On measurable semi-compact and semi-Lindelof Quotient Radon measure manifolds and their intrinsic structures

S. C. P. Halakatti\textsuperscript{a,}*, Soubhagya Baddi\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, Karnatak University, Dharwad - 580003, India.

Abstract

In this paper the invariance of Radon measure structure is studied on measurable semi-compact and measurable semi-Lindelof Quotient measure manifolds under measurable homeomorphism and Radon measure structure-invariant map to generate categories of measurable semi-compact and measurable semi-Lindelof Quotient Radon measure manifolds.

Keywords: Quotient Radon measure manifold, measurable semi-compact property, measurable semi-Lindelof property, measurable semi-compact Quotient Radon measure manifold, measurable semi-Lindelof Quotient Radon measure manifold.

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1. Introduction

Historically, the research has been carried either on measure space or on the subsets of the manifolds \cite{23}. Unlike introducing all the properties on subsets of manifolds, in 2014, a new theory of Measure Manifold \cite{5}-\cite{23} was introduced and developed by Halakatti to study different intrinsic properties of manifolds other than metric based properties. Measure Manifold \((M, \tau, \Sigma, \mu)\) has been introduced by inducing additional structures called \(\sigma\)-algebraic structure \(\Sigma\) and measure structure induced by \(\mu\) on differentiable manifold using Inverse Function Theorem on measure manifold and pull back measure function with a new vision \cite{5}. These structures enable us to describe mathematically the micro and the macro structures of the measure manifold. Keeping the back ground for applications in the field of Quantum physics, life science particularly neuroscience and cosmology to address the real world problems, the concept of Measure Manifold has been introduced by Halakatti.

The topological space \((\mathbb{R}^n, \tau)\) is approximated into a measurable space \((\mathbb{R}^n, \tau, \Sigma)\) \cite{5}-\cite{23}. A measurable space \((\mathbb{R}^n, \tau, \Sigma)\) admitting the measure function \(\mu\) is called a measure space \((\mathbb{R}^n, \tau, \Sigma, \mu)\). The topological properties which are well defined on topological space are re-designated as extended/measurable topological properties on measure space \((\mathbb{R}^n, \tau, \Sigma, \mu)\). The significance of such new concepts is that the extended/measurable topological properties like measurable regular property, measurable normal property, measurable Heine

\textsuperscript{*}Corresponding author

\textit{Email addresses:} scphalakatti07@gmail.com (S. C. P. Halakatti), soubhagyabaddi@gmail.com (Soubhagya Baddi)

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Borel property, measurable Lindelof property, measurable countably compactness property, etc., [5]-[23], that are well defined on measure space \((\mathbb{R}^n, \tau, \Sigma, \mu)\) are measurable and measure - invariant under measurable homeomorphism and measure - invariant function[13].

By modeling a Hausdorff second countable topological space \((M, \tau)\) onto a measure space \((\mathbb{R}^n, \tau, \Sigma, \mu)\) with the help of measurable homeomorphism \(\phi : (M, \tau) \rightarrow (\mathbb{R}^n, \tau, \Sigma, \mu)\), Halakatti has introduced a new concept of Measure Manifold \((M, \tau, \Sigma, \mu)\) in terms of measure charts and measure atlases [12][13] along with equivalence relations, viz. \(\mathcal{A}_1 \sim \mathcal{A}_2 \) iff \(\mathcal{A}_1 \cup \mathcal{A}_2 \in \mathcal{A}^4(M)\) induces differentiable structure on \((M, \tau, \Sigma, \mu)\) and \(\mathcal{A}_1 \sim \mathcal{A}_2 \) iff \(\mu(\mathcal{A}_1) = \mu(\mathcal{A}_2) \in \Lambda^4(M)\), induces a new structure called measure structure induced by \(\mu\) on \((M, \tau, \Sigma, \mu)\). Such structured space \((M, \tau, \Sigma, \mu)\) is named as a Measure Manifold. According to Halakatti measure structure is compatible with measurable topological properties on measure manifold. On such measure manifold \((M, \tau, \Sigma, \mu)\) any measurable topological property, say, \(P\) holds \(\mu - a.e.\) on those non-empty Borel subsets \(A \in (M, \tau, \Sigma, \mu)\) where the measure of that Borel subset \(A\) is positive, that is, \(\mu(A) > 0\). If the measure of that Borel subset \(A\) containing any measurable topological property, say, \(P\) is zero, that is, \(\mu(A) = 0\) then \(A \in (M, \tau, \Sigma, \mu)\) is identified as a dark region of the measure manifold \((M, \tau, \Sigma, \mu)\) [5]-[23].

The theory of measure manifold introduced and developed by Halakatti with eight different methodologies, with more than hundred concepts and more than hundred fifty results with different measurable topological and measurable group properties, is potential enough to generate different categories of measure manifolds like Quotient Measure Manifolds, Network Measure Manifolds, Complete Network Measure Manifolds, 4-dimensional Complete Measure Manifolds, Radon measure manifolds, Quotient Radon measure Manifolds, Network Radon measure Manifolds and measurable fibre bundles [10] which acts as a blue print to describe the physical universe through topological, analytical, algebraic and geometrical structures as \(\mu - a.e.,\) properties. This theory provides a unified mathematical framework to describe the micro and macro structures of the measure manifold of dimension \(n\).

The importance of this new approach is that, the major concepts and properties of topology, measure theory, differentiable manifold and topological manifolds are adopted with a new perception to develop advanced concepts like measurable homeomorphism, measure structure-invariant maps, Inverse Function Theorem on measure manifolds, measure charts, measure atlas, measure manifold, measurable homomorphism; and new equivalence relations like local path connectedness, internal path connectedness and maximal path connectedness; new measurable topological properties on measure manifolds; new measurable group structures and geometric structures - to advance the theory of manifold in a new direction [5]-[23].

As an application to the theory of measure manifold, Radon measure manifold, Quotient Radon measure manifold and their different categories have been introduced and developed by Halakatti [17]-[23] by applying Radon measure conditions on specific measurable compact subsets.

In this paper we extend our study on Quotient Radon measure manifolds with two specific cases namely measurable semi-compact Quotient Radon measure manifolds and measurable semi-Lindelof Quotient Radon measure manifolds. Accordingly the concepts of measurable semi-compact Quotient Radon measure chart, measurable semi-compact Quotient Radon measure atlas, measurable semi-Lindelof Quotient Radon measure chart and measurable semi-Lindelof Quotient Radon measure atlases have been introduced and the corresponding Quotient Radon measure manifolds like Measurable semi-compact Quotient Radon measure manifold and measurable semi-Lindelof Quotient Radon measure manifolds have been generated by Halakatti.

2. Preliminaries

According to the theory of measure manifold developed by Halakatti, following are the basic definitions:

**Definition 2.1.** [12, 13] A non-empty measurable subspace \(((U, \tau_{1, U}, \Sigma_{1, U}), \phi) \subseteq (M, \tau_1, \Sigma_1)\) is called a measurable chart of \((M, \tau_1, \Sigma_1)\) if...
(i) $\phi$ is homeomorphism and
(ii) $\phi$ is measurable.

