On semi separation axioms on \( L \)-fuzzifying topological spaces

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Abstract

In this paper we introduce and study the concepts of semi \( - T_0 \), semi \( - T_1 \), semi \( - T_2 \) (Hausdorff), semi \( - T_3 \) (regularity), semi \( - T_4 \) (normality), semi \( - R_0 \), and semi \( - R_1 \) separation axioms in \( L \)-fuzzifying topological space where \( L \) is a complete residuated lattice. Sometimes we add more conditions on \( L \) such as the completely distributive law or the double negation law

Keywords: \( L \)-fuzzifying topological space, complete residuated lattice, semi \( - T_2 \), double negation law

1. Introduction

Since Chang \cite{Chang}, Hutton \cite{Hutton}, Lowen \cite{Lowen}, Pu and Liu \cite{Pu}, Wong \cite{Wong}, etc., introduced fuzzy theory into topology, many authors discussed various aspects of fuzzy topology. In a Chang-Goguen fuzzy topology, the open sets were fuzzy, but the topology comprising those open sets was a crisp subset of the power \( I_X \). On the other hand, fuzzification of openness was first initiated by Höhle \cite{Hohle} and later developed to subsets of \( I_X \). Then Kubiak \cite{Kubiak}, Šostak \cite{Sostak} jointly and independently extended Höhle’s notion to fuzzy subsets of \( I_X \) (see also \cite{Hohle}). Notice that this kind of topology, according to the final agreement with all authors of Höhle (\cite{Hohle, Sostak}), is called a fuzzy topology, and the kind of topology defined by Chang and Goguen, will be called a fuzzy topology. Moreover, the corresponding Chang-Goguen, spaces are called fuzzy topological spaces. It has been developed in many direction (Chattopadhyay et. al., \cite{Chattopadhyay}, Demirci \cite{Demirci}, Fang Jinming \cite{Fang}, Höhle and Rodabaugh \cite{HohleRodabaugh}, Höhle and Šostak \cite{HohleSostak}, Šostak \cite{Sostak}). In 1991, from a logical point of view, Ying \cite{Ying} studied Höhle’s topology and a fuzzifying topology.

In the present paper we introduce semi \( - R_0 \) and semi \( - R_1 \)-separation axioms in fuzzifying topological space and study their relations with semi \( - T_1 \) and semi \( - T_2 \)-separation axioms, respectively. Furthermore, we introduce and study semi \( - T_0 \), semi \( - T_3 \) (semi-regularity) and semi \( - T_4 \) (semi-normality)-separation axioms in fuzzifying topological spaces and give some of their characterizations as well as the relations of these axioms and other semi separation axioms in fuzzifying topological space introduced. In Section 1, we explain some results in some types in separation axioms in \( L \)-fuzzifying topological space. In Section 2, semi separation axioms in \( L \)-fuzzifying topological space are introduced and studied. In Section 3, relations among semi separation axioms are discussed.

For more details of many concepts used in this paper see (\cite{Soininen}, \cite{Papakonstantinou}, \cite{Ying}).

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Definition 1.1 [6]. The double negation law in a complete residuated lattice \( L \) is given as follows: \( \forall a, b \in L, \ (a \rightarrow \bot) \rightarrow \bot = a \).

Definition 1.2 (\[6\], \[15\]). A structure \((L, \vee, \wedge, ^*, \bot, \top)\) is called a strictly two-sided commutative quantale iff

1. \((L, \vee, \wedge, \bot, \top)\) is a complete lattice whose greatest and least element are \( \top, \bot \) respectively,
2. \((L, ^*, \top)\) is a commutative monoid,
3. \((L, \wedge, \bot)\) is distributive over arbitrary joins, i.e., \( a \vee \bigwedge_{j \in J} b_j = \bigvee_{j \in J} (a \wedge b_j) \ \forall a \in L, \vee \{b_j | j \in J\} \subseteq L \),

4. \((L, \wedge, \bot)\) is a binary operation on \( L \) defined by: \( a \rightarrow b = \bigvee_{a \leq \lambda \leq b} \lambda \ \forall a, b \in L \).

Definition 1.3 \[20\].

A structure \((L, \vee, \wedge, ^*, \rightarrow, \bot, \top)\) is called a complete residuated lattice iff

1. \((L, \vee, \wedge, \bot, \top)\) is a complete lattice whose greatest and least element are \( \top, \bot \) respectively,
2. \((L, ^*, \top)\) is a commutative monoid, i.e.,
3. \((L, \wedge, \bot)\) is a commutative and associative binary operation on \( L \), and
4. \( \forall a \in L, a \ast \top = \top \ast a = a \),
5. \( \rightarrow \) is a binary operation on \( L \) which is antitone in the first and isotone in the second variable,
6. \( \ast \rightarrow \) is couple with \( \ast \) as: \( a \ast b \leq c \text{ iff } a \leq b \rightarrow c \ \forall a, b, c \in L \).

Definition 1.4 \[20\]. Let \((X, \tau)\) be an \( L \)-fuzzifying topological space, let \( x \in X \). The fuzzifying neighbourhood system of \( x \), denoted by \( N_x \in \mathcal{P}(X) \), is defined as follows: \( N_x(A) = \bigvee_{x \in \mathcal{B} \in A} \tau(B) \).

Definition 1.5 \[20\]. Let \( f, g \in L^X \). Then,

1. The \( L \)-equality between \( f \) and \( g \) is denoted by \([f, g]\) and is given as follows:

\[
[f, g] = \bigwedge_{x \in X} (f(x) \rightarrow g(x)) \wedge (g(x) \rightarrow f(x)) \] \[7\].

2. The \( L \)-inclusion of \( f \) in \( g \) is denoted by \([f, g] \] and is given as follows:

\[
[f, g] \] = \bigwedge_{x \in X} (f(x) \rightarrow g(x)).

