an element of \( L(X, Y) \), the set of all linear fuzzy continuous mappings of \( X \) into \( Y \). The mapping \( f \) is fuzzy differentiable if it is fuzzy differentiable at every point of \( X \).

Definition 2.11. [2] Let \( X \) be a crisp set. A fuzzy ordered relation on \( X \) is a fuzzy subset \( R \) of \( X \times X \) with the following properties:

1. \( \forall x \in X, r(x, x) \in [0, 1] \) (reflexivity),
2. \( \forall x, y \in X, r(x, y) + r(y, x) > 1 \) implies \( x = y \) (antisymmetry),
3. \( \forall (x, y, z) \in X^3, [r(x, y) \geq r(y, x) \text{ and } r(y, z) \geq r(z, y)], \) imply \( r(x, z) \geq r(z, x) \) (f-transitivity).

A set with fuzzy order defined on it is called a fuzzy ordered set.

Definition 2.12. [2] Let \( X \) be a set with fuzzy ordered relation \( R \). Then the fuzzy order \( R \) is said to be total if \( \forall x \neq y \) we have either \( r(x, y) > r(y, x) \) or \( r(y, x) > r(x, y) \).

Theorem 2.13 (Fuzzy Zorn’s lemma [2]). Let \( X \) be a fuzzy ordered set with fuzzy order \( R \). If every fuzzy ordered set \( X \) in which every totally ordered subset has an upper bound must contain a maximal element.

Definition 2.14. [14] A fuzzy differentiable function \( F : N \rightarrow M \) is called a fuzzy immersion if its rank is equal to the dimension of \( N \) at each point of its domain. If its domain is the whole of \( N \) then, \( F \) is said to be a fuzzy immersion of \( N \) into \( M \).

Definition 2.15. [14] A fuzzy subset \( N \) of fuzzy manifold \( M \) is said to be fuzzy submanifold of \( M \) if the natural fuzzy injection \( i : N \rightarrow M \) is a fuzzy immersion.

3. Tangent bundle on fuzzy Banach manifold

In 2003, Guner has defined the tangent bundle on \( C^1 \) fuzzy manifold [7], using their approach we define fuzzy tangent bundle on fuzzy Banach manifold.

Let \( M \) be fuzzy Banach manifold. Consider triplet \( (U_l, \phi_l, x) \) where \( (U_l, \phi_l) \) is a fuzzy Banach chart at \( p \in M \) and \( x \) is fuzzy point of fuzzy Banach space in which \( \phi(U_l) \) lies.

Two such triplets \( (U_l, \phi_l, x), (U_j, \phi_j, y) \) are said to be related, written \( (U_l, \phi_l, x) \sim (U_j, \phi_j, y) \), if the fuzzy derivative of \( \phi_j \circ \phi_l^{-1} \) at \( \phi_l(p) \) maps \( x \) on to \( y \). That is,
\[
(\phi_j \circ \phi_l)'(\phi_l(p))x = y.
\]

The relation \( (U_l, \phi_l, x) \sim (U_j, \phi_j, y) \) is an equivalence relation. An equivalence class of \( (U_l, \phi_l, x) \) is called a tangent vector of the fuzzy Banach manifold \( M \) at \( p \) and this equivalence class is denoted by \( [U_l, \phi_l, x]_p \).

The tangent space at \( p \in M \) is denoted by \( T_p(M) \) and the disjoint union of tangent spaces \( T_p(M) \) \( \forall p \in M \) is denoted by \( TM \) and is called as fuzzy tangent bundle, i.e.,
\[
TM = \bigvee_{p \in M} T_p(M).
\]

Now we shall show that fuzzy tangent bundles \( TM \) admits structure of fuzzy Banach manifold.
**Proposition 3.1.** A fuzzy Banach atlas $A$ on $M$ induces a fuzzy Banach atlas $\tilde{A}$ on $TM$.