**Definition 2.2.** [12, 13] By an $\mathbb{R}^n$-measurable atlas of class $C^k(k \geq 1)$ on $(M, \tau, \Sigma)$, we mean a countable collection $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}})$ of $n$-dimensional measurable charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}), \phi_{i/\mathcal{A}})$ for all $i \in I$ on $(M, \tau, \Sigma)$ satisfying the following conditions:

(a1) $\bigcup_{i \in I} (U_i, \tau_{/U_i}, \Sigma_{/U_i}) = (M, \tau, \Sigma)$. That is, the countable union of all measurable charts in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}})$ cover $(M, \tau, \Sigma)$.

(a2) For any pair of measurable charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}), \phi_{i/\mathcal{A}})$ and $((U_j, \tau_{/U_j}, \Sigma_{/U_j}), \phi_{j/\mathcal{A}})$ in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}})$, the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are:

1. **differentiable maps** of class $C^k(k \geq 1)$, that is, $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \subseteq (\mathbb{R}^n, \tau_1, \Sigma_1)$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j) \subseteq (\mathbb{R}^n, \tau_1, \Sigma_1)$ are differentiable maps of class $C^k(k \geq 1)$.

2. **measurable**, that is, the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are measurable functions if,
   a) any Borel subset $K \subseteq \phi_i(U_i \cap U_j)$ is measurable in $(\mathbb{R}^n, \tau_1, \Sigma_1)$, then $(\phi_i \circ \phi_j^{-1})^{-1}(K) \in \phi_j(U_i \cap U_j)$ is also measurable,
   b) $\phi_j \circ \phi_i^{-1}$ is measurable if $S \subseteq \phi_i(U_i \cap U_j)$ is measurable in $(\mathbb{R}^n, \tau_1, \Sigma_1)$, then $(\phi_j \circ \phi_i^{-1})^{-1}(S) \in \phi_i(U_i \cap U_j)$ is also measurable.

**Definition 2.3.** [12, 13] A second countable Hausdorff topological space $M$ equipped with differentiable structure, topological structure $\tau$ and $\sigma$-algebraic structure $\Sigma$ is called a **measurable manifold** denoted by $(M, \tau, \Sigma)$.

**Definition 2.4.** [12, 13] A measurable chart $((U, \tau_{/U}, \Sigma_{/U}), \phi)$ with the induced measure function $\mu_{/U}$ is called a **measure chart** $((U, \tau_{/U}, \Sigma_{/U}, \mu_{/U}), \phi)$, satisfying the following conditions:

(i) $\phi$ is measurable homeomorphism i.e $\phi^{-1}(V) = U \in (M, \tau, \Sigma)$, $V \in (\mathbb{R}^n, \tau_1, \Sigma_1)$ and $(\mathbb{R}^n, \tau_1, \Sigma_1) = (M, \tau, \Sigma)$, and

(ii) $\phi$ is measure invariant function.

**Definition 2.5.** [12, 13] By an $\mathbb{R}^n$-measure atlas of class $C^k(k \geq 1)$ on measure manifold $(M, \tau, \Sigma)$, we mean a countable collection $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ of $n$-dimensional measure charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{/U_i}), \phi_{i/\mathcal{A}})$ for all $i \in I$ on $(M, \tau, \Sigma)$ satisfying the following conditions:

(a1) $\bigcup_{i \in I} (U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{/U_i}) = (M, \tau, \Sigma)$. That is, the countable union of all measure charts in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$ cover $(M, \tau, \Sigma)$.

(a2) For any pair of measure charts $((U_i, \tau_{/U_i}, \Sigma_{/U_i}, \mu_{/U_i}), \phi_{i/\mathcal{A}})$ and $((U_j, \tau_{/U_j}, \Sigma_{/U_j}, \mu_{/U_j}), \phi_{j/\mathcal{A}})$ in $(\mathcal{A}, \tau_{/\mathcal{A}}, \Sigma_{/\mathcal{A}}, \mu_{/\mathcal{A}})$, the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are:

1. **differentiable maps** of class $C^k(k \geq 1)$, that is, $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \subseteq (\mathbb{R}^n, \tau_1, \Sigma_1, \mu_1)$ and $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j) \subseteq (\mathbb{R}^n, \tau_1, \Sigma_1, \mu_1)$ are differentiable maps of class $C^k(k \geq 1)$.

2. **measurable**, that is, the transition maps $\phi_i \circ \phi_j^{-1}$ and $\phi_j \circ \phi_i^{-1}$ are measurable functions if,
   a) any Borel subset $K \subseteq \phi_i(U_i \cap U_j)$ is measurable in $(\mathbb{R}^n, \tau_1, \Sigma_1, \mu_1)$, then $(\phi_i \circ \phi_j^{-1})^{-1}(K) \in \phi_j(U_i \cap U_j)$ is also measurable,
   b) $\phi_j \circ \phi_i^{-1}$ is measurable if $S \subseteq \phi_i(U_i \cap U_j)$ is measurable in $(\mathbb{R}^n, \tau_1, \Sigma_1, \mu_1)$, then $(\phi_j \circ \phi_i^{-1})^{-1}(S) \in \phi_i(U_i \cap U_j)$ is also measurable.

(a3) Any two atlases $(\mathcal{A}_1, \tau_{/\mathcal{A}_1}, \Sigma_{/\mathcal{A}_1}, \mu_{/\mathcal{A}_1}), (\mathcal{A}_2, \tau_{/\mathcal{A}_2}, \Sigma_{/\mathcal{A}_2}, \mu_{/\mathcal{A}_2})$ are compatible on $(M, \tau, \Sigma)$ if they satisfy the following equivalence relations:

i) $\mathcal{A}_1 \sim \mathcal{A}_2$, iff $\mathcal{A}_1 \cup \mathcal{A}_2 \in \mathcal{A}^k(M)$,
Observation 2.6. [12, 13] The above two equivalence relations induce partition on \((M, \tau, \Sigma)\).

Definition 2.7. [12, 13] A measure space \((M, \tau, \Sigma, \mu) = \bigcup_{i=1}^{\infty} (U_i, \phi_i)\) admitting two structures, namely differentiable structure of class \(C^k\) and a measure structure induced by \(\mu\) is called a measure manifold \((M, \tau, \Sigma, \mu)\) of class \(C^k\).

