On $\pi ge$-closed sets and related topics

Burcu Sümbül Ayhan, Murad Özkoc

$^a$Muğla Sıtki Koçman University, Faculty of Science, Department of Mathematics, 48000 Menteşe, Muğla, Turkey.

Abstract

The aim of this paper is to introduce and investigate $\pi ge$-continuous functions and $\pi ge$-irresolute functions via $\pi ge$-closed sets which are defined by us. The notion of $\pi ge$-continuity is a weaker than $ge$-continuity introduced by Ayhan and Özkoc [3]. We obtain some properties about $\pi ge$-closed sets and $\pi ge$-continuous functions and $\pi ge$-irresolute functions and $\pi ge$-compactness.

Keywords: $\pi ge$-closed, $\pi ge$-continuity, $\pi ge$-irresoluteness, $\pi ge$-compactness.

2010 MSC: 54C08, 54C10, 54D10, 54D30.

1. Introduction

Continuity on topological spaces, as significant and fundamental subject in the study of topology, has been researched by several mathematicians. Many investigations related to generalized closed sets have been published various forms of generalized continuity types have been introduced. The study of generalized closed sets in a topological space was initiated by Levine [12] in 1970. Next, $ge$-closed sets and $ge$-continuous functions were defined and studied by Ayhan and Özkoc [3] in 2014. In 1968, Zaitsev [22] defined the concept of $\pi$-closed sets. Later Dontchev and Noiri [5] introduced the notion of $\pi g$-closed sets.

2. Preliminaries

Throughout the present paper, $X$ and $Y$ represent topological spaces. For a subset $A$ of a space $X$, $cl(A)$ and $int(A)$ denote the closure of $A$ and the interior of $A$, respectively. The family of all closed sets of $X$ is denoted $C(X)$. A subset $A$ of a space $(X, \tau)$ is called regular open [20] (resp. regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The family of all regular open (resp. regular closed) sets of $X$ will be denoted by $RO(X)$ (resp. $RC(X)$). The finite union of regular open sets is called $\pi$-open [22]. The complement of a $\pi$-open set is called $\pi$-closed. The family of all $\pi$-open (resp. $\pi$-closed) sets of $X$ will be denoted by $\pi O(X)$ (resp. $\pi C(X)$). A subset $A$ is said to be $\delta$-open [21] if for each $x \in A$ there exists a regular open set $B$ such that $x \in B \subset A$. A point $x$ of $X$ is called a $\delta$-cluster point of $A$ if $A \cap int(cl(U)) \neq \emptyset$ for each open set $U$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$ and is denoted by $cl_\delta(A)$. The set $\{x|(\exists U \in RO(X)(x \in U \subset A))\}$ is called the $\delta$-interior of $A$ and is denoted by $int_\delta(A)$.

Received: 10 November 2015   Accepted: 31 December 2015

http://dx.doi.org/10.20454/jast.2016.1021

2090-8288 ©2016 Modern Science Publishers. All rights reserved.
Definition 2.1. A subset $A$ of a space $X$ is called:

1. $\alpha$-closed [16] if $\text{cl}(\text{int}(\pi(A))) \subset A$,
2. preclosed [15] if $\text{cl}(\text{int}(A)) \subset A$,
3. $b$-closed [1] if $\text{cl}(\pi(A)) \cap \text{int}(\pi(A)) \subset A$,
4. $e$-closed [8] if $\text{cl}(\text{int}_b(A)) \cap \text{int}(\text{cl}_b(A)) \subset A$.

The intersection of all $e$-closed sets containing $A$ is called the $e$-closure of $A$ and is denoted by $e\text{-cl}(A)$. The union of all $e$-open sets contained in $A$ is called the $e$-interior of $A$ and is denoted by $e\text{-int}(A)$. The family of all $e$-closed (resp. $e$-open) subsets of $X$ is denoted by $eC(X)$ (resp. $eO(X)$).