Definition 1.6. \[21\]. Let \( \Omega \) be the class of all \( L \)-fuzzifying topological spaces. The unary \( L \)-predicates \( T_i, R_j \in L^\Omega, i = 0, 1, 2, 3, 4 \) and \( j = 0, 1 \) are defined as follows:

\[
T_0(X, \tau) = \bigwedge_{x \neq y} \left( \bigvee_{y \in A} N_x(A) \right) \cup \left( \bigvee_{x \in A} N_y(A) \right),
\]
\[
T_1(X, \tau) = \bigwedge_{x \neq y} \left( \bigvee_{y \in B} N_x(B) \right) \cup \left( \bigvee_{x \in C} N_y(C) \right),
\]
\[
T_2(X, \tau) = \bigwedge_{x \neq y} \left( \bigvee_{C \cap B = \phi} (N_x(B) \wedge N_y(C)) \right),
\]
\[
T_3(X, \tau) = \bigwedge_{x \neq y} \left( \bigvee_{A \cap B = \phi} (N_x(A) \wedge N_y(C)) \right),
\]
\[
T_4(X, \tau) = \bigwedge_{A \cap B = \phi} \left( (F_\tau(A) \wedge F_\tau(B)) \rightarrow \left( \bigvee_{A \subseteq C, B \subseteq D, C \cap D = \phi} (\tau(C) \wedge \tau(D)) \right) \right),
\]
2. Semi separation axioms in **L**-fuzzifying topological space

**Definition 2.1** Let \((X, \tau)\) be an \(L\)-fuzzifying topological space.

(1) The semi-neighborhood system of a point \(x \in X\) is denoted by \(SN_x \in L^{P(X)}\) and defined as follows: 
\[
SN_x(A) = \bigvee_{x \in B \subseteq A} S\tau(B).
\]

(2) The semi interior of a set \(A \in P(X)\) is denoted by \(semi-int_\tau (A) \in L^X\) and defined as follows: 
\[
semi-int_\tau (A)(x) = SN_x(A).
\]

(3) The semi closure of a set \(A \in P(X)\) is denoted by \(semi-cl_\tau (A) \in L^X\) and defined as follows: 
\[
semi-cl_\tau (A)(x) = SN_x(X - A) \rightarrow \bot.
\]

**Definition 2.2** Let \((X, \tau)\) be an \(L\)-fuzzifying topological space.

(1) The map \(S\tau \in L^{P(X)}\) is called semi-open and defined as 
\[
S\tau(A) = \bigwedge_{x \in A} semi-cl_\tau (semi-int_\tau (A))(x).
\]

(2) The map \(SF \in L^{P(X)}\), is called semi-closed and defined as 
\[
SF(A) = S\tau(X - A).
\]

**Proposition 2.1** Let \((X, \tau)\) be an \(L\)-fuzzifying topological space. Then:

(1) If \(L\) satisfies the double negation law, then 
\[
SN_x(A) = semi-cl_\tau (X - A)(x) \rightarrow \bot,
\]

(2) \(semi-cl_\tau (\emptyset) = 1_\emptyset\), where \(1_\emptyset \in L^X\) is defined as follows: 
\[
1_\emptyset(x) = \bot, \quad \forall x \in X,
\]

(3) \(A \subseteq semi-cl_\tau (A),\)

(4) \(A \subseteq B \Rightarrow semi-cl_\tau (A) \leq semi-cl_\tau (B),\)

(5) \(semi-cl_\tau (A \cup B) = (semi-cl_\tau (A)) \lor (semi-cl_\tau (B)).\)

**Definition 2.3** Let \(\Omega\) be the class of all \(L\)-fuzzifying topological spaces. The unary \(L\)-predicates \(semi - T_i, semi - R_j \in L^\Omega, i = 0, 1, 2, 3, 4\) and \(j = 0, 1\) are defined as follows:

\[
semi - T_0(X, \tau) = \bigwedge_{x \# y} \left( (\bigvee_{y \not\in A} SN_x(A)) \lor (\bigvee_{x \not\in A} SN_y(A)) \right),
\]
semi - \( T_1(X, \tau) = \bigwedge_{x \neq y} \left( \bigvee_{y \notin B} SN_x(B) \land \bigvee_{x \notin C} SN_y(C) \right) \),

semi - \( T_2(X, \tau) = \bigwedge_{x \neq y} \left( \bigvee_{C \cap B = \emptyset} (SN_x(B) \land SN_y(C)) \right) \),

semi - \( T_3(X, \tau) = \bigwedge_{x \neq y} (\bigcup D(D) \to \left( \bigvee_{A \subseteq C, B \subseteq D} (SN_x(A) \land \tau(B)) \right) \),

semi - \( T_4(X, \tau) = \bigwedge_{A \cap B = \emptyset} ((SF_\tau(D) \land SF_\tau(B)) \to \left( \bigvee_{A \subseteq C, B \subseteq D} (SN_x(A) \land \tau(B)) \right) \),

semi - \( R_0(X, \tau) = \bigwedge_{x \neq y} (\left( \bigvee_{x \notin A} SN_x(A) \lor \bigvee_{y \notin A} SN_y(A) \right) \to \left( \bigvee_{y \notin B} SN_x(B) \land \bigvee_{x \notin C} SN_y(C) \right) ),

semi - \( R_1(X, \tau) = \bigwedge_{x \neq y} (\left( \bigvee_{x \notin A} SN_x(A) \lor \bigvee_{y \notin A} SN_y(A) \right) \to \left( \bigvee_{C \cap B = \emptyset} (SN_x(B) \land SN_y(C)) \right) ).