In the following Halakatti has induced a maximal path connectedness relation:

Definition 2.8. [7] A measure manifold \((M, \tau, \Sigma, \mu)\) is maximally path connected if \(\exists\) a measurable \(C^\infty\) path \(\gamma_i : [0, 1] \rightarrow (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \in \Lambda^k(M)\) such that:

\[
\gamma_i(0) = p_i \in (U_i, \phi_i) \in \mathcal{A}_i, \text{ for which } \mu_\tau(U_i, \phi_i) > 0, \mu(\mathcal{A}_i) > 0 \text{ and,}
\]

\[
\gamma_i(\frac{1}{2}) = p_j \in (U_j, \phi_j) \in \mathcal{A}_j, \text{ for which } \mu_\tau(U_j, \phi_j) > 0, \mu(\mathcal{A}_j) > 0 \text{ and,}
\]

\[
\gamma_i(1) = p_k \in (U_k, \phi_k) \in \mathcal{A}_k, \text{ for which } \mu_\tau(U_k, \phi_k) > 0, \mu(\mathcal{A}_k) > 0.
\]

In other words, every \(p_i \in (U_i, \phi_i) \in \mathcal{A}_i\) is maximally path connected to every \(p_j \in (U_j, \phi_j) \in \mathcal{A}_j\) for \((\mathcal{A}_i \cup \mathcal{A}_j) \in \Lambda^0(M), \mu(\mathcal{A}_i \cup \mathcal{A}_j) > 0\) and every \(p_j \in (U_j, \phi_j) \in \mathcal{A}_j\) is maximally path connected to every \(p_k \in (U_k, \phi_k) \in \mathcal{A}_k\) for \((\mathcal{A}_j \cup \mathcal{A}_k) \in \Lambda^0(M), \mu(\mathcal{A}_j \cup \mathcal{A}_k) > 0\) of \((M, \tau, \Sigma, \mu)\) such that maximal path connected measurable \(C^\infty\) paths \(\gamma_i\) form a non-empty set, say, \(G = \{\gamma_1, \gamma_2, ..., \gamma_n\} \in (\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) \subseteq (M, \tau, \Sigma, \mu).\) Then \((M, \tau, \Sigma, \mu)\) is maximally path connected for which \(\mu(\mathcal{A}_i \cup \mathcal{A}_j \cup \mathcal{A}_k) > 0\).

This maximal path connectedness relation is an equivalence relation:


Observation 2.10. The maximal path connectedness relation on \((M, \tau, \Sigma, \mu)\) is an equivalence relation and hence induces a partition on \((M, \tau, \Sigma, \mu)\).

Observation 2.11. This equivalence relation induces a Quotient Measure Manifold.

Definition 2.12. [10] A measure manifold \((M, \tau, \Sigma, \mu)\) with a maximal path connectedness equivalence relation \(\sim\) that is closed under a measurable canonical projection map \(q : (M, \tau, \Sigma, \mu) \longrightarrow (M, \tau, \Sigma, \mu)/\sim, \forall (U, \phi) \in (M, \tau, \Sigma, \mu)/\sim \exists q^{-1}(U, \phi) \in (M, \tau, \Sigma, \mu)\) is called a Quotient measure manifold denoted by \((M, \tau, \Sigma, \mu)/\sim = (M_1, \tau_1, \Sigma_1, \mu_1)\).

Measurable homeomorphism, measure-invariant function and inverse function theorem on measure manifold are induced to develop some results on Quotient measure manifolds.

Definition 2.13. [13] Let \((M_1, \tau_1, \Sigma_1, \mu_1)\) and \((M_2, \tau_2, \Sigma_2, \mu_2)\) be two measure manifolds. Then the function \(F : M_1 \rightarrow M_2\) is called measurable homeomorphism if,

(i) \(F\) is bijective and bi continuous.
(ii) \(F\) and \(F^{-1}\) are measurable.

Definition 2.14. [13] Let \((M_1, \tau_1, \Sigma_1, \mu_1)\) and \((M_2, \tau_2, \Sigma_2, \mu_2)\) be measure manifolds and \(F : M_1 \rightarrow M_2\) be measurable homeomorphism. Then \(F\) is said to be measure-invariant function if for all measure charts \((U, \phi) \in M_2\) we have, \(\mu_2(F^{-1}(U)) = \mu_2(U)\) where \(F^{-1}(U) \subseteq M_1\).

Theorem 2.15. [13] Let \(F : (M, \tau, \Sigma, \mu) \longrightarrow (M_1, \tau_1, \Sigma_1, \mu_1)\) be a \(C^\infty\) measurable homeomorphism and measure - invariant map of measure manifolds and suppose that \(F_{\tau p} : T_p(M) \longrightarrow T_p(M_1)\) is a linear isomorphism at some point \(p\) of \(M\). Then there exists a measure chart \((U, \phi)\) of \(p\) in \(M\) such that the restriction of \(F\) to \((U, \phi)\) is a measurable diffeomorphism onto a measure chart \((V, \psi)\) of \(F(p)\) in \(M_1\). This implies for every \(C^\infty\) function \(F\) which is measurable homeomorphism and measure-invariant has a \(C^\infty\) \(F^{-1}\) which is also measurable homeomorphism and measure-invariant.
As an application to the theory of measure manifold and Quotient measure manifold, Halakatti has developed Radon measure manifold and Quotient Radon measure manifolds by using the following extended concepts of charts and atlases:

When Radon measure conditions are applied on measure manifold, the following concepts are generated:

**Definition 2.16.** [18, 21] A Radon measure on a measurable chart \((U, \phi)\) of a measure manifold \((M, \tau, \Sigma, \mu)\) is a positive Borel measure \(\mu : B \rightarrow [0, \infty]\) which is finite on compact Borel subsets and is inner regular in the sense that for every Borel charts \((U, \phi) \subset (M, \tau, \Sigma, \mu)\), we have

\[
\mu_\phi(U) = \sup\{\mu_\phi(K) : K \subseteq U; K \in \mathcal{K}\},
\]

where \(\mathcal{K}\) denote the family of all compact Borel subsets and \(\mu_\phi\) is outer regular on a family \(\mathcal{F}\) of Borel charts if for every \((U, \phi) \subset (M, \tau, \Sigma, \mu)\) we have,

\[
\mu_\phi(U) = \inf\{\mu_\phi(O) : O \supseteq U; O \in \mathcal{O}\},
\]

where \(\mathcal{O}\) denote the family of all open Borel charts.

**Definition 2.17.** [18, 21, 22] A measurable chart \((U, \tau/\Sigma, \phi)\) of \((M, \tau, \Sigma, \mu)\) equipped with a Radon measure \(\mu_\phi\) satisfying the Radon measure conditions (2.1) and (2.2) is called a Radon measure chart denoted by \((U, \tau, \Sigma, \phi, \mu_\phi)\).

Since \(\cup_{i=1}^\infty (U_i, \tau/\Sigma, \phi_i/\mu_{\phi_i}) = (M, \tau, \Sigma, \mu)\), we can measure measurable manifold \((M, \tau, \Sigma)\) by Radon measure.

**Definition 2.18.** [18, 21, 22] A measurable atlas \(\mathcal{A}\) is Radon measurable if it satisfies the following Radon measure conditions:

(i) Let \(\mathcal{F} = \{\cup_{i=1}^\infty (U_i, \phi_i)\}\) be a family of all Radon measure charts of \((M, \tau, \Sigma, \mu)\). A Radon measure \(\mu_\mathcal{A}\) of a measurable atlas \(\mathcal{A}\) is a positive Borel measure \(\mu : \mathcal{B} \rightarrow [0, \infty]\) which is finite on measurable compact charts \((U_i, \phi_i)\) and is inner regular in the sense that for every measurable atlas \(\mathcal{A}\), we have

\[
\mu_\mathcal{A}(\mathcal{A}) = \sup\{\mu_\phi(U_i) : \forall i \in I; U_i \subseteq \mathcal{A}; \forall U_i \in \mathcal{F}\},
\]

(ii) Also \(\mu_\mathcal{A}\) is outer regular on a family \(\mathcal{O}\) of measurable charts if, for every measurable atlas \(\mathcal{A} \in A^k(M)\) we have,

\[
\mu_\mathcal{A}(\mathcal{A}) = \inf\{\mu_\phi(O) : O \supseteq \{\phi_i, O \in \mathcal{A}\}\}.
\]