Definition 2.2. A subset $A$ of a space $X$ is called:

1. generalized closed [12] (br. $g$-closed) if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is open,
2. $\pi$-generalized closed [5] (br. $\pi g$-closed) if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is $\pi$-open,
3. $\pi$-generalized $b$-closed [19] (br. $\pi gb$-closed) if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is $\pi$-open,
4. $\alpha$-generalized closed [13] (br. $ag$-closed) if $\text{acl}(A) \subset U$ whenever $A \subset U$ and $U$ is open,
5. generalized preclosed [14] (br. $gp$-closed) if $\text{pcl}(A) \subset U$ whenever $A \subset U$ and $U$ is open,
6. generalized $b$-closed [17] (br. $gb$-closed) if $\text{bcl}(A) \subset U$ whenever $A \subset U$ and $U$ is open,
7. generalized $e$-closed [3] (br. $ge$-closed) if $e\text{-cl}(A) \subset U$ whenever $A \subset U$ and $U$ is open.

Definition 2.3. A function $f : X \to Y$ is called:

1. $b$-continuous [9] if $f^{-1}[V]$ is $b$-closed in $X$ for every closed set $V$ of $Y$,
2. $\pi$-generalized $b$-continuous [17] (br. $\pi gb$-continuous) if $f^{-1}[V]$ is $gb$-closed in $X$ for every closed set $V$ of $Y$,
3. $\pi$-generalized $b$-continuous [19] (br. $\pi gb$-continuous) if $f^{-1}[V]$ is $gb$-closed in $X$ for every closed set $V$ of $Y$,
4. precontinuous [15] if $f^{-1}[V]$ is preclosed in $X$ for every closed set $V$ of $Y$,
5. generalized precontinuous [2] (br. $gp$-continuous) if $f^{-1}[V]$ is $gp$-closed in $X$ for every closed set $V$ of $Y$,
6. $e$-continuous [8] if $f^{-1}[V]$ is $e$-closed in $X$ for every closed set $V$ of $Y$,
7. generalized $e$-continuous [3] (br. $ge$-continuous) if $f^{-1}[V]$ is $ge$-closed in $X$ for every closed set $V$ of $Y$,
8. $e$-irresolute [7] if $f^{-1}[V]$ is $e$-closed in $X$ for every $e$-closed set $V$ of $Y$,

The following basic properties of $e$-closure are useful in the sequel:

Lemma 2.4. [8] For any subsets $A$ and $B$ of a space $X$, the following hold:

1. $e\text{-cl}(A) = A \cup [\text{int}(\text{cl}_b(A)) \cap \text{cl}(\text{int}_b(A))]$,
2. $e\text{-cl}(X \setminus A) = X \setminus e\text{-int}(A)$,
3. $x \in e\text{-cl}(A)$ if and only if $A \cap U \neq \emptyset$ for every $U \in eO(X, x)$,
4. $A \in eC(X)$ if and only if $A = e\text{-cl}(A)$.

3. $\pi ge$-closed sets

Definition 3.1. Let $X$ be a topological space. A subset $A$ of $X$ is called $\pi$-generalized $e$-closed set (briefly $\pi ge$-closed) if $e\text{-cl}(A) \subset U$ whenever $A \subset U$ and $U$ is $\pi$-open. The family of all $\pi ge$-closed subsets of $X$ will be denoted by $\pi geC(X)$.

Theorem 3.2. For a topological space $X$ the followings hold:

1. Every closed set is $\pi ge$-closed.
2. Every $g$-closed set is $\pi ge$-closed.
(3) Every $a$-closed set is $\pi ge$-closed.
(4) Every $ag$-closed set is $\pi ge$-closed.
(5) Every preclosed set is $\pi ge$-closed.
(6) Every $gp$-closed set is $\pi ge$-closed.
(7) Every $e$-closed set is $\pi ge$-closed.
(8) Every $ge$-closed set is $\pi ge$-closed.

Proof. Obvious.

Remark 3.3. From the Definition 3.1 and Theorem 3.2 we have the following diagram:

\[
\begin{array}{ccccccc}
\text{closed} & \rightarrow & a\text{-closed} & \rightarrow & \text{preclosed} & \rightarrow & e\text{-closed} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{g\text{-closed}} & \rightarrow & ag\text{-closed} & \rightarrow & gp\text{-closed} & \rightarrow & ge\text{-closed} & \rightarrow & \pi ge\text{-closed} \\
& & & \downarrow & & \\
& & & gb\text{-closed} & \rightarrow & \pi gb\text{-closed}
\end{array}
\]

However, none of these implications is reversible as shown by the following examples.