**Lemma 2.1.** Let \((X, \tau) \in \Omega\). Then for any \(x, y \in X\),

1. \( \bigvee_{x \neq y} (N_x(B) \land N_y(C)) \leq \bigvee_{C \cap B = \emptyset} (SN_x(B) \land SN_y(C)) \),

2. \( \bigvee_{x \neq A} N_x(A) \lor \bigvee_{y \neq A} N_y(A) \leq \bigvee_{x \neq A} N_x(A) \lor \bigvee_{y \neq A} N_y(A) \),

3. \( \bigvee_{y \neq B} N_x(B) \land \bigvee_{x \neq C} N_y(C) \leq \bigvee_{y \neq B} N_x(B) \land \bigvee_{x \neq C} N_y(C) \),

4. \( \bigvee_{A \cap B = \emptyset, D \subseteq B} (N_x(A) \land \tau(B)) \leq \bigvee_{A \cap D = \emptyset, D \subseteq B} (SN_x(A) \land \tau(B)) \),

5. \( \bigvee_{A \subseteq C, B \subseteq D, C \cap D = \emptyset} (\tau(C) \land \tau(D)) \leq \bigvee_{A \subseteq C, B \subseteq D, C \cap D = \emptyset} (\tau(C) \land \tau(D)) \).

**Proof.** Since \( \tau \leq \tau \), one can deduce that \( N_x(A) \leq SN_x(A) \) for any \( A \in P(X) \), the proof is immediate.

**Theorem 2.1.** For any \((X, \tau) \in \Omega\), \( T_i(X, \tau) \leq semi - T_i(X, \tau) \), where \( i = 0, 1, 2, 3, 4 \).

**Proof.** It is obtained from lemma 2.1.
Lemma 2.2 Let \((X, \tau) \in \Omega\). Then for any \(x, y \in X\),

\[
\begin{align*}
(1) & \quad \bigvee_{C \cap B=\emptyset} (SN_x(B) \land SN_y(C)) \leq (\bigvee_{y \in B} SN_x(B)) \land (\bigvee_{x \in C} SN_y(C)), \\
(2) & \quad (\bigvee_{y \notin B} SN_x(B)) \land (\bigvee_{x \notin C} SN_y(C)) \leq (\bigvee_{y \notin A} SN_x(A)) \lor (\bigvee_{x \notin A} SN_y(A)), \\
(3) & \quad \bigvee_{C \cap B=\emptyset} (SN_x(B) \land SN_y(C)) \leq (\bigvee_{y \notin A} SN_x(A)) \lor (\bigvee_{x \notin A} SN_y(A)).
\end{align*}
\]

Proof. It is easy.

Theorem 2.2. For any \((X, \tau) \in \Omega\), \(semi - R_1(X, \tau) \leq semi - R_0(X, \tau)\).

Proof. \(semi - R_1(X, \tau) = \bigwedge_{x \neq y} ((\bigvee_{y \notin A} SN_x(A)) \lor (\bigvee_{x \notin A} SN_y(A)) \rightarrow \bigvee_{C \cap B=\emptyset} (SN_x(B) \land SN_y(C)))\).

Since \(\rightarrow\) is isotone in the second, then

\[
semi - R_1(X, \tau) \leq \bigwedge_{x \neq y} ((\bigvee_{y \notin A} SN_x(A)) \lor (\bigvee_{x \notin A} SN_y(A)) \rightarrow (\bigvee_{y \notin B} SN_x(B)) \land (\bigvee_{x \notin C} SN_y(C))).
\]

\[
= semi - R_0(X, \tau)
\]

Theorem 2.3. For any \((X, \tau) \in \Omega\), the following statements are satisfied:

\[
\begin{align*}
(1) & \quad semi - T_1(X, \tau) \leq semi - R_0(X, \tau), \\
(2) & \quad semi - T_1(X, \tau) \leq semi - T_0(X, \tau), \\
(3) & \quad semi - T_1(X, \tau) \leq (semi - R_0(X, \tau)) \land (semi - T_0(X, \tau)), \\
(4) & \quad If semi - T_0(X, \tau) = \top, then
\end{align*}
\]

\[
semi - T_1(X, \tau) = (semi - R_0(X, \tau)) \land (semi - T_0(X, \tau)).
\]

Proof.

\[
(1) \quad Since (\bigvee_{y \notin B} SN_x(B)) \land (\bigvee_{x \notin C} SN_y(C)) \leq (\bigvee_{y \notin A} SN_x(A)) \lor (\bigvee_{x \notin A} SN_y(A)) \rightarrow (\bigvee_{y \notin B} SN_x(B)) \land (\bigvee_{x \notin C} SN_y(C)),
\]

we have \(semi - T_1(X, \tau) \leq semi - R_0(X, \tau)\).

\[
(2) \quad Since (\bigvee_{y \notin B} SN_x(B)) \land (\bigvee_{x \notin C} SN_y(C)) \leq (\bigvee_{y \notin A} SN_x(A)) \lor (\bigvee_{x \notin A} SN_y(A)),
\]
then \( \text{semi} - T_1(X, \tau) \leq \text{semi} - T_0(X, \tau) \).

(3) The proof follows from (1) and (2) above.

(4) Since \( \top \rightarrow \alpha = \alpha \forall \alpha \in L \) (Indeed \( \top \rightarrow \alpha = \bigvee_{\lambda \leq \alpha} \lambda = \bigvee_{\lambda \leq \alpha} \lambda = \alpha \).) then,

\[
\text{semi} - T_1(X, \tau) = \bigwedge_{x \neq y} \left( (\bigvee_{y \in B} SN_x(B)) \land (\bigvee_{x \in C} SN_y(C)) \right)
\]

\[
= \left( \bigwedge_{x \neq y} \left( (\bigvee_{y \in A} N_x(A)) \lor (\bigvee_{x \in A} N_y(A)) \right) \rightarrow \left( (\bigvee_{y \in B} SN_x(B)) \land (\bigvee_{x \in C} SN_y(C)) \right) \right) \land \top
\]

\[
= \text{semi} - R_0(X, \tau) \land \text{semi} - T_0(X, \tau) \forall x, y \in X \text{ s.t. } x \neq y.
\]

**Theorem 2.4.** For any \((X, \tau) \in \Omega\), the following statements are satisfied If \( \text{semi} - T_0(X, \tau) = \top \), then

(1) \( R_0(X, \tau) \leq \text{semi} - R_0(X, \tau) \).