**Proof.** Let $A = (U_i, \phi_i)_{i \in I}$ be a fuzzy Banach atlas of $M$ and $\pi : TM \rightarrow M$ be a natural projection defined as $\pi([U_i, \phi_i, v]) = p \forall i \in I$. Let $(U_i, \phi_i)$ be a fuzzy Banach atlas in $A$. We define $T_\phi : \pi^{-1}(U_i) \rightarrow E \times E$ given by

$$T_\phi([U_i, \phi_i, x]) = (x, y).$$

Notice that $T_\phi$ is bijection, further,

$$T_\phi(\pi^{-1}(U_i)) = \pi^{-1}(U_i) \times E$$

is an open fuzzy subset of $E \times E$. Therefore the pair $(\pi^{-1}(U_i), T_\phi)$ is called a fuzzy Banach chart of $TM$. Now we shall show that

$$\tilde{A} = (\pi^{-1}(U_i), T_\phi) : (U_i, \phi_i) \in \tilde{A}$$

is a fuzzy Banach atlas on $TM$. We know that the pair $(\pi^{-1}(U_i), T_\phi)$ is a fuzzy Banach chart on $TM$ and the union of co-domains of $T_\phi$ is equal to $T(M)$, i.e,

$$\bigcup_{(U_i, \phi_i) \in \tilde{A}} (\pi^{-1}(U_i)) = \pi^{-1} \left( \bigcup_{(U_i, \phi_i) \in \tilde{A}} U_i \right) = \pi^{-1}(M) = TM.$$

Let $(\pi^{-1}(U_i), T_\phi)$ and $(\pi^{-1}(U_j), T_\phi)$ be ant two charts in $\tilde{A}$ then we have,

$$T_\phi(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)) = \phi_i(U_i \cap U_j) \times E = E \times E$$

. Similarly,

$$T_\phi(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)) = \phi_j(U_i \cap U_j) \times E = E \times E.$$

It is clear that each of the sets are open fuzzy sets in $E \times E$. Suppose that for $i, j \in I$ the co-domains of $T_\phi$ and $T_\phi$ overlap, that is, $\pi^{-1}(U_i) \cap \pi^{-1}(U_j) \neq \emptyset$. Since $\pi^{-1}(U_i) \cap \pi^{-1}(U_j) = \pi^{-1}(U_i \cap U_j)$, we have $U_i \cap U_j \neq \emptyset$.

Let $(u, v) \in T_\phi(\pi^{-1}(U_i) \cap \pi^{-1}(U_j))$, then

$$T_\phi \circ T_\phi^{-1}(x, y) = T_\phi([U_i, \phi_i, y]_{\phi_i^{-1}(x)})$$

$$= T_\phi([U_j, \phi_j, (\phi_j \circ \phi_i^{-1})(x)]_{\phi_i^{-1}(\phi_j \circ \phi_i^{-1}(x)))}$$

$$= ((\phi_j \circ \phi_i^{-1})(x), (\phi_j \circ \phi_i^{-1})(y)).$$

Now consider,

$$T_\phi(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)) = T_\phi(\pi^{-1}(U_i \cap U_j))$$

$$= \phi^{-1}(U_i \cap U_j) \times E.$$

So, fuzzy transition maps $T_\phi \circ T_\phi^{-1}$ on $T_\phi(\pi^{-1}(U_i) \cap \pi^{-1}(U_j))$ are fuzzy diffeomorphisms which follows from the fuzzy diffeomorphism of $\phi_j \circ \phi_i^{-1}$ on $\phi_i(U_i \cap U_j)$. Hence a family $\{(\pi^{-1}(U_i)), T_\phi\}_{i \in I}$ is fuzzy Banach atlas on $T(M)$. A differentiable structure on $M$ induces a unique differentiable structure on $T(M)$. Hence $T(M)$ is also a fuzzy Banach manifold. \hfill \Box

It can be easily shown that the atlas of $TM$ induces a fuzzy topological structure on $TM$ with the help of above proposition and [6, Proposition 3.2].

**Notation:** For simplicity we use notation $u, v, w, ...$ for tangent vectors $[U_i, \phi_i, x], [U_j, \phi_j, y], [U_k, \phi_k, z], ...$ belonging to $T(M)$ on fuzzy Banach manifold where $x, y, z \in E$ and $(U_i, \phi_i), (U_j, \phi_j), (U_k, \phi_k)$ are charts of $M$. 
**Proposition 3.2.** The fuzzy topology induced on $TM$ is independent of all the fuzzy Banach atlas $A$ representing the fixed differentiable structure on $M$.