Example 3.4. $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, [a], [c], [d], [a, c, d], [b, c, d]\}$. Then the set $\{a, c, d\}$ is $\pi ge$-closed but it is not $ge$-closed.

Example 3.5. $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, [a], [c], [a, c], [c, d], [a, c, d]\}$. Then the set $\{a, d\}$ is $\pi gb$-closed but it is not $\pi ge$-closed.

Theorem 3.6. Let $X$ be a topological space. If $A$ is $\pi$-open and $\pi ge$-closed, then $A$ is $e$-closed.

Proof. Let $A$ be $\pi$-open and $\pi ge$-closed. Let $A \subseteq B$ where $A$ is $\pi$-open. Since $A$ is $\pi ge$-closed, $e\text{-}\text{cl}(A) \subseteq A$. Then $A = e\text{-}\text{cl}(A)$. Hence is $e$-closed.

Theorem 3.7. Let $A$ be $\pi ge$-closed in $X$. Then $e\text{-}\text{cl}(A) \setminus A$ does not contain any non-empty $\pi$-closed set.

Proof. Let $F$ be $\pi$-closed set such that $F \subseteq e\text{-}\text{cl}(A) \setminus A$. Then $F \subseteq X \setminus A$ implies $A \subseteq X \setminus F$. Therefore $e\text{-}\text{cl}(A) \subseteq X \setminus F$. That is $F \subseteq X \setminus e\text{-}\text{cl}(A)$. Hence $F \subseteq e\text{-}\text{cl}(A) \cap (X \setminus e\text{-}\text{cl}(A))$. This shows that $F = \emptyset$ which is a contradiction.

Corollary 3.8. Let $A$ be $\pi ge$-closed in $X$. Then $A$ is $e$-closed iff $e\text{-}\text{cl}(A) \setminus A$ is $\pi$-closed.

Proof. Necessity: Let $A$ be $e$-closed. Then $e\text{-}\text{cl}(A) = A$. This implies $e\text{-}\text{cl}(A) \setminus A = \emptyset$ which is $\pi$-closed.

Sufficiency: Assume $e\text{-}\text{cl}(A) \setminus A$ is $\pi$-closed and $A$ is $\pi ge$-closed. By Theorem 3.7, $e\text{-}\text{cl}(A) \setminus A = \emptyset$. Hence $e\text{-}\text{cl}(A) = A$.

Remark 3.9. Let $X$ be a topological space. Finite intersection of $\pi ge$-closed sets need not be $\pi ge$-closed.

Example 3.10. Consider $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, [a], [b], [a, b], [a, b, c], [a, b, d]\}$. Let $A = \{a, b, c\}$, $B = \{a, b, d\}$. Here $A$ and $B$ are $\pi ge$-closed but $A \cap B = \{a, b\}$ is not $\pi ge$-closed.

Remark 3.11. Let $X$ be a topological space. Finite union of $\pi ge$-closed sets need not be $\pi ge$-closed.

Example 3.12. Consider $X = \{a, b, c\}$, $\tau = \{\emptyset, X, [a], [b], [a, b]\}$. Let $A = \{a\}$, $B = \{b\}$. Here $A$ and $B$ are $\pi ge$-closed but $A \cup B = \{a, b\}$ is not $\pi ge$-closed.

Lemma 3.13. [3] If $D(A) = D_e(A)$, then we have $\text{cl}(A) = e\text{-}\text{cl}(A)$. 
Theorem 3.14. Let $A$ and $B$ be $\pi ge$-closed in $X$ such that $D(A) \subset D_1(A)$ and $D(B) \subset D_2(B)$. Then $A \cup B$ is $\pi ge$-closed.