(2) \( R_1(X, \tau) \leq \text{semi} - R_1(X, \tau) \).

**Proof.** Since \( \text{semi} - T_0(X, \tau) = \top \) and for any \( x, y \in X, x \neq y \), we have

\[
T_0(X, \tau) = \bigwedge_{x \neq y} \left( (\bigvee_{y \in A} N_x(A)) \lor (\bigvee_{x \in A} N_y(A)) \right) = \top \text{ and so, } \left( (\bigvee_{y \in B} SN_x(B)) \land (\bigvee_{x \in C} SN_y(C)) \right) = \top.
\]

(1) (Indeed \( \top \rightarrow \alpha = \bigvee_{\lambda \leq \alpha} \lambda = \bigvee_{\lambda \leq \alpha} \lambda = \alpha \).) then

\[
R_0(X, \tau) = \bigwedge_{x \neq y} \left( (\bigvee_{y \in A} N_x(A)) \lor (\bigvee_{x \in A} N_y(A)) \right) \rightarrow \left( \bigvee_{C \cap B = \emptyset} (N_x(B) \land N_y(C)) \right),
\]

\[
\leq \bigwedge_{x \neq y} \left( (\bigvee_{y \in B} SN_x(B)) \land (\bigvee_{x \in C} SN_y(C)) \right) \rightarrow \left( \bigvee_{y \in B} SN_x(B) \land (\bigvee_{x \in C} SN_y(C)) \right)
\]

\[
= \text{semi} - R_0(X, \tau).
\]

(2)

\[
R_1(X, \tau) = \bigwedge_{x \neq y} \left( (\bigvee_{y \in A} N_x(A)) \lor (\bigvee_{x \in A} N_y(A)) \right) \rightarrow \left( \bigvee_{C \cap B = \emptyset} (N_x(B) \land N_y(C)) \right),
\]

\[
\leq \bigwedge_{x \neq y} \left( (\bigvee_{y \in B} \varphi_x(A)) \lor (\bigvee_{x \in A} \varphi_y(A)) \right) \rightarrow \left( \bigvee_{C \cap B = \emptyset} (SN_x(B) \land SN_y(C)) \right)
\]

\[
= \bigwedge_{x \neq y} \left( (\bigvee_{y \in B} SN_x(B)) \land (\bigvee_{x \in C} SN_y(C)) \right) \rightarrow \left( \bigvee_{C \cap B = \emptyset} (SN_x(B) \land SN_y(C)) \right)
\]

\[
= \text{semi} - R_1(X, \tau).
\]

**Corollary 2.1** \( \text{semi} - T_2(X, \tau) \leq \text{semi} - T_1(X, \tau) \).
Lemma 2.3 Let \((X, \tau) \in \Omega\), Then \(S\tau(A) = \bigwedge_{y \in A} SN_y(A)\).

Theorem 2.5 If \(L\) satisfies the completely distributive law, then for any \((X, \tau) \in \Omega\),

\[
\text{semi} - T_1(X, \tau) = \bigwedge_{x \in X} SF_\tau(|x|).
\]

Proof. Let \(x_1, x_2 \in X\) s.t. \(x_1 \neq x_2\).

Then, \(\bigwedge_{x \in X} SF_\tau(|x|) = \bigwedge_{x \in X} S\tau(X - \{x\}) = \bigwedge_{x \in X} (\bigwedge_{y \in X - \{x\}} SN_y(X - \{x\}))\)

\[
\leq \bigwedge_{y \in X - \{x_2\}} SN_y(X - \{x_2\}) \leq SN_{x_1}(X - \{x_2\}) = \bigvee_{x \in A} SN_{x_1}(A).
\]

Similarly, we have \(\bigwedge_{x \in X} SF_\tau(|x|) \leq \bigvee_{x \in B} SN_{x_2}(B)\). Then

\[
\bigwedge_{x \in X} SF_\tau(|x|) \leq \bigvee_{x_1 \neq x_2} \bigwedge_{x \in B, x_2 \in A} (SN_{x_1}(A) \land SN_{x_2}(B)) = \text{semi} - T_1(X, \tau).
\]

For the other hand,

\[
\text{semi} - T_1(X, \tau) = \bigwedge_{x_1, x_2 \in X, x_1 \neq x_2} ((\bigvee_{x \in A} SN_{x_1}(A)) \land (\bigvee_{x \in B} SN_{x_2}(B)))
\]

\[
= \bigwedge_{x_1 \neq x_2} (SN_{x_1}(X - \{x_2\}) \land (SN_{x_2}(X - \{x_1\}))
\]

\[
\leq \bigwedge_{x_1 \neq x_2} SN_{x_1}(X - \{x_2\})
\]

\[
= \bigwedge_{x_2 \in X} \bigwedge_{x_1 \in X - \{x_2\}} SN_{x_1}(X - \{x_2\})
\]

\[
= \bigwedge_{x_2 \in X} S\tau(X - \{x_2\})
\]

\[
= \bigwedge_{x \in X} S\tau(X - \{x\}) = \bigwedge_{x \in X} SF_\tau(|x|).
\]

Theorem 2.6 Let \((X, \tau) \in \Omega\). If \(L\) satisfies the double negation law, then,

\[
\text{semi} - T_0(X, \tau) = \bigwedge_{x \neq y} ((\text{semi} - \text{semi} - cl_\tau(|y|)(x) \rightarrow \bot) \lor (\text{semi} - cl_\tau(|x|)(y) \rightarrow \bot)).
\]

Proof.