**Proof.** Let $B = (V_j, \psi_j)_{j \in J}$ be another equivalent fuzzy Banach atlases $A$ on $M$ and $(\pi^{-1}(V_j), T_{\psi_j})_{j \in J}$ be on $TM$. Let $O \subset TM$ is an open fuzzy subset of $B$, i.e., $T_{\psi_j}(O \cap \pi^{-1}(V_j))$ is an open fuzzy subset of $E \times E \forall j \in J$. We have,

$$T_{\psi_i}(O \cap \pi^{-1}(U_i)) = \bigvee_{j \in J} T_{\psi_j}(T_{\psi_j}(O \cap \pi^{-1}(U_i) \cap \pi^{-1}(V_j))).$$

Consider $j \in J$ such that $\pi^{-1}((U_i) \cap \pi^{-1}(U_j)) = \emptyset$. Since $T_{\psi_i}$ is bijection we get,

$$T_{\psi_j}(O \cap \pi^{-1}(U_i) \cap \pi^{-1}(V_j)) = T_{\psi_j}(O \cap \pi^{-1}(V_j)) \cap T_{\psi_j}(\pi^{-1}(U_i) \cap \pi^{-1}(V_j)).$$

Since the first term on the right-hand side is open fuzzy subset and the second term $T_{\psi_j}(\pi^{-1}(U_i) \cap \pi^{-1}(V_j))$ equals $\psi_j((U_i \cap V_j) \times E)$ is also open fuzzy set in $E \times E$. Using the fact that the fuzzy Banach charts $(U_i, \phi_i)_{i \in I}$ and $(V_j, \psi_j)_{j \in J}$ are equivalent, by above proposition we can easily show that the maps:

$$T_{\phi_i} \circ T_{\psi_j}^{-1} : T_{\psi_j}(\pi^{-1}(U_i) \cap \pi^{-1}(V_j)) \longrightarrow T_{\phi_i}(\pi^{-1}(U_i) \cap \pi^{-1}(V_j)),$$

are fuzzy diffeomorphism. Further we have,

$$T_{\phi_i}(T_{\psi_j}^{-1}(T_{\psi_j}(O \cap \pi^{-1}(U_i) \cap \pi^{-1}(V_j)))),$$

and hence

$$\bigvee_{j \in J} T_{\phi_i}(T_{\psi_j}^{-1}(T_{\psi_j}(O \cap \pi^{-1}(U_i) \cap \pi^{-1}(V_j)))),$$

are open sets in $E \times E$. Therefore the result holds for all $i \in I$ and the fuzzy topology on $TM$ thus only depends on the differentiable structure on $M$. \hfill \Box

**Definition 3.3.** Let $M$ and $N$ be fuzzy Banach manifolds and $f : M \longrightarrow N$ be a fuzzy smooth map. Consider $p \in M$ with $q = f(p) \in N$, choose any fuzzy Banach chart $(V, \psi)$ containing $q = f(p)$ so that for every $u \in T_pM$ we have the representative $(U_i, \phi_i, x) = u$, then the fuzzy tangent map $T_pf : T_pM \longrightarrow T(q)N$ is defined by letting the representative of $T_pf \cdot u_p$ in the fuzzy Banach chart $(V, \psi)$ be given by $(V_j, \psi_j, y)$ where

$$y = (\psi \circ f \circ \phi^{-1}) \cdot x,$$

which uniquely determines $T_pf \cdot x$.

**Proposition 3.4.** If $f : M \longrightarrow N$ is fuzzy smooth map then the tangent map $Tf : TM \longrightarrow TN$ is fuzzy smooth.

**Proof.** Let $M$ and $N$ be fuzzy Banach manifolds and $f : M \longrightarrow N$ be fuzzy smooth map. By Proposition 3.1 we know that the fuzzy Banach charts $(U_i, \phi_i)$, $(V_j, \psi_j)$ on $M$ and $N$ respectively lead to the coordinate charts $(\pi^{-1}(U_i), T_{\phi_i})$ and $(\pi^{-1}(V_j), T_{\psi_j})$ on $TM$ and $TN$ respectively therefore we have,

$$(T_{\phi_i} \circ Tf \circ T_{\phi_i}^{-1})(x, y) = ((\psi \circ f \circ \phi^{-1}) \cdot x, (\psi \circ f \circ \phi^{-1}) \cdot y) \forall (x, y) \in T_{\phi_i}(\pi^{-1}(U_i)) \in E \times E.$$ 

Both the components of above equation are fuzzy smooth thus $Tf$ is fuzzy smooth function. \hfill \Box

**Corollary 3.5.** If $f : M \longrightarrow N$ and $g : N \longrightarrow P$ are fuzzy smooth maps then $gf : M \longrightarrow P$ is fuzzy smooth and $T(g \circ f) = Tg \circ Tf$.