Proof. Let $U$ be an $\pi$-open set such that $A \cup B \subset U$. Then since $A$ and $B$ be $\pi ge$-closed sets we have $e-cl(A) \subset U$ and $e-cl(B) \subset U$. Since $D(A) \subset D_1(A)$, $D(B) \subset D_2(B)$. By Lemma 3.13, $cl(A) = e-cl(A)$ and $cl(B) = e-cl(B)$. Thus $e-cl(A \cup B) \subset cl(A \cup B) = cl(A) \cup cl(B) = e-cl(A) \cup e-cl(B) \subset U$, which implies that $A \cup B$ is $\pi ge$-closed.

Theorem 3.15. Let $X$ be a topological space. If $A$ is $\pi ge$-closed set and $B$ is any such that $A \subset B \subset e-cl(A)$, then $B$ is $\pi ge$-closed set.

Proof. Let $B \subset U$ and $U$ is $\pi$-open set. Given $A \subset B$. Then $A \subset U$. Since $A$ is $\pi ge$-closed, $A \subset U$ implies $e-cl(A) \subset U$. By assumption it follows that $e-cl(B) \subset e-cl(A) \subset U$. Hence $B$ is a $\pi ge$-closed set.

4. $\pi ge$-open Sets

Definition 4.1. Let $X$ be a topological space. A subset $A$ of $X$ is called $\pi$-generalized $e$-open set (briefly $\pi ge$-open) if and only if its complement is $\pi ge$-closed. The family of all $\pi ge$-open subsets of $X$ will be denoted by $\pi geO(X)$.

Theorem 4.2. If $A \subset X$ is $\pi ge$-open iff $F \subset e-int(A)$ whenever $F$ is $\pi$-closed and $F \subset A$.

Proof. Necessity: Let $A$ be a $\pi ge$-open. Let $F$ be $\pi$-closed and $F \subset A$. Then $X \setminus A \subset X \setminus F$ where $X \setminus F$ is $\pi$-open. By assumption, $e-cl(X \setminus A) \subset X \setminus F$. Then $X \setminus e-int(A) \subset X \setminus F$. Thus $F \subset e-int(A)$.

Sufficiency: Suppose $F$ is $\pi$-closed and $F \subset A$ such that $F \subset e-int(A)$. Let $X \setminus A \subset U$ where $U$ is $\pi$-open. Then $X \setminus U \subset A$ where $X \setminus U$ is $\pi$-closed. By hypothesis, $X \setminus U \subset e-int(A)$ and so $X \setminus e-int(A) \subset U$. Hence $e-cl(X \setminus A) \subset U$. Thus $X \setminus A$ is $\pi ge$-closed and $A$ is $\pi ge$-open.

Theorem 4.3. Let $X$ be a topological space and $A, B \subset X$. If $e-int(A) \subset B \subset A$ and $A$ is $\pi ge$-open, then $B$ is $\pi ge$-open.

Proof. Let $e-int(A) \subset B \subset A$. Thus $X \setminus A \subset X \setminus B \subset e-cl(X \setminus A)$. Since $X \setminus A$ is $\pi ge$-closed. By Theorem 3.15, $(X \setminus A) \subset (X \setminus B) \subset e-cl(X \setminus A)$ implies $X \setminus B$ is $\pi ge$-closed.

Lemma 4.4. Let $X$ be a topological space and $A \subset X$. $e-int(e-cl(A) \setminus A) = \emptyset$.

Theorem 4.5. Let $(X, \tau)$ be a topological space. If $A \subset X$ is $\pi ge$-closed, then $e-cl(A) \setminus A$ is $\pi ge$-open.

Proof. Assume $A$ is $\pi ge$-closed. Let $F$ be $\pi$-closed set and $F \subset e-cl(A) \setminus A$. By Theorem 3.7, $F = \emptyset$. By Lemma 4.4, $e-int(e-cl(A) \setminus A) = \emptyset$. Thus $F \subset e-int(e-cl(A) \setminus A)$. Hence $e-cl(A) \setminus A$ is $\pi ge$-open.

Definition 4.6. A topological space $X$ is called a $\pi ge-T_{1/2}$ space if every $\pi ge$-closed set is $e$-closed.

Theorem 4.7. Let $X$ be a topological space.