**Definition 2.19.** [18, 21, 22] By an \(\mathbb{R}^n\)-Radon measure atlas of class \(C^k(k \geq 1)\) on a measure manifold \((M, \tau, \Sigma, \mu)\), we mean a countable collection \((\mathcal{A}, \tau/\mathcal{A}, \Sigma/\mathcal{A}, \mu/\mathcal{A})\) of n-dimensional Radon measure charts \((U_i, \tau/\Sigma, \phi_i/\mu_{\phi_i})\) for all \(i \in I\) on \((M, \tau, \Sigma, \mu)\) satisfying the following conditions:

(a1) \(\bigcup_{i \in I} (U_i, \tau/\Sigma, \phi_i/\mu_{\phi_i}) = (M, \tau, \Sigma, \mu)\). That is, the countable union of all Radon measure charts in \((\mathcal{A}, \tau/\mathcal{A}, \Sigma/\mathcal{A}, \mu/\mathcal{A})\) cover \((M, \tau, \Sigma, \mu)\).

(a2) For any pair of Radon measure charts \((U_i, \tau/\Sigma, \phi_i/\mu_{\phi_i})\) and \((U_j, \tau/\Sigma, \phi_j/\mu_{\phi_j})\) in \((\mathcal{A}, \tau/\mathcal{A}, \Sigma/\mathcal{A}, \mu/\mathcal{A})\), the transition maps \(\phi_i \circ \phi_j^{-1}\) and \(\phi_j \circ \phi_i^{-1}\) are:

1. differentiable maps of class \(C^k(k \geq 1)\), i.e., \(\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j) \subseteq (\mathbb{R}^n, \tau_1, \Sigma_1, \mu_{\phi_i})\)

and \(\phi_j \circ \phi_i^{-1} : \phi_i(U_j \cap U_i) \rightarrow \phi_j(U_j \cap U_i) \subseteq (\mathbb{R}^n, \tau_1, \Sigma_1, \mu_{\phi_j})\) are differentiable maps of class \(C^k(k \geq 1)\).

2. Radon measurable: Transition maps \(\phi_i \circ \phi_j^{-1}\) and \(\phi_j \circ \phi_i^{-1}\) are Radon measurable functions if,

a) any Borel subset \(K \subseteq \phi_j(U_i \cap U_j)\) is Radon measurable in \((\mathbb{R}^n, \tau_1, \Sigma_1, \mu_{\phi_i})\), then \((\phi_j \circ \phi_i^{-1})^{-1}(K) \in \phi_j(U_i \cap U_j)\) is also Radon measurable.
b) $\phi_j \circ \phi^{-1}_i$ is Radon measurable if $S \subseteq \phi_j(U_i \cap U_j)$ is Radon measurable in $(R^n, \tau_1, \Sigma_1, \mu_{R_1})$, then $(\phi_j \circ \phi^{-1}_i)^{-1}(S) \in \phi_j(U_i \cap U_j)$ is also Radon measurable.

(a3) Any two measure atlases $(A_1, \tau_1, \Sigma_1, \mu_{A_1}), (A_2, \tau_2, \Sigma_2, \mu_{A_2})$ are compatible on $(M, \tau, \Sigma, \mu)$ satisfying the two equivalence relations:
i) $A_1 \sim A_2$, iff $A_1 \cup A_2 \in A_k(M)$
ii) $A_1 \sim A_2$, iff $\mu_k(A_1) = \mu_k(A_2)$.

Definition 2.20. [22] For any two Radon measure atlases $A_1, A_2 \in A_k(M) \subset (M, \tau, \Sigma, \mu)$, we say that $A_1 \sim A_2$ is an equivalence relation on $(M, \tau, \Sigma, \mu)$ if and only if $\mu_k(A_1) = \mu_k(A_2)$. This equivalence relation induces a Radon measure-structure on $(M, \tau, \Sigma, \mu)$.

Definition 2.21. [18, 21, 22] A Radon measure space $(M, \tau, \Sigma, \mu)$ with differentiable structure of class $C^k$ and a Radon measure structure induced by $\mu_k$, on $(M, \tau, \Sigma, \mu)$ is a Radon measure manifold of class $C^k$.

That is, i) $A_1 \sim A_2$, iff $A_1 \cup A_2 \in A_k(M)$ and
ii) $A_1 \sim A_2$, iff $\mu_k(A_1) = \mu_k(A_2) \in A_k(M)$.

Following are the specific examples of Radon measure manifolds:

Exercise 2.22. [17] Lebesgue measure on measurable manifold $(M, \tau, \Sigma)$ is a Radon measure manifold.

Exercise 2.23. [18] Measurable compact unit interval $I = [0, 1] \subset (R^1, \tau_1, \Sigma_1, \mu_{R_1})$ is a Radon measure manifold of dimension 1.

Exercise 2.24. [8, 19] The unit circle of radius 1, defined as $S^1 = \{(x, y) \in (R^2, \tau, \Sigma, \mu) : x^2 + y^2 = 1\}$ in $(R^2, \tau, \Sigma, \mu)$ is a Normal Radon measure manifold.

Solution: Since $S^1 = \{(x, y) \in (R^2, \tau, \Sigma, \mu) : x^2 + y^2 = 1\}$ is a Normal measure manifold where $S^1$ is covered by four measurable charts which are Lebesgue measurable and hence Radon measurable since Lebesgue measure on measurable manifold $(M, \tau, \Sigma)$ is a Radon measure manifold [8, 17]. Hence $S^1$ is a Normal Radon measure manifold.

Exercise 2.25. [8, 18] The sphere of radius $\frac{1}{\sqrt{2}}$, defined as $S^2 = \{(x, y, z) \in (R^3, \tau, \Sigma, \mu) : x^2 + y^2 + z^2 = \frac{1}{2}\}$ in $(R^3, \tau, \Sigma, \mu)$ is a Normal Radon measure manifold.

Solution: Since $S^2 = \{(x, y, z) \in (R^3, \tau, \Sigma, \mu) : x^2 + y^2 + z^2 = \frac{1}{2}\}$ is a Normal measure manifold where $S^2$ is covered by six measurable charts which are six hemispheres of $S^2$ that are Lebesgue measurable and hence Radon measurable since Lebesgue measure on measurable manifold $(M, \tau, \Sigma)$ is a Radon measure manifold.

The following concept of Radon measure structure - invariance is used to prove the results in this paper.

Definition 2.26. [22] Suppose $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ are Radon measure manifolds. Let $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be a measurable homeomorphism then $F$ is Radon measure structure-invariant if $A_1 \sim A_2$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ with Radon measure invariant condition $\mu_{R_1}(A_1) = \mu_{R_1}(A_2)$ then $F(A_1) \sim F(A_2)$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ with Radon measure invariant condition $\mu_{R_2}(F(A_1)) = \mu_{R_2}(F(A_2))$.

Halakatti has also developed Quotient Radon measure manifold [22] by using the developed concepts like Quotient Radon measure chart and Quotient Radon measure atlas.