**Proof.** It is an obvious consequence of Proposition 3.4. \hfill \Box
Since every $TM$ is a vector space one can define fuzzy tangent norm with reference to tangent norm defined on Banach manifold [3].

**Definition 3.6.** Let $M$ be a fuzzy Banach manifold with fuzzy tangent bundle $T(M)$. A fuzzy tangent norm on $M$ is a fuzzy set on $T(M) \times (0, \infty)$, i.e., $N : T(M) \times (0, \infty) \rightarrow [0, 1]$ such that for every $p \in M$ the restriction $N^p_T : T_p(M) \times (0, \infty) \rightarrow [0, 1]$ is a fuzzy norm on $T_p(M)$ satisfying the following condition of fuzzy norm [7], $\forall u, v \in T_p(M), t, s > 0$:

1. $N'(u, t) > 0$.
2. $N'(u, t) = 1$ iff $u = 0$.
3. $N'(au, t) = N'(u, \frac{t}{|a|})$.
4. $N'(u, t) \ast N'(v, s) \leq N'(u + v, t + s)$.
5. $N'(u, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.
6. $\lim_{t \rightarrow \infty} N'(u, t) = 1$.

**Definition 3.7.** A fuzzy tangent norm $N'$ on a fuzzy Banach manifold $M$ is called compatible if for every $p \in M$ there exists a fuzzy Banach chart $(U, \phi)$ of $M$ about $p$ and constant $0 \leq r_1 \leq r_2$ such that every $v \in T_p(M)$ satisfies

$$r_1 N'(T_p(M)v, t) \leq N'(v, t) \leq r_2 N'(T_p(M)v, t), \forall t > 0.$$

A fuzzy Banach manifold $M$ endowed with a compatible fuzzy tangent norm $N'$ is a fuzzy normed fuzzy Banach manifold $(M, N')$.

Now we shall study cotangent bundle on $M$.

### 4. Cotangent bundle on fuzzy Banach manifold

In this section we define fuzzy cotangent space and fuzzy differential 1-forms on fuzzy Banach manifold by referring [13, 19, 12, 17, 4]. Also we study the Hahn-Banach theorem on fuzzy Banach tangent bundle such that the fuzzy differential 1-forms are extended from submanifold $T(N)$ to whole of $T(M)$ in norm preserving manner. The following results are developed by Halakatti.

Now we shall define fuzzy differential 1-forms on $TM$.

**Definition 4.1.** Let $f : M \rightarrow R$ be a fuzzy continuous function. The map $df : T_pM \rightarrow R$ is defined by $p \rightarrow df(p)$ where $df(p)$ is the fuzzy differential at $p$ given by, $df(p).v_p = v_p \cdot f, \forall v_p \in T_p(M)$.

Clearly, $df(p)$ is an element of the dual space $T^*(M)$ called as fuzzy differential 1-form which is usually denoted by $a, \omega, \eta, ...$.

**Proposition 4.2.** A fuzzy Banach atlas $A$ on $M$ induces a fuzzy Banach atlas on $\tilde{A}^*$ on $TM$

**Proof.** Let $A$ be a fuzzy Banach atlas of $M$. We know that, $T^*(M)$ be union of all cotangent spaces $T^*_p(M) \forall p \in M$, i.e.,

$$T^*(M) = \bigcup_{p \in M} T^*_p(M).$$

Let $\pi^* : T^*(M) \rightarrow M$ be projection map defined as $\pi^*(\eta) = p, \forall p \in M$ and $\eta \in T^*_p(M)$. Let $(U_i, \phi_i) \forall i \in I$ be fuzzy Banach chart of $M$. We define, $T^*_{\phi_i} : \pi^{*-1}(U_i) \rightarrow E \times E$ by,

$$T^*_{\phi_i}(\eta) = (\phi_i(p), \eta).$$
Clearly \( T^*_{\phi_i} \) is bijection and further \( T^*_{\phi_i}(\pi^{-1}(U_i)) \) is an open fuzzy subset of \( E \times E \). Hence the pair \((\pi^{-1}(U_i), T^*_{\phi_i})\) is fuzzy Banach chart of \( T^*(M) \). Notice that union of all codomain of \( T^{-1}_{\phi_i} \) cover \( T^*(M) \), i.e.,

\[
\bigcup_{(U_i,\phi_i)\in A} (\pi^{-1}(U_i)) = \pi^{-1} \left( \bigcup_{(U_i,\phi_i)\in A} U_i \right) = \pi^{-1}(\pi^{-1}(M)) = T^*M.
\]