1. $eO(X) \subset \pi geO(X)$,
2. A space $X$ is $\pi ge-T_{1/2}$ iff $eO(X) = \pi geO(X)$.

Proof. (1) Let $A$ be $e$-open, then $X \setminus A$ is $e$-closed so $X \setminus A$ is $\pi ge$-closed. Thus $A$ is $\pi ge$-open. Hence $eO(X) \subset \pi geO(X)$.

(2) Necessity: Let $(X, \tau)$ be $\pi ge-T_{1/2}$ space. Let $A$ be $\pi ge$-open. Then $X \setminus A$ is $\pi ge$-closed. By hypothesis, $X \setminus A$ is $e$-closed. Thus $A$ is $e$-open. Therefore $eO(X) = \pi geO(X)$. Sufficiency: Let $eO(X) = \pi geO(X)$. Let $A$ be $\pi ge$-closed. Then $X \setminus A$ is $\pi ge$-open. $X \setminus A$ is $e$-open. Hence $A$ is $e$-closed. This implies $(X, \tau)$ is $\pi ge-T_{1/2}$ space.
Lemma 4.8. Let $A$ be a subset of $X$ and $x \in X$. Then $x \in e\text{-cl}(A)$ iff $V \cap \{x\} \neq \emptyset$ for every $e$-open set $V$ containing $x$.

Theorem 4.9. For a topological space $X$ the following are equivalent:

(1) $X$ is $\pi ge$-$T_{1/2}$ space.
(2) Every singleton set is either $\pi$-closed or $e$-open.

Proof. (1) $\Rightarrow$ (2): Let $X$ be a $\pi ge$-$T_{1/2}$ space. Let $x \in X$ and assuming that $\{x\}$ is not $\pi$-closed. Then clearly $X \setminus \{x\}$ is not $\pi$-open. Hence $X \setminus \{x\}$ is trivially a $\pi ge$-closed. Since $X$ is $\pi ge$-$T_{1/2}$ space, $X \setminus \{x\}$ is $e$-closed. Therefore $\{x\}$ is $e$-open.

Case I: Assume every singleton of $X$ is either $\pi$-closed or $e$-open. Let $A$ be a $\pi ge$-closed set. Let $[x] \in e\text{-cl}(A)$. Let $x \in X \setminus A$. By Theorem 3.7, $[x] \in A$. Hence $e\text{-cl}(A) \subseteq A$.

Case II: Let $[x]$ be $e$-open. Since $[x] \in e\text{-cl}(A)$, we have $A \cap [x] \neq \emptyset$ implies $[x] \in A$. Therefore $e\text{-cl}(A) \subseteq A$.

5. $\pi ge$-continuous and $\pi ge$-irresolute Functions

Definition 5.1. A function $f : X \rightarrow Y$ is called $\pi ge$-continuous if $f^{-1}[V]$ is $\pi ge$-closed in $X$ for every closed set $V$ of $Y$.

Definition 5.2. A function $f : X \rightarrow Y$ is called $\pi ge$-irresolute if $f^{-1}[V]$ is $\pi ge$-closed in $X$ for every $\pi ge$-closed set $V$ of $Y$.

Proposition 5.3. Every $\pi ge$-irresolute functions is $\pi ge$-continuous.

Remark 5.4. From Definition 5.1, Definition 5.2 and Proposition 5.3 we have following the diagram:

\[
\begin{array}{cccc}
\text{b-continuous} & \rightarrow & \text{gb-continuous} & \rightarrow \\
\uparrow & \searrow & \uparrow & \searrow \\
\text{g-continuous} & \leftarrow & \text{continuous} & \rightarrow & \text{precontinuous} & \rightarrow & \text{gp-continuous} \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
\text{e-continuous} & \rightarrow & \text{ge-continuous} & \leftarrow & \text{ge-irresolute} \\
\uparrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\text{e-irresolute} & \rightarrow & \text{pg-continuous} & \leftarrow & \pi ge-irresolute \\
\end{array}
\]

The converses of these implications are not true in general as shown in the following examples.