\[
\text{semi} - T_0(X, \tau) = \bigwedge_{x \neq y} ((\bigvee_{x \in A} SN_x(A)) \lor (\bigvee_{x \in A} SN_y(A)))
\]

\[
= \bigwedge_{x \neq y} ((SN_x(X - \{y\}) \lor (SN_y(X - \{x\}))
\]

\[
= \bigwedge_{x \neq y} ((\text{semi} - cl_\tau(|y|)(x) \rightarrow \bot) \lor (\text{semi} - cl_\tau(|x|)(y) \rightarrow \bot)).
\]
Theorem 2.7. Let \((X, \tau) \in \Omega\) and let \(A\) be a finite subset of \(X\). If \(L\) satisfies the completely distributive, then,

\[
\text{semi} - T_1(X, \tau) \leq \bigwedge_{y \in X} SN_y((X - A) \cup \{y\}).
\]

Proof. Now,

\[
\bigwedge_{y \in X - A} SN_y((X - A) \cup \{y\}) = \bigwedge_{y \in X - A} SN_y(X - A)
\]

\[
= \bigwedge_{y \in X - A} SN_y(X - \bigcup_{x \in A} \{x\}) = \bigwedge_{y \in X - A} SN_y(\bigcap_{x \in A} (X - \{x\}))
\]

\[
= \bigwedge_{y \in X - A} (\bigwedge_{x \in A} SN_y(X - \{x\})) \geq \bigwedge_{x \not\in y} SN_y(X - \{x\}) , \text{ and}
\]

\[
\bigwedge_{y \in A} SN_y((X - A) \cup \{y\}) = \bigwedge_{y \in A} SN_y(X - (A - \{y\}))
\]

\[
= \bigwedge_{y \in A} SN_y(X - (\bigcup_{x \in A - \{y\}} \{x\})) = \bigwedge_{y \in A} SN_y(\bigcap_{x \in A - \{y\}} (X - \{x\}))
\]

\[
= \bigwedge_{y \in A} (\bigwedge_{x \in A - \{y\}} SN_y(X - \{x\})) \geq \bigwedge_{x \not\in y} SN_y(X - \{x\}).
\]

Then \(\bigwedge_{y \in X} SN_y((X - A) \cup \{y\}) \geq \bigwedge_{x \not\in y} SN_y(X - \{x\})\)

\[
= \bigwedge_{x \in X} (\bigwedge_{y \in X - \{x\}} SN_y(X - \{x\})) = \bigwedge_{x \in X} SF_y(\{x\}) = \bigwedge_{x \in X} \tau(x) = \text{semi} - T_1(X, \tau).
\]

Definition 2.4. Let \((X, \tau) \in \Omega\) and let \(A \subseteq X\). The \(L\)-fuzzifying derived set of \(A\) is denoted by \(\text{semi} - d_\tau(A) \in L^X\) and defined as follows:

\[
\text{semi} - d_\tau(A)(x) = SN_x((X - A) \cup \{x\}) \rightarrow \perp \forall x \in X.
\]

Theorem 2.8. Let \((X, \tau) \in \Omega\) and let \(A\) be a finite subset of \(X\). If \(L\) satisfies the completely distributive and the double negation law, then \(\text{semi} - T_1(X, \tau) \leq [[\text{semi} - d_\tau(A), 1_\varphi]]\).

Proof. It follows from Theorem 2.7 and since

\[
[[\text{semi} - d_\tau(A), 1_\varphi]] = \bigwedge_{y \in X} (\text{semi} - d_\tau(A)(y) \rightarrow 1_\varphi(y)) \land (1_\varphi(y) \rightarrow \text{semi} - d_\tau(A)(y))
\]

\[
= \bigwedge_{y \in X} (\text{semi} - d_\tau(A)(y) \rightarrow 1_\varphi(y)) = \bigwedge_{y \in X} SN_y((X - A) \cup \{y\}).
\]
Definition 2.5 Let \((X, \tau) \in \Omega\) and let \(x \in X\). Then \(S_\beta x \in L^p(X)\) is called a semi-local base for \(\tau\) at \(x \in X\) iff the following conditions are satisfied:

1. \([[S_\beta x, SN_x]] = \top\),
2. \(SN_x(A) \leq \bigvee_{x \in H} S_\beta x(H)\),
3. If \(A \subseteq B\), then \(S_\beta x(A) \leq S_\beta x(B)\) \(\forall x \in X\).

Lemma 2.4 Let \((X, \tau) \in \Omega\) and let \(S_\beta x\) be a semi-local base for \(\tau\) at \(x \in X\).

Then for any \(A \subseteq X\), \(SN_x(A) = \bigvee_{x \in H \subseteq A} S_\beta x(H)\).

Proof. From condition (2) in Definition 2.5, we have \(SN_x(A) \leq \bigvee_{x \in H \subseteq A} S_\beta x(H)\).

Now, we need to prove that \(SN_x(A) \geq \bigvee_{x \in H \subseteq A} S_\beta x(H)\).

From condition (1) in Definition 2.5, we have \(S_\beta x(K) \leq SN_x(K) \forall K \in P(X)\). So, \(\forall H \in P(X)\)

\(s.t. \; x \in H \subseteq A, SN_x(A) \geq SN_x(H)\) so that, \(SN_x(A) \geq \bigvee_{x \in H \subseteq A} S_\beta x(H)\).

Theorem 2.9. Let \((X, \tau) \in \Omega, x \in X\) and let \(S_\beta x\) be a semi-local base for \(\tau\) at \(x \in X\). If \(L\) satisfies the completely distributive, then, \(semi - T_1(X, \tau) = \bigwedge_{xy} (\bigvee_{y \in X - A} S_\beta x(A))\).