Definition 2.27. [22] A Radon measure chart $(U_i, \phi_i)$ with path connectedness equivalence relation $\sim$ that is closed under a measurable canonical projection map $q : (U, \phi) \rightarrow (U, \phi)/\sim$, $\forall (U, \phi) \in (M, \tau, \Sigma, \mu)/\sim \exists \ q^{-1}(U, \phi) \in (M, \tau, \Sigma, \mu)$ is called a Quotient Radon measure chart denoted by $(U, \phi)/\sim$. 
Definition 2.28. [22] By an $R^i$-Quotient Radon measure atlas of class $C^k (k \geq 1)$ on a measure manifold $(M, \tau, \Sigma, \mu_R)$, we mean a countable collection $(\mathcal{A}, \tau/\mathcal{A}, \Sigma/\mathcal{A}, \mu/\mathcal{A})$ of $n$-dimensional Quotient Radon measure charts $((U_i, \tau_{U_i}, \Sigma_{U_i}, \mu_{U_i}, \phi_{i0}))$ for all $i \in I$ on $(M, \tau, \Sigma, \mu_R)$ satisfying the following conditions:

(a1) $\cup_{i \in I}(U_i, \tau_{U_i}, \Sigma_{U_i}, \mu_{U_i}, \phi_{i0}) = (M, \tau, \Sigma, \mu_R)$. That is, the countable union of all Quotient Radon measure charts in $(\mathcal{A}, \tau/\mathcal{A}, \Sigma/\mathcal{A}, \mu/\mathcal{A})$ cover $(M, \tau, \Sigma, \mu_R)$.

(a2) For any pair of Quotient Radon measure charts $(U_i, \phi_i)$ and $(U_j, \phi_j)$ there exists transition maps $\phi_j \circ \phi^{-1}_i : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ and $\phi_i \circ \phi^{-1}_j : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are differentiable and Radon measurable.

(a3) Any two Radon measure atlases are compatible on $(M, \tau, \Sigma, \mu_R)$ satisfying two equivalence relations:

(i) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $(\mathcal{A}_1 \cup \mathcal{A}_2) \in A^k(M)$.

(ii) $\mathcal{A}_1 \sim \mathcal{A}_2$ iff $\mu_R(\mathcal{A}_1) = \mu_R(\mathcal{A}_2)$.

Definition 2.29. [22] Let $(M, \tau, \Sigma, \mu_R) \subseteq \bigcup_{i \in I}(\mathcal{A}_i)$ be a metrizable Radon measure manifold and $(U_i, \phi_i)$ and $\mathcal{A}_i$ (where $i \in I$) are Quotient Radon measure charts and Quotient Radon measure atlas respectively. If $(M, \tau, \Sigma, \mu_R)$ admits the local, internal and maximal path connectedness relations under the measurable $C^\infty$ canonical surjective projection map $q : (M, \tau, \Sigma, \mu_R) \to (M, \tau, \Sigma, \mu_R)$ such that for all $(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$, we have $q^{-1}(U_i, \phi_i) \in (M, \tau, \Sigma, \mu_R)$, then $(M, \tau, \Sigma, \mu_R)$ is called a Quotient Radon measure Manifold.

In this paper we have chosen to study two specific Quotient Radon measure manifolds like measurable semi-compact Quotient Radon measure manifold and measurable semi-Lindelof Quotient Radon measure manifold as categories of Quotient Radon measure manifolds.

3. Main Results

In this paper, we continue to study different categories of Quotient Radon measure manifolds and their properties:

Case 3.1. Measurable semi-compact chart on Quotient Radon measure manifold

Definition 3.2. A Radon measure on a measurable semi-compact chart $(U, \phi)$ of a Quotient measure manifold $(M, \tau, \Sigma, \mu)$ is a positive Borel measure $\mu : B \to [0, \infty]$ which is finite on compact Borel subsets and is inner regular in the sense that for every measurable semi-compact chart $(U, \phi) \subset (M, \tau, \Sigma, \mu)$, we have

$$\mu_r(U) = \sup\{\mu_r(K) : K \subseteq U; K \in \mathcal{K}\},$$

(3.1)

where $\mathcal{K}$ denote the family of all compact Borel subsets and $\mu_r$ is outer regular on a family $\mathcal{F}$ of measurable semi-compact chart $(U, \phi) \subset (M, \tau, \Sigma, \mu)$ we have,

$$\mu_r(U) = \inf\{\mu_r(O) : O \supseteq U; O \in \mathcal{O}\},$$

(3.2)

where $\mathcal{O}$ denote the family of all open Borel charts.

Definition 3.3. We say $(U_i, \phi_i)$ is measurable semi-compact chart, if for every Borel semi-open cover, say, $[\cup_{i \in I}S_i]$ of $(U_i, \phi_i)$ if $\exists$ a finite Borel semi-open sub cover, say, $[\cup_{j \in J}S_j]$ for $j \in J, j \subset I$ such that $(U_i, \phi_i) \subseteq [\cup_{j \in J}S_j]$ where $S_j$ are Borel semi-open subsets of $(U_i, \phi_i)$ and is Radon measurable if it satisfies the Radon measure conditions (3.1) and (3.2). Then $(U_i, \phi_i)$ is a semi-compact Radon measure chart of $(M, \tau, \Sigma, \mu_r)$.

Definition 3.4. By a measurable semi-compact Radon measure atlas of class $C^k$, $(k \geq 1)$ on a Quotient Radon measure manifold, we mean a collection of $n$-dimensional measurable semi-compact Radon measure charts $(U_i, \phi_i)$ satisfying the following conditions:

(a1) $\cup_{i \in I}(U_i, \phi_i) \subset A^k(M) \subset (M, \tau, \Sigma, \mu_r)$,
(a2) For any pair of \((U_i, \phi_i)\) and \((U_j, \phi_j)\) there exists transition maps \(\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)\) and \(\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)\) that are differentiable and Radon measurable.

(a3) Any two measure atlases are compatible on \((M, \tau, \Sigma, \mu)\) satisfying two equivalence relations:
   (i) \(\mathcal{A}_1 \sim \mathcal{A}_2\) iff \((\mathcal{A}_1 \cup \mathcal{A}_2) \in \mathcal{A}^k(M)\).
   (ii) \(\mathcal{A}_1 \sim \mathcal{A}_2\) iff \(\mu_\kappa(\mathcal{A}_1) = \mu_\kappa(\mathcal{A}_2)\).

(a4) For every Borel semi-open cover, say, \(\{U_{i\in I}(U_i, \phi_i)\}\) for \(\mathcal{A}\) i.e., \(\mathcal{A} \subseteq \{U_{i\in I}(U_i, \phi_i)\}\) if there exists a finite Borel semi-open sub cover, say, \(\{\bigcup_{j\in I}^\kappa \mathcal{A}_j\}\) such that \(\tilde{A}^k(M) \subseteq \{\bigcup_{j=1}^\kappa \mathcal{A}_j\}\), then \(\mathcal{A}\) is a semi-compact measure atlas of \((M, \tau, \Sigma, \mu)\).

If \(\mathcal{A}\) is Radon measurable then \(\mathcal{A}\) is a measurable semi-compact Radon measure atlas of \((M, \tau, \Sigma, \mu)\).