Further for any \((x, y) \in \phi(U_i \cap U_j) \times E\), we have

\[
T^*_{\phi_j} \circ T^{-1}_{\phi_i}(x, y) = T^*_{\phi_j}(\eta)
\]

\[
= (\phi_j \circ \phi_i^{-1}(x), \eta_{\phi_j})
\]

where \( \eta \in T^*_{\phi_i^{-1}((\eta))}(M) \). From above expression it is clear that fuzzy transition maps are fuzzy diffeomorphism. Hence the family \( \{(\phi^{-1}_i(U_i)), T^*_{\phi_i}\}_{i \in I} \) is a fuzzy Banach atlas on \( T^*(M) \). So, equivalence atlases of \( M \) induces equivalent atlases of \( T^*(M) \) and thus differential structure on \( M \) induces unique differential structure on \( T^*(M) \). Therefore \( T^*(M) \) is called as fuzzy cotangent bundle of fuzzy Banach manifold, which admits fuzzy differential 1-forms as linear functional. \( \Box \)

**Definition 4.3.** For \( \eta \in T^*(M) \) let \( N^* \) be fuzzy set on \( T^*(M) \times (0, \infty) \) given by

\[
N^*(\eta, t) = \sup_{v \in N^*} \left| \eta(v) \right|
\]

\[
= \sup_{N^* \cap (v, t)} \left| \eta(v) \right|, \forall \eta \in T^*(M)
\]

where \( N^* \) is fuzzy tangent norm. We observe that \( T(M) \) admits fuzzy differential 1-form as linear functionals so, one can study the Hahn-Banach theorem on \( T(M) \) in terms of fuzzy differential 1-form.

**Theorem 4.4.** Let \( T(M) \) be fuzzy Banach tangent bundle and \( T(N) \) be submanifold of \( T(M) \). If \( \eta \in T^*(N) \) then \( \exists \omega \in T^*(M) \) such that \( \omega = \eta \) on \( T(N) \) and \( N^*(\omega, t) = N^*(\eta, t) \) \( \forall \ t \in (0, \infty) \).

**Proof.** To prove the theorem first we shall define a fuzzy function \( p \) on \( T(M) \) as follows

\[
p(u) = N^*(\eta, t) \cdot N^*(u, t).
\]

We observe that, \( p(u + v) \leq p(u) + p(v) \) and \( p(cu) \leq cp(u) \), \( \forall \ u, v \in T(M) \) and \( c \geq 0 \). Also we observe that, \( \eta(u) \leq p(u), \forall \ u \in T(N) \). Consider a fixed tangent vector \( w \in (\pi^{-1}(V_j), T_{\phi_i}) \in T(M) \setminus T(N) \). For all \( u, v \in T(N) \), we have,

\[
\eta(u) - \eta(v) = \eta(u - v)
\]

\[
\leq p(u - v)
\]

\[
= p(u + w) + p(-w - v).
\]

Hence,

\[
-p(-w - v) - \eta(v) \leq p(u + w) - \eta(u).
\]

Therefore, \( v \in T(N) \) implies that,

\[
s = \sup_{v \in T(N)} \{-p(-w - v) - \eta(v)\} \leq p(u + w) - \eta(u),
\]

and hence,

\[
s \leq \inf_{u \in T(N)} \{p(u + w) - \eta(u)\}.
\]
For the \( w \) specified above, we define the submanifold \( T(N)_w \) of \( T(M) \).

\[
T(N)_w = \{ u + cw : u \in (\pi^{-1}(U_i), T_{\phi_i}), c \in \mathbb{R} \}.
\]