Proof. Let \(H \in P(X)\) \(s.t. \; x \in H \subseteq X - \{y\}\). Then \(y \notin H\) so that \(\bigvee_{y \notin A} S_\beta x(A) \geq S_\beta x(H)\)

so that \(\bigvee_{y \notin A} S_\beta x(A) \geq \bigvee_{x \in H \subseteq X - \{y\}} S_\beta x(H) = SN_x(X - \{y\})\) so that

\[
\bigwedge_{xy} (\bigvee_{y \notin A} S_\beta x(A)) \geq \bigwedge_{xy} SN_x(X - \{y\}) = \bigwedge_{y \notin X} \bigwedge_{x \in X - \{y\}} SN_x(X - \{y\})
\]

\[
= \bigwedge_{y \notin X} SN_x(X - \{y\}) = \bigwedge_{y \notin X} SF_{\tau}(\{y\}) = semi - T_1(X, \tau).
\]

Now, we need only to prove that

\(semi - T_1(X, \tau) \geq \bigwedge_{xy} (\bigvee_{y \in X - A} S_\beta x(A))\). From condition (1) in Definition 2.5,
we have \((\bigvee_{y \in X - A} S\beta_x(A)) \land (\bigvee_{x \in X - M} S\beta_y(M))\)

\[
\leq (\bigvee_{y \in X - A} SN_x(A)) \land (\bigvee_{x \in X - M} SN_y(M)) = SH(x, y).
\]

Then \(\bigwedge_{x \in X} (\bigvee_{y \in X - A} S\beta_x(A)) \leq \bigwedge_{x \in X} SH(x, y) = semi - T_1(X, \tau)\).

**Theorem 2.10.** Let \((X, \tau) \in \Omega, x \in X\), and let \(S\beta_x\) be a semi-local base for \(\tau\) at \(x\). If the meet is distributive over arbitrary joins, then \(semi - T_2(X, \tau) = \bigwedge_{x \in X} \bigvee_{H \in P(X)} (S\beta_x(H) \land SN_y(X - H))\).

**Proof.**

\[
\bigwedge_{x \in X} \bigvee_{H \in P(X)} (S\beta_x(H) \land SN_y(X - H)) = \bigwedge_{x \in X} \bigvee_{H \in \mathcal{P}(X)} (S\beta_x(H) \land (\bigvee_{y \in \mathcal{C} \subseteq X - H} S\beta_y(C)))
\]

\[
= \bigwedge_{x \in X} \bigvee_{H \in \mathcal{P}(X)} \bigwedge_{y \in \mathcal{C} \subseteq X - H} (S\beta_x(H) \land S\beta_y(C)) = \bigwedge_{x \in X} \bigvee_{H \cap C \neq \emptyset} \bigvee_{x \in C \subseteq H} \bigwedge_{y \in \mathcal{C} \subseteq C} (S\beta_x(D) \land S\beta_y(E))
\]

\[
= \bigwedge_{x \in X} \bigvee_{H \cap C \neq \emptyset} ((SN_x(H) \land SN_y(C)) = semi - T_2(X, \tau).
\]

**Theorem 2.11.** Let \((X, \tau) \in \Omega, x \in X\), and let \(S\beta_x\) be a semi-local base for \(\tau\) at \(x\). If the meet is distributive over arbitrary joins and the double negation law is satisfied in \(L\), then \(semi - T_2(X, \tau) = \bigwedge_{x \in X} \bigvee_{H \in P(X)} (S\beta_x(H) \land (semi - cl_\tau(H)(y) \rightarrow \bot))\).

**Proof.**

It follows from Theorem 2.10 and \([\text{by Proposition 2.1. (1). } SN_x(A) = semi - cl_\tau(X - A)(x) \rightarrow \bot]\).

**Definition 2.6.** The unary \(L\)-predicates \(semi - T_3^i \in L^\Omega\) where \(i = 1, 2\) are defined as follows:

\[
semi - T_3^1(X, \tau) = \bigwedge_{x \in D} (SF_\tau(D) \rightarrow \bigvee_{A \in P(X)} (SN_x(A) \land (\bigwedge_{y \in D} (semi - cl_\tau(A)(y) \rightarrow \bot))))
\]

\[
semi - T_3^2(X, \tau) = \bigwedge_{x \in A} (S_\tau(A) \rightarrow \bigvee_{B \in P(X)} (SN_x(B) \land [[semi - cl_\tau(B), A]])
\]

**Theorem 2.12.** If \(L\) satisfies the completely distributive law and the double negation law, then for each \((X, \tau) \in \Omega, semi - T_3(X, \tau) = semi - T_3^1(X, \tau)\).

**Proof.** Now,
semi - \( T_3^1(X, \tau) = \bigwedge_{x \in D}(SF_\tau(D) \rightarrow \bigvee_{A \in P(X)}(SN_x(A) \land \left( \land_{y \in D}(\text{semi} - I_\tau(y) \rightarrow\bot) \right)) \)

= \bigwedge_{x \in D}(S\tau(X - D) \rightarrow \bigvee_{A \in P(X)}(SN_x(A) \land \left( \land_{y \in D} SN_y(X - A) \right))) \text{ and }

semi - \( T_3(X, \tau) = \bigwedge_{x \in D}(S\tau(X - D) \rightarrow \bigvee_{A \in P(X)}(SN_x(A) \land S\tau(B))) \).

So, the result holds if we prove that

\[ \bigvee_{A \in P(X)}(SN_x(A) \land \left( \land_{y \in D} SN_y(X - A) \right)) = \bigvee_{A \in P(X)}(SN_x(A) \land S\tau(B)). \] (*)

In the left side of (*) if \( A \cap D \neq \phi \), \( \exists y \in D \) s.t. \( y \notin X - A \)

so that \( \land_{y \in D} SN_y(X - A) = \bot \). Second,

\[ \bigvee_{A \in P(X)}(SN_x(A) \land \left( \land_{y \in D} SN_y(X - A) \right)) = \bigvee_{A \in P(X)}(SN_x(A) \land \left( \land_{y \in D} SN_y(\tau(B)) \right)). \]

Now we prove that

\[ \bigwedge_{y \in D} \bigvee_{y \in B \subseteq X - A} S\tau(B) = \bigvee_{A \in P(X)}(SN_x(A) \land \left( \land_{y \in D} \bigvee_{y \in B \subseteq X - A} S\tau(B) \right)). \]