**Definition 3.5.** Let \((M, \tau, \Sigma, \mu)\) be a Quotient Radon measure manifold. Let \(\tilde{A}^k(M) \subseteq (M, \tau, \Sigma, \mu)\). For every Borel semi-open cover, say, \(\{U_{i\in I}(U_i, \phi_i)\}\) i.e., \(\tilde{A}^k(M) \subseteq \{U_{i\in I}(U_i, \phi_i)\}\), if \(\exists\) a finite Borel semi-open sub cover, say, \(\{\bigcup_{j\in I}^{\kappa} \mathcal{A}_j\}\) such that \(\tilde{A}^k(M) \subseteq \{\bigcup_{j=1}^\kappa \mathcal{A}_j\}\), then \((M, \tau, \Sigma, \mu)\) satisfies the Radon measure conditions (3.1) and (3.2) is called a measurable semi-compact Quotient Radon measure manifold.

**Case 3.6.** Measurable semi-compact chart on Quotient Radon measure manifold.

**Definition 3.7.** A Radon measure on a measurable semi-Lindelof chart \((U, \phi)\) of a Quotient measure manifold \((M, \tau, \Sigma, \mu)\) is a positive Borel measure \(\mu : B \to [0, \infty]\) which is finite on compact Borel subsets and is inner regular in the sense that for every measurable semi-Lindelof charts \((U, \phi) \subset (M, \tau, \Sigma, \mu)\), we have

\[
\mu (U) = \sup \{\mu (K) : K \subseteq U; K \in \mathcal{K}\}
\]

where \(\mathcal{K}\) denote the family of all compact Borel subsets and \(\mu\) is outer regular on a family \(\mathcal{F}\) of Borel charts if for every \((U, \phi) \subset (M, \tau, \Sigma, \mu)\), we have

\[
\mu (U) = \inf \{\mu (O) : O \supseteq U; O \in \mathcal{O}\}
\]

where \(\mathcal{O}\) denote the family of all open Borel charts.

**Definition 3.8.** We say \((U_i, \phi_i)\) is a measurable Semi-Lindelof, if for every Borel semi-open cover, say, \(\{U_{i\in I}(U_i, \phi_i)\}\) of \((U_i, \phi_i)\) if \(\exists\) a countable Borel semi-open sub cover, say, \(\{\bigcup_{j\in J}^\kappa \mathcal{A}_j\}\) for \(J \subset I\) such that \((U_i, \phi_i) \subseteq \{\bigcup_{j=1}^\kappa \mathcal{A}_j\}\) where \(\mathcal{A}_i\) are semi-open Borel subsets of \((U_i, \phi_i)\) and is Radon measurable if it satisfies the Radon measure conditions (3.3) and (3.4). Then \((U_i, \phi_i)\) is a measurable semi-Lindelof Radon measure chart of \((M, \tau, \Sigma, \mu)\).

**Definition 3.9.** By a measurable semi-Lindelof Radon measure atlas of class \(\mathcal{C}_k\) \((k \geq 1)\) on a Quotient Radon measure manifold, we mean a collection of n-dimensional measurable semi-Lindelof Radon measure charts \((U_i, \phi_i)\) satisfying the following conditions:

(a1) \(\bigcup_{i\in I}(U_i, \phi_i) \in \tilde{A}^k(M) \subseteq (M, \tau, \Sigma, \mu)\).

(a2) For any pair of measurable semi-Lindelof Radon measure charts \((U_i, \phi_i)\) and \((U_j, \phi_j)\) there exists transition maps \(\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)\) and \(\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)\) that are differentiable and Radon measurable.

(a3) Any two measure atlases are compatible on \((M, \tau, \Sigma, \mu)\) satisfying two equivalence relations
   (i) \(\mathcal{A}_1 \sim \mathcal{A}_2\) iff \((\mathcal{A}_1 \cup \mathcal{A}_2) \in \mathcal{A}^k(M)\).
   (ii) \(\mathcal{A}_1 \sim \mathcal{A}_2\) iff \(\mu_\kappa(\mathcal{A}_1) = \mu_\kappa(\mathcal{A}_2)\).

(a4) For every Borel semi-open cover \(\{U_{i\in I}(U_i, \phi_i)\}\) for \(\mathcal{A}\) i.e., \(\mathcal{A} \subseteq \{U_{i\in I}(U_i, \phi_i)\}\) if there exists a countable Borel semi-open sub cover, say, \(\{\bigcup_{j\in J}^{\kappa} \mathcal{A}_j\}\) for \(J \subset I\) such that \(\tilde{A}^k(M) \subseteq \{\bigcup_{j=1}^{\kappa} \mathcal{A}_j\}\), then \(\mathcal{A}\) is a Lindelof measure atlas of \((M, \tau, \Sigma, \mu)\). If \(\mathcal{A}\) is Radon measurable then \(\mathcal{A}\) is a semi-Lindelof Radon measure atlas of \((M, \tau, \Sigma, \mu)\).

**Definition 3.10.** Let \((M, \tau, \Sigma, \mu)\) be a Quotient Radon measure manifold. Let \(\tilde{A}^k(M) \subseteq (M, \tau, \Sigma, \mu)\). For every Borel semi-open cover, say, \(\{\bigcup_{j=1}^{\kappa} \mathcal{A}_j\}\) i.e., \(\tilde{A}^k(M) \subseteq \{\bigcup_{j=1}^{\kappa} \mathcal{A}_j\}\), if \(\exists\) a countable Borel semi-open sub cover, say, \(\{\bigcup_{j\in J}^{\kappa} \mathcal{A}_j\}\) for \(J \subset I\) such that \(\tilde{A}^k(M) \subseteq \{\bigcup_{j=1}^{\kappa} \mathcal{A}_j\}\), then \((M, \tau, \Sigma, \mu)\) such that \(\tilde{A}^k(M) \subseteq (M, \tau, \Sigma, \mu)\) satisfying the Radon measure conditions (3.3) and (3.4) is called a measurable semi-Lindelof Quotient Radon measure manifold.
We now prove some of the following results on these two Quotient Radon measure manifolds.

**Theorem 3.11.** Let \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) and \((M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) be Quotient Radon measure manifolds. If measurable semi-compactness property holds \(\mu_{\Sigma_1} - a.e.,\) on \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) then the property also holds \(\mu_{\Sigma_2} - a.e.,\) on \((M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) under measurable homeomorphism and Radon measure structure - invariant map
\[
F : (M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1}) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2}).
\]