Example 5.5. $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$. Define $f : (X, \tau) \rightarrow (X, \sigma)$ by identity function. Then $f$ is $\pi ge$-continuous but it is not $\pi ge$-irresolute.

Example 5.6. $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{c\}, \{d\}, \{a, c, d\}, \{b, c, d\}\}$, $\sigma = \{\emptyset, X, \{b\}\}$. Define $f : (X, \tau) \rightarrow (X, \sigma)$ by identity function. Then $f$ is $\pi ge$-continuous but it is not $ge$-continuous.

Example 5.7. $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, X, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$, $\sigma = \{\emptyset, X, \{b, c, e\}\}$. Define $f : (X, \tau) \rightarrow (X, \sigma)$ by identity function. Then $f$ is $\pi gb$-continuous but it is not $\pi ge$-continuous.

Remark 5.8. The composition of two $\pi ge$-continuous functions need not be $\pi ge$-continuous.

Example 5.9. Let $X = \{x, y, z, w\}$, $\tau = \{\emptyset, X, \{y\}, \{z\}, \{y, z\}\}$ and $\sigma = \{\emptyset, X, \{x, y, w\}\}$ and $\eta = \{\emptyset, X, \{x, w\}\}$. Define $f : (X, \tau) \rightarrow (X, \sigma)$ by $f = \{(x, x), (y, z), (z, y), (w, w)\}$. Define $g : (X, \sigma) \rightarrow (X, \eta)$ by $g = \{(x, w), (y, y), (z, z), (w, x)\}$. Then $f$ and $g$ are $\pi ge$-continuous but $g \circ f$ is not $\pi ge$-continuous.
Theorem 5.10. Let $f : X \to Y$ be a function.

(1) If $f$ is $\pi ge$-irresolute and $X$ is $\pi ge$-$T_{1/2}$ space, then $f$ is $e$-irresolute.

(2) If $f$ is $\pi ge$-continuous and $X$ is $\pi ge$-$T_{1/2}$ space, then $f$ is $e$-continuous.

Proof. (1) Let $V$ be $e$-closed in $Y$. Since $f$ is $\pi ge$-irresolute, $f^{-1}[V]$ is $\pi ge$-closed in $X$. Since $X$ is $\pi ge$-$T_{1/2}$ space, $f^{-1}[V]$ is $e$-closed in $X$. Hence $f$ is $e$-irresolute.

(2) Let $V$ be closed in $Y$. Since $f$ is $\pi ge$-continuous, $f^{-1}[V]$ is $\pi ge$-closed in $X$. By assumption, it is $e$-closed. Therefore $f$ is $e$-continuous.

Definition 5.11. A function $f : X \to Y$ is called $\pi$-irresolute if $f^{-1}[U]$ is $\pi$-closed in $X$ for each $\pi$-closed set $U$ of $Y$.


Theorem 5.13. Let $f : X \to Y$ be $\pi$-irresolute and pre-$e$-closed map. Then $f[A]$ is $\pi ge$-closed in $Y$ for every $\pi ge$-closed set $A$ of $X$.

Proof. Let $A$ be $\pi ge$-closed in $X$. Let $f[A] \subseteq V$ where $V$ is $\pi$-open in $Y$. Then $A \subseteq f^{-1}[V]$ and $A$ is $\pi ge$-closed in $X$ implies $e-cl(A) \subseteq f^{-1}[V]$. Hence $e-cl(f[A]) \subseteq e-cl(f[e-cl(A)]) = f[e-cl(A)] \subseteq V$. Therefore $f[A]$ is $\pi ge$-closed in $Y$.

Definition 5.14. [11] A function $f : X \to Y$ is $\pi$-open map if $f[V]$ is $\pi$-open set in $Y$ for every $\pi$-open set $V$ of $X$.

Theorem 5.15. If $f : X \to Y$ is $e$-irresolute and $\pi$-open bijection, then $f$ is $\pi ge$-irresolute.