Let \( y \in D \). Assume \( S = \{ H \in P(X) \mid H \cap A = \phi \text{ and } D \subseteq H \} \)

and \( \varphi_y = \{ M \in P(X) \mid y \in M \subseteq X - A \} \). Then \( S \subseteq \varphi_y \)

so that \( \bigvee_{B \in \varphi_y} S\tau(B) \geq \bigvee_{B \in S} S\tau(B) \) so that \( \land_{y \in D} \bigvee_{y \in B \subseteq X - A} S\tau(B) \geq \bigvee_{A \in P(X)}(SN_x(A) \land \left( \land_{y \in D} \bigvee_{y \in B \subseteq X - A} S\tau(B) \right)). \)

Let \( \varphi^*_y = \{ \tau(M) \mid M \in \varphi_y \} \). Then \( \land_{y \in D} \bigvee \varphi^*_y = \bigvee_{y \in D} \bigvee_{B \in \varphi_y} S\tau(B) \)

= \( \bigvee_{f \in \prod_{y \in D} \varphi^*_y} \land_{y \in D} f(y) \). Then for each \( f \in \prod_{y \in D} \varphi^*_y \), \( \exists K = \cup \{ f(y) \mid f(y) \in \varphi_y, \ y \in D \} \)

s.t. \( D \subseteq K \subseteq X - A \) and \( \land_{y \in D} f(y) \leq \tau(\cup \{ f(y) \mid f(y) \in \varphi_y, \ y \in D \}) = S\tau(K) \)
so that $\bigwedge_{y \in D} f(y) \leq S\tau(K) \leq \bigvee_{A \cap B = \emptyset, D \subseteq B} S\tau(B)$ so that $\bigwedge_{y \in D} \bigvee_{y \in B \subseteq X - A} S\tau(B)$

$= \bigvee_{f \in \Pi_{y \in D} \Phi(f)} \bigwedge_{y \in D} f(y) \leq \bigvee_{A \cap B = \emptyset, D \subseteq B} S\tau(B)$.

**Theorem 2.13** If $L$ satisfies the completely distributive law, then for any $(X, \tau) \in \Omega$,

semi - $T_3(X, \tau)$ = semi - $T_3^2(X, \tau)$.

**Proof.**

semi - $T_3(X, \tau)$ = semi - $T_3^1(X, \tau)$

$= \bigwedge_{x \in D}(S\tau(D) \rightarrow \bigvee_{A \in P(X)} (SN_x(A) \land (\bigwedge_{y \in D} (semi - cl_\tau(A)(y) \rightarrow \bot))))$

$= \bigwedge_{x \in X - B}(S\tau(X - B) \rightarrow \bigvee_{A \in P(X)} (SN_x(A) \land (\bigwedge_{y \in X - B} (semi - cl_\tau(A)(y) \rightarrow \bot))))$

$= \bigwedge_{x \in B}(S\tau(B) \rightarrow \bigvee_{A \in P(X)} (SN_x(A) \land (\bigwedge_{y \in X} (semi - cl_\tau(A)(y) \rightarrow B(y)))))$

$= \bigwedge_{x \in B}(S\tau(B) \rightarrow \bigvee_{A \in P(X)} (SN_x(A) \land ([semi - cl_\tau(A), B[I]]) = T_3^2(X, \tau))$.

Note that

$\bigwedge_{y \in X} (semi - cl_\tau(A)(y) \rightarrow B(y))$

$= (\bigwedge_{y \in B} (semi - cl_\tau(A)(y) \rightarrow \top)) \land (\bigwedge_{y \in X - B} (semi - cl_\tau(A)(y) \rightarrow \bot))$

$= \top \land \bigwedge_{y \in X - B} (semi - cl_\tau(A)(y) \rightarrow \bot) = \bigwedge_{y \in X - B} (semi - cl_\tau(A)(y) \rightarrow \bot)$.

3. Relations among separation axioms

**Theorem 3.1.** If $L$ satisfies the completely distributive law, then for any $(X, \tau) \in \Omega$,

semi - $T_3(X, \tau) \ast semi - T_1(X, \tau) \leq semi - T_2(X, \tau)$.

**Proof.** Now,
\[ \wedge_{x \neq y}(S\tau(x - \{y\}) \rightarrow \lor_{A \cap B = \emptyset, y \in B}(\wedge_{y \in B}(SN_x(A) \land SN_y(B)))) \]

\[ \leq \wedge_{x \neq y}(\lor_{y \in X} S\tau(x - \{y\}) \rightarrow \lor_{A \cap B = \emptyset, y \in B}(\wedge_{y \in B}(SN_x(A) \land SN_y(B)))) \]

\[ = \wedge_{x \neq y}(\text{semi} - T_1(X, \tau)) \rightarrow \lor_{A \cap B = \emptyset, y \in B}(\wedge_{y \in B}(SN_x(A) \land SN_y(B)))) \]

\[ = \text{semi} - T_1(X, \tau) \rightarrow \wedge_{x \neq y}(\lor_{A \cap B = \emptyset, y \in B}(\wedge_{y \in B}(SN_x(A) \land SN_y(B)))) \]

\[ \leq \text{semi} - T_1(X, \tau) \rightarrow \wedge_{x \neq y}(\lor_{A \cap B = \emptyset}(SN_x(A) \land SN_y(B))) \]

\[ = \text{semi} - T_1(X, \tau) \rightarrow \text{semi} - T_2(X, \tau). \]

Since \( \text{semi} - T_3(X, \tau) = \wedge_{x \neq y}(S\tau(x - D) \rightarrow \lor_{A \cap B = \emptyset, y \in B}(SN_x(A) \land S\tau(B))) \]

\[ = \wedge_{x \neq y}(S\tau(x - D) \rightarrow \lor_{A \cap B = \emptyset, D \subseteq B}(SN_x(A) \land (\lor_{y \in B}(SN_y(B)))) \]

\[ \leq \wedge_{x \neq y}(\lor_{y \in D}(S\tau(x - \{y\}) \rightarrow \lor_{A \cap B = \emptyset, y \in B}(\wedge_{y \in B}(SN_x(A) \land SN_y(B)))) \]

\[ = \wedge_{x \neq y}(\lor_{y \in D}(S\tau(x - \{y\}) \rightarrow \lor_{A \cap B = \emptyset, y \in B}(\wedge_{y \in B}(SN_x(A) \land SN_y(B)))) \]

then from above, \( \text{semi} - T_3(X, \tau) \leq \text{semi} - T_1(X, \tau) \rightarrow \text{semi} - T_2(X, \tau) \)

so that \( \text{semi} - T_3(X, \tau) \ast \text{semi} - T_1(X, \tau) \leq \text{semi} - T_2(X, \tau) \).