**Proof.** Let \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) and \((M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) be Quotient Radon measure manifolds and \(F : (M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1}) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) be measurable homeomorphism and Radon measure structure - invariant map. We show that if measurable semi-compactness property say \(P_1\) holds \(\mu_{\Sigma_1} - a.e.,\) on \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) then \(P_1\) also holds \(\mu_{\Sigma_2} - a.e.,\) on \((M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\). Since by Theorem 2.4.5 [22], measurable semi-compactness property holds \(\mu_{\Sigma_1} - a.e.,\) on \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) then it also holds \(\mu_{\Sigma_2} - a.e.,\) on \((M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) under measurable homeomorphism and Radon measure structure - invariant map \(F : (M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1}) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) where \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) and \((M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) are Radon measure manifolds. Then, if for any two measurable semi-compact Radon measure atlases \(A_1, A_2 \in A^k(M_1) \subset (M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) \(\exists F(A_1), F(A_2) \in A^k(M_2) \subset (M_2, \tau_1, \Sigma_1, \mu_{\Sigma_2})\): \(A_1 \sim A_2\) implies \(F(A_1) \sim F(A_2)\) with Radon measure structure condition:
\[
\mu_{\Sigma_1}(A_1) = \mu_{\Sigma_2}(A_2) \implies \mu_{\Sigma_2}(F(A_1)) = \mu_{\Sigma_2}(F(A_2)) \tag{3.5}
\]
Therefore, if for any two Quotient Radon measure atlases \(A_1, A_2 \in A^k(M_1) \subset (M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) \(\exists F(A_1), F(A_2) \in A^k(M_2) \subset (M_2, \tau_1, \Sigma_1, \mu_{\Sigma_2})\): \(A_1 \sim A_2\) implies \(F(A_1) \sim F(A_2)\) with Radon measure structure condition:
\[
\mu_{\Sigma_1}(A_1) = \mu_{\Sigma_2}(A_2) \implies \mu_{\Sigma_2}(F(A_1)) = \mu_{\Sigma_2}(F(A_2)) \tag{3.6}
\]
Therefore, from (3.5) and (3.6), if measurable semi-compactness property holds \(\mu_{\Sigma_1} - a.e.,\) on \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) then it also holds \(\mu_{\Sigma_2} - a.e.,\) on \((M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) under measurable homeomorphism and Radon measure structure - invariant map where \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) and \((M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) are Quotient Radon measure manifolds. \(\square\)

**Theorem 3.12.** Let \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1}), (M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) and \((M_3, \tau_3, \Sigma_3, \mu_{\Sigma_3})\) be Quotient Radon measure manifolds. Let \(F : (M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1}) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) and \(G : (M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2}) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_{\Sigma_3})\) be \(C^\infty\) measurable homeomorphisms and Radon measure structure - invariant maps. Then if measurable semi-compactness property holds \(\mu_{\Sigma_1} - a.e.,\) on \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) then the property also holds \(\mu_{\Sigma_3} - a.e.,\) on \((M_3, \tau_3, \Sigma_3, \mu_{\Sigma_3})\) under the composition mapping \(G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1}) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_{\Sigma_3})\).

**Proof.** Let \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1}), (M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\) and \((M_3, \tau_3, \Sigma_3, \mu_{\Sigma_3})\) be Quotient Radon measure manifolds. We show that if measurable semi-compactness property say \(P_1\) holds \(\mu_{\Sigma_1} - a.e.,\) on \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) then \(P_1\) also holds \(\mu_{\Sigma_3} - a.e.,\) on \((M_3, \tau_3, \Sigma_3, \mu_{\Sigma_3})\). From Theorem 2.5.6 in [22], if measurable compactness property say \(P_1\) holds \(\mu_{\Sigma_1} - a.e.,\) on \((M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) then it also holds \(\mu_{\Sigma_2} - a.e.,\) on \((M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\): That is, if for any two Quotient Radon measure atlases \(A_1, A_2 \in A^k(M_1) \subset (M_1, \tau_1, \Sigma_1, \mu_{\Sigma_1})\) \(\exists F(A_1), F(A_2) \in A^k(M_2) \subset (M_2, \tau_1, \Sigma_1, \mu_{\Sigma_2})\): \(A_1 \sim A_2\) implies \(F(A_1) \sim F(A_2)\) with Radon measure structure condition:
\[
\mu_{\Sigma_1}(A_1) = \mu_{\Sigma_2}(A_2) \implies \mu_{\Sigma_2}(F(A_1)) = \mu_{\Sigma_2}(F(A_2)) \tag{3.7}
\]
Similarly, since \(G : (M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2}) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_{\Sigma_3})\) is \(C^\infty\) measurable homeomorphisms and Radon measure structure - invariant map, if for any two Quotient Radon measure atlases \(F(A_1), F(A_2) \in A^k(M_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{\Sigma_2})\): \(A_1 \sim A_2\) with Radon measure structure condition:
\[
\mu_{\Sigma_2}(F(A_1)) = \mu_{\Sigma_2}(F(A_2)) \implies \mu_{\Sigma_3}((G \circ F)(A_1)) = \mu_{\Sigma_3}((G \circ F)(A_2)) \tag{3.8}
\]
This implies under $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{\tau_1}) \rightarrow (M_2, \tau_3, \Sigma_3, \mu_{\tau_3})$, if $P_1$ holds $\mu_{\tau_1} - a.e.$, on $\mathcal{A}_1$, $\mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ then it also holds $\mu_{\tau_3} - a.e.$ on $(G \circ F)(\mathcal{A}_1), (G \circ F)(\mathcal{A}_2) \subset (M_3, \tau_3, \Sigma_3, \mu_{\tau_3})$; $\mathcal{A}_1 \sim \mathcal{A}_2$ implies $(G \circ F(\mathcal{A}_1)) \sim (G \circ F(\mathcal{A}_2))$ with Radon measure structure condition:

$$\mu_{\tau_1}(\mathcal{A}_1) = \mu_{\tau_3}(\mathcal{A}_2) \implies \mu_{\tau_3}(G \circ F(\mathcal{A}_1)) = \mu_{\tau_3}(G \circ F(\mathcal{A}_2))$$

Therefore, from (3.7), (3.8) and (3.9), if measurable semi-compactness property holds $\mu_{\tau_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ then it also holds $\mu_{\tau_2} - a.e.$, on $(M_3, \tau_3, \Sigma_3, \mu_{\tau_3})$ under measurable homeomorphism and Radon measure structure - invariant map $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{\tau_1}) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_{\tau_3})$ where $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1}), (M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{\tau_3})$ are Quotient Radon measure manifolds.

**Remark 3.13.** By using above method, one can show that the measurable semi-compactness property remains invariant $\mu_{\tau_1} - a.e.$, on the non-empty set of Quotient Radon measure manifolds $M_{\tau_1}, M_{\tau_2}, ..., M_{\tau_n}$ under the composition of functions $F_1 : M_{\tau_1} \rightarrow M_{\tau_2}, F_2 : M_{\tau_2} \rightarrow M_{\tau_3}, ..., F_n : M_{\tau_n-1} \rightarrow M_{\tau_n}$ and by using this, one can generate a new category of measurable semi-compact Quotient Radon measure manifold $(\Pi, (G, \circ))$ where $\Pi = \{M_1 \times M_2 \times ... \times M_n\}$ which is closed under the group action $(G, \circ) = \{F_1, F_2, ..., F_n\}$ of $C^\infty$ measurable homeomorphisms and Radon measure structure-invariant maps.

**Theorem 3.14.** Let $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ be Quotient Radon measure manifolds. If measurable semi-Lindelof property holds $\mu_{\tau_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ then the property also holds $\mu_{\tau_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{\tau_1}) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$.