Proof. Let $V$ be $\pi ge$-closed in $Y$. Let $f^{-1}[V] \subseteq U$ where $U$ is $\pi$-open in $X$. Hence $V \subseteq f[U]$ and $f[U]$ is $\pi$-open implies $e-cl(V) \subseteq f[U]$. Since $f$ is $e$-irresolute, $f^{-1}[e-cl(V)]$ is $e$-closed in $X$. Hence $e-cl(f^{-1}[V]) \subseteq e-cl(f^{-1}[e-cl(V)]) = f^{-1}[e-cl(V)] \subseteq U$. Therefore $f^{-1}[V]$ is $\pi ge$-closed and thus $f$ is $\pi ge$-irresolute.

Theorem 5.16. Let $f : X \to Y$ be pre-$e$-closed and $\pi ge$-irresolute surjection. If $X$ is $\pi ge$-$T_{1/2}$ space, then $Y$ is also a $\pi ge$-$T_{1/2}$ space.

Proof. Let $F$ be $\pi ge$-closed set in $Y$. Since $f$ is $\pi ge$-irresolute, $f^{-1}[F]$ is $\pi ge$-closed in $X$. Since $X$ is $\pi ge$-$T_{1/2}$ space, $f^{-1}[F]$ is $e$-closed in $X$ and hence $f[f^{-1}[F]] = F$ is $e$-closed in $Y$. This shows that $Y$ is $\pi ge$-$T_{1/2}$ space.

6. Covering Properties

Definition 6.1. A topological space $X$ is said to be:

(1) nearly compact [18] if every regular open cover of $X$ has a finite subcover.

(2) countably compact [4] if every open countable cover of $X$ has a finite subcover.

(3) nearly countably compact [10] if every countable cover by regular open sets has a finite subcover.

(4) nearly Lindelöf [6] if every cover by regular open sets has a countable subcover.

(5) $\pi ge$-compact if every $\pi ge$-open cover of $X$ has a finite subcover.

(6) $\pi ge$-Lindelöf if every cover by $\pi ge$-open sets has a countable subcover.

(7) countably $\pi ge$-compact if every $\pi ge$-open countable cover of $X$ has a finite subcover.

Corollary 6.2. For a topological space $X$ the followings hold:

(1) If $X$ is $\pi ge$-Lindelöf, then $X$ is Lindelöf.

(2) If $X$ is $\pi ge$-compact, then $X$ is compact.
Definition 6.6. A function \( \pi \) is \( \pi \)e-continuous (resp. \( \pi \)e-irresolute) if \( \pi \) is \( \pi \)e-continuous (resp. \( \pi \)e-irresolute).

Theorem 6.3. Every \( \pi \)e-compact subset of a \( \pi \)e-compact space is \( \pi \)e-compact space relative to \( X \).

Proof. Let \( \mathcal{A} \subset \pi \)eO\((X)\) and \( A \subset \bigcup \mathcal{A} \).

\[
\{ (\mathcal{A} \subset \pi \)eO\((X))(A \subset \bigcup \mathcal{A}) : A \in \pi \)eC\((X) \Rightarrow X \setminus A \in \pi \)eO\((X) \} \Rightarrow (X = (\bigcup \mathcal{A}) \cup (\downarrow A))(\mathcal{A}_1 := \mathcal{A} \cup (\downarrow A) \subset \pi \)eO\((X) \} \Rightarrow (\exists \mathcal{A}_1 = \{ A_1, A_2, \ldots, A_n \} \subset \mathcal{A}_1(X = \bigcup \mathcal{A}_1) \} \Rightarrow (\exists \mathcal{A}_1^* = \{ A_1, A_2, \ldots, A_n \} \subset \mathcal{A}_1)(A \subset \bigcup \mathcal{A}_1^*).
\]

Theorem 6.4. Let \( f : X \to Y \) be a function. If \( f \) is \( \pi \)e-continuous surjection (resp. almost \( \pi \)e-continuous) and \( X \) is \( \pi \)e-compact space, then \( Y \) is \( \pi \)e-compact space (resp. nearly \( \pi \)e-compact).