**Theorem 3.2.** If \( L \) satisfies the completely distributive law, then for any \((X, \tau) \in \Omega, \)

\[ \text{semi} - T_4(X, \tau) \ast \text{semi} - T_1(X, \tau) \leq \text{semi} - T_3(X, \tau). \]

**Proof.**

\[ \text{semi} - T_4(X, \tau) = \wedge_{E \subseteq D = \emptyset}((SF_{\tau}(E) \land SF_{\tau}(D)) \rightarrow \lor_{E \subseteq A, D \subseteq B, A \cap B = \emptyset}(S\tau(A) \land S\tau(B))) \]

\[ \leq \wedge_{x \neq D}(SF_{\tau}(\{x\}) \land SF_{\tau}(D)) \rightarrow \lor_{x \in A, D \subseteq B, A \cap B = \emptyset}(S\tau(A) \land S\tau(B)) \]

\[ = \wedge_{x \neq D}(SF_{\tau}(\{x\}) \land SF_{\tau}(D)) \rightarrow \lor_{D \subseteq B, A \cap B = \emptyset}(\lor_{x \in A} S\tau(A) \land S\tau(B)) \]

\[ \leq \wedge_{x \neq D}(SF_{\tau}(\{x\}) \land SF_{\tau}(D)) \rightarrow \lor_{D \subseteq B, A \cap B = \emptyset}(\lor_{x \in K \subseteq A} S\tau(K) \land S\tau(B)) \]
\[= \bigwedge_{x \in D} (SF_\tau(\{x\}) \wedge SF_\tau(D)) \to \bigvee_{D \subseteq B, A \cap B = \emptyset} (SN_x(A) \wedge S\tau(B)) \]

\[\leq \bigwedge_{x \in D} (\left((\bigwedge_{x \in X} SF_\tau(\{x\})) \wedge SF_\tau(D)\right) \to \bigvee_{D \subseteq B, A \cap B = \emptyset} (SN_x(A) \wedge S\tau(B)) \]

\[= \bigwedge_{x \in D} (\left(((\text{semi} - T_1(X, \tau) \wedge SF_\tau(D)) \to \bigvee_{D \subseteq B, A \cap B = \emptyset} (SN_x(A) \wedge S\tau(B)) \right) \)

\[\leq \bigwedge_{x \in D} (\left((\text{semi} - T_1(X, \tau) \ast SF_\tau(D)) \to \bigvee_{D \subseteq B, A \cap B = \emptyset} (SN_x(A) \wedge S\tau(B)) \right) \)

\[\leq \text{semi} - T_1(X, \tau) \to \bigwedge_{x \in D} (SF_\tau(D) \to \bigvee_{D \subseteq B, A \cap B = \emptyset} (SN_x(A) \wedge S\tau(B)) \)

\[= \text{semi} - T_1(X, \tau) \to \text{semi} - T_3(X, \tau) \]

so that \(\text{semi} - T_4(X, \tau) \ast \text{semi} - T_1(X, \tau) \leq \text{semi} - T_3(X, \tau)\).

(Indeed, put \(\text{semi} - T_1(X, \tau) = \alpha, \ j = (x, D)\),

\[J = \{(x, D) \mid x \in X, \ D \in P(X), \ x \notin D\}, \]

\[B_{(x,D)} = (SF_\tau(D), \]

\[M_{(x,D)} = \bigvee_{D \subseteq B, A \cap B = \emptyset} (SN_x(A) \wedge S\tau(B)) \]

and \(A_j = \{\lambda \mid \lambda \ast \alpha \leq B_j \to M_j\}. \)

Then \(\bigwedge_{j \in J} ((\alpha \ast B_j) \to M_j) = \bigwedge_{j \in J} \bigvee_{\lambda \ast (\alpha \ast B_j) \leq M_j} \lambda \)

\[= \bigwedge_{j \in J} \bigvee_{\lambda \ast \alpha \leq B_j \leq M_j} \lambda = \bigwedge_{j \in J} \bigvee_{\lambda \ast \alpha \leq (B_j \rightarrow M_j)} \lambda \]

\[= \bigwedge_{j \in J} \bigvee A_j = \bigvee_{f \in \prod_{j \in J} A_j} \bigwedge_{j \in J} f(j). \]

Now, \(\forall f \in \prod_{j \in J} A_j\) there exists
\[ K_f = \bigwedge_{j \in J} f(j) \text{ s.t. } K_f \ast \alpha = (\bigwedge_{j \in J} f(j)) \ast \alpha \]

\[ \leq \bigwedge_{j \in J} (f(j) \ast \alpha) \leq \bigwedge_{j \in J} (B_j \rightarrow M_j) \]. Then \[ \bigwedge_{j \in J} ((\alpha \ast B_j) \rightarrow M_j) \leq \bigvee_{f \in \prod_{j \in J} A_j} K_f \]

\[ \leq \bigvee_{\lambda \in S} \bigwedge_{j \in J} (B_j \rightarrow M_j) \lambda = \alpha \rightarrow \bigwedge_{j \in J} (B_j \rightarrow M_j) \).

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References