**Proof:** Let $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ be Quotient Radon measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_{\tau_1}) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ be measurable homeomorphism and Radon measure structure - invariant map. We show that if measurable semi-Lindelof property say $P_2$ holds $\mu_{\tau_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ then it also holds $\mu_{\tau_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$. Since by Theorem 2.5.6 [22], measurable semi-Lindelof property holds $\mu_{\tau_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ then it also holds $\mu_{\tau_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ under measurable homeomorphism and Radon measure structure - invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{\tau_1}) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ where $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ are Radon measure manifolds. Then, if for any two measurable semi-Lindelof Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{A}^k(M_1) \subset (M_1, \tau_1, \Sigma_1, \mu_{\tau_1}) \exists F(\mathcal{A}_1), F(\mathcal{A}_2) \in \mathcal{A}^k(M_2) \subset (M_2, \tau_1, \Sigma_1, \mu_{\tau_1})$:

$$\mathcal{A}_1 \sim \mathcal{A}_2 \implies F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$$

with Radon measure structure condition:

$$\mu_{\tau_1}(\mathcal{A}_1) = \mu_{\tau_2}(\mathcal{A}_2) \implies \mu_{\tau_2}(F(\mathcal{A}_1)) = \mu_{\tau_2}(F(\mathcal{A}_2))$$

Therefore, if for any two Quotient Radon measure atlases $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{A}^k(M_1) \subset (M_1, \tau_1, \Sigma_1, \mu_{\tau_1}) \exists F(\mathcal{A}_1), F(\mathcal{A}_2) \in \mathcal{A}^k(M_2) \subset (M_2, \tau_1, \Sigma_1, \mu_{\tau_1})$: $\mathcal{A}_1 \sim \mathcal{A}_2$ implies $F(\mathcal{A}_1) \sim F(\mathcal{A}_2)$ with Radon measure structure condition:

$$\mu_{\tau_1}(\mathcal{A}_1) = \mu_{\tau_2}(\mathcal{A}_2) \implies \mu_{\tau_2}(F(\mathcal{A}_1)) = \mu_{\tau_2}(F(\mathcal{A}_2))$$

Therefore, from (3.10) and (3.11), if measurable semi-Lindelof property holds $\mu_{\tau_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ then it also holds $\mu_{\tau_2} - a.e.$, on $(M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ under measurable homeomorphism and Radon measure structure - invariant map where $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ are Quotient Radon measure manifolds.

**Theorem 3.15.** Let $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1}), (M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{\tau_3})$ be Quotient Radon measure manifolds. Let $F : (M_1, \tau_1, \Sigma_1, \mu_{\tau_1}) \rightarrow (M_2, \tau_2, \Sigma_2, \mu_{\tau_2})$ and $G : (M_2, \tau_2, \Sigma_2, \mu_{\tau_2}) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_{\tau_3})$ be $C^\infty$ measurable homeomorphisms and Radon measure structure - invariant maps. Then if measurable semi-Lindelof property holds $\mu_{\tau_1} - a.e.$, on $(M_1, \tau_1, \Sigma_1, \mu_{\tau_1})$ then the property also holds $\mu_{\tau_3} - a.e.$, on $(M_3, \tau_3, \Sigma_3, \mu_{\tau_3})$ under the composition mapping $G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{\tau_1}) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_{\tau_3})$. 

**Proof:**
Proof. Let \((M_1, \tau_1, \Sigma_1, \mu_{R_1}), (M_2, \tau_2, \Sigma_2, \mu_{R_2}), \ldots, (M_n, \tau_n, \Sigma_n, \mu_{R_n})\) be Quotient Radon measure manifolds. We show that if measurable semi-Lindelof property say \(P_2\) holds \(\mu_{R_1} - \text{a.e.},\) on \((M_1, \tau_1, \Sigma_1, \mu_{R_1})\) then \(P_2\) also holds \(\mu_{R_3} - \text{a.e.},\) on \((M_3, \tau_3, \Sigma_3, \mu_{R_3})\). From Theorem 4.5.6 in [22], if measurable semi-Lindelof property say \(P_2\) holds \(\mu_{R_1} - \text{a.e.},\) on \((M_1, \tau_1, \Sigma_1, \mu_{R_1})\) then it also holds \(\mu_{R_2} - \text{a.e.},\) on \((M_2, \tau_2, \Sigma_2, \mu_{R_2})\): That is, if for any two measurable Quotient Radon measure atlases \(\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{A}_1(M) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})\) \(\exists F(\mathcal{A}_1), F(\mathcal{A}_2) \in \mathcal{A}^1(M) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})\): \(\mathcal{A}_1 \sim \mathcal{A}_2\) implies \(F(\mathcal{A}_1) \sim F(\mathcal{A}_2)\) with Radon measure structure condition:

\[
\mu_{R_1}(\mathcal{A}_1) = \mu_{R_2}(\mathcal{A}_2) \implies \mu_{R_1}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2)) \text{ according to Definition 2.14.}
\]

Similarly, since \(G : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})\) is \(C^\infty\) measurable homeomorphisms and Radon measure structure - invariant map, if for any two Quotient Radon measure atlases \(F(\mathcal{A}_1), F(\mathcal{A}_2) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})\) \(\exists (G \circ F)(\mathcal{A}_1), (G \circ F)(\mathcal{A}_2) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})\) with Radon measure structure condition:

\[
\mu_{R_2}(F(\mathcal{A}_1)) = \mu_{R_2}(F(\mathcal{A}_2)) \implies \mu_{R_3}(G \circ F(\mathcal{A}_1)) = \mu_{R_3}(G \circ F(\mathcal{A}_2)).
\]

This implies under \(G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})\), if \(P_2\) holds \(\mu_{R_2} - \text{a.e.},\) on \(\mathcal{A}_1, \mathcal{A}_2 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})\) then it also holds \(\mu_{R_3} - \text{a.e.},\) on \((G \circ F)(\mathcal{A}_1), (G \circ F)(\mathcal{A}_2) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})\):

\[
\mathcal{A}_1 \sim \mathcal{A}_2 \implies F(\mathcal{A}_1) \sim F(\mathcal{A}_2)\text{ with Radon measure structure condition:}
\]

\[
\mu_{R_1}(\mathcal{A}_1) = \mu_{R_2}(\mathcal{A}_2) \implies \mu_{R_3}(G \circ F(\mathcal{A}_1)) = \mu_{R_3}(G \circ F(\mathcal{A}_2)) \text{ according to Definition 2.14.}
\]

Therefore, from (3.12), (3.13) and (3.14), if measurable semi-Lindelof property holds \(\mu_{R_1} - \text{a.e.},\) on \((M_1, \tau_1, \Sigma_1, \mu_{R_1})\) then it also holds \(\mu_{R_3} - \text{a.e.},\) on \((M_3, \tau_3, \Sigma_3, \mu_{R_3})\) under measurable homeomorphism and Radon measure structure - invariant map \(G \circ F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \rightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})\) where \((M_1, \tau_1, \Sigma_1, \mu_{R_1}), (M_2, \tau_2, \Sigma_2, \mu_{R_2})\) and \((M_3, \tau_3, \Sigma_3, \mu_{R_3})\) are Quotient Radon measure manifolds.

Remark 3.16. By using above method, one can show that the measurable semi-Lindelof property remains invariant \(\mu_{R} - \text{a.e.},\) on the non-empty set of Quotient Radon measure manifolds \(M_{R_1}, M_{R_2}, \ldots, M_{R_n}\) under the composition of functions \(F_1 : M_{R_1} \rightarrow M_{R_2}, F_2 : M_{R_2} \rightarrow M_{R_3}, \ldots, F_n