Proof. Let \( B \subset \tau_2 \) and \( Y = \bigcup B \).

\[
\{ (B \subset \tau_2)(Y = \bigcup B) : \mathcal{A} := \{ f^{-1}[B] \mid B \in B \} \Rightarrow (\mathcal{A} \subset \pi \)eO\((X))(X = \bigcup \mathcal{A}) \} \Rightarrow (\exists \mathcal{A}^* \subset \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(X = \bigcup \mathcal{A}^*) \} \Rightarrow (\exists \mathcal{A}^* \subset \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(Y = \bigcup f[B]) \}.
\]

Theorem 6.5. Let \( f : X \to Y \) be a function and \( A \subset X \). If \( f \) is \( \pi \)e-irresolute and \( A \) is \( \pi \)e-compact, then \( f[A] \) is \( \pi \)e-compact.

Proof. Let \( B \subset \pi \)eO\((Y)\) and \( f[A] \subset \bigcup B \).

\[
\{ (B \subset \pi \)eO\((Y))(f[A] \subset \bigcup B) : \mathcal{A} := \{ f^{-1}[B] \mid B \in B \} \Rightarrow (\mathcal{A} \subset \pi \)eO\((X))(A \subset \bigcup \mathcal{A}) \} \Rightarrow (\exists \mathcal{A}^* \subset \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(A \subset \bigcup \mathcal{A}^*) \} \Rightarrow (\exists \mathcal{B}^* \subset \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(f[A] \subset \bigcup \mathcal{B}^*) \}.
\]

Definition 6.6. A function \( f : X \to Y \) is \( \pi \)e-open if \( f[U] \) is \( \pi \)e-closed in \( Y \) for each \( \pi \)e-closed set in \( X \).

Theorem 6.7. Let \( f : X \to Y \) be a function. If \( f \) is \( \pi \)e-open bijection and \( Y \) is \( \pi \)e-compact, then \( X \) is \( \pi \)e-compact.

Proof. Let \( \mathcal{A} \subset \pi \)eO\((X)\) and \( X = \bigcup \mathcal{A} \).

\[
\{ (\mathcal{A} \subset \pi \)eO\((X))(X = \bigcup \mathcal{A}) : \mathcal{B} := \{ f[A] \mid A \in \mathcal{A} \} \Rightarrow (\mathcal{B} \subset \pi \)eO\((Y))(f[X] = Y = f \bigcup \mathcal{A} = \bigcup \mathcal{B}) \} \Rightarrow (\exists \mathcal{B}^* \subset \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(Y = \bigcup \mathcal{B}^*) \} \Rightarrow (\exists \mathcal{B}^* \subset \mathcal{B})(|\mathcal{B}^*| < \aleph_0)(X = f^{-1}[Y] = f^{-1}[\bigcup \mathcal{B}^*]) \} \Rightarrow (\exists \mathcal{A}^* \subset \mathcal{A})(|\mathcal{A}^*| < \aleph_0)(X = \bigcup \mathcal{A}^*) \}.
\]
Definition 6.8. A function $f : X \to Y$ is called almost $\pi ge$-continuous if $f^{-1}[V]$ is $\pi ge$-closed in $X$ for every regular closed set $V$ of $Y$.

Remark 6.9. Every $\pi ge$-continuous function is almost $\pi ge$-continuous function. However the converse need not be true as shown by the following example.

Example 6.10. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, [a, b], [d], [a, b, d]\}$, $\sigma = \{X, [a, b], [d]\}$. We define the function $f : (X, \tau) \to (X, \sigma)$ such as $f(a) = a$, $f(b) = b$, $f(c) = d$, $f(d) = c$. $f$ is almost $\pi ge$-continuous but it is not $\pi ge$-continuous since for the regular closed set $[a, b, c]$ of $(X, \sigma)$, we have $f^{-1}[\{a, b, c\}] = \{a, b, d\}$ is not $\pi ge$-closed in $(X, \tau)$.

Theorem 6.11. Let $f : X \to Y$ be an almost $\pi ge$-continuous surjection.

1. If $X$ is $\pi ge$-compact, then $Y$ is nearly compact.
2. If $X$ is $\pi ge$-Lindelöf, then $Y$ is nearly Lindelöf.
3. If $X$ is countably $\pi ge$-compact, then $Y$ is nearly countably compact.

Proof. Straightforward.

References