It is clear that, \( x = u + cw \) is unique for \( w \in T(N)_w \). Since, \( w \in (\pi^{-1}(V_i)), T_{\phi_i} \) is arbitrary and charts of \( T(M) \) are compatible, we can say that \( T(N)_w \) is a submanifold of \( T(M) \). Now we shall define \( \overline{\eta}(u) = \eta(u) + cs \) on \( T(N)_w \). From above equation it is clear that, \( \overline{\eta}(u) = \eta(u) \forall u \in T(N) \). So, we have thus extended \( \eta \) from \( T(N) \) to a bigger submanifold \( T(N)_w \) of \( T(M) \). If \( T(N)_w \) is equal to \( T(M) \) then proof is complete. Since \( \frac{u}{c} \) in \( T(M) \) for every \( c \neq 0 \), we see that,

\[
-p(-w - \frac{u}{c}) - \eta(\frac{u}{c}) \leq \frac{s}{c} \\
\leq p(\frac{u}{c} + w) - \eta(\frac{u}{c})
\]

From above equation it is clear that, \( \overline{\eta}(x) \leq p(x) \forall x \in T(N)_w \). If \( T(N)_w \neq T(M) \), we proceed further by using fuzzy Zorn’s lemma to prove the theorem. Let \( X \) denotes the set of all pair \( (T(N), \eta) \) where \( T(N) \) is submanifold of \( T(M) \) containing \( T(N) \) and \( \eta' \) is an extension of \( \eta \) from \( T(N) \) to \( T(N)' \) satisfying \( \eta' \leq p \) on \( T(N)' \). Define a fuzzy ordered relation \( \geq \) on \( X \) by

\[
r(T(N)'', \eta'') \geq r(T(N)', \eta'),
\]

if \( T(N)' \) is a submanifold of \( T(N)'' \) and \( \eta = \eta' \) on \( T(N)' \). This defines a fuzzy ordered relation on \( X \).

Let \( S = \{(T(N)_w, \eta_i)\}_{i \in I} \) be a totally fuzzy ordered subset of \( X \). The pair \( (\cup_{i \in I} T(N)_w, \overline{\eta}) \) where \( \overline{\eta}(u) = \eta_i(u) \) for \( u \in T(N)_w \) is an element of \( X \), and is an upper bound for \( S \). Since \( S \) is an arbitrary totally fuzzy ordered subset of \( X \), by fuzzy Zorn’s lemma for fuzzy ordered relation implies that \( X \) has a maximal element which we will call \( (T(N)_{\omega}, \eta_{\omega}) \) observe that \( \eta_{\omega} \) is an extension of \( \eta \) from \( T(N) \) to \( T(N)_{\omega} \) that satisfies \( \eta_{\omega} \leq p \) on \( T(N)_{\omega} \). Now we shall show that \( T(N)_{\omega} \) is all of \( T(M) \).

If it is not, then we apply the process of extension used in processing from \( T(N) \) to \( T(N)_w \) to create the element \( (T(N)_{\omega}, \eta_{\omega}) \) of \( X \) satisfying the following condition:

\[
r(T(N)_{\omega}, \eta_{\omega}) \geq r(T(N)_{\omega}, \eta_{\omega}).
\]

Since \( (T(N)_{\omega}, \eta_{\omega}) \) is maximal in \( X \). We must have that \( T(N)_{\omega} = T(N)_{\omega}' \), contradicting the definition of “\( \geq \)”. If we now let \( \omega = \eta_{\omega} \), we have an extension of \( \eta \) to all of \( T(M) \) satisfying \( \omega(u) \leq p(u) \forall u \in T(M) \). Replacing \( u \) by \(-u\) we get,

\[
|\omega(u)| \leq p(u) = N'(\eta) \cdot N'(u), \quad \forall u \in T(M) \\
\Rightarrow N'(\omega, t) \leq N'(\eta, t).
\]

To show the equality of norms, let for each \( \epsilon > 0 \) choose \( u \in T(N) \) such that \( N'(u, t) = 1 \) and \( |\eta(u)| > N'(\eta, t) - \epsilon \), then \( |\omega(u)| > N'(\eta, t) - \epsilon \) consequently,

\[
N'(\omega, t) \geq N'(\eta, t) \quad \text{(2)}
\]

from (1) and (2),

\[
N'(\omega, t) = N'(\eta, t).
\]

\[
\square
\]

5. Conclusion

The present paper is a study of fuzzy tangent bundle, fuzzy cotangent bundle and some of their fuzzy topological and analytical properties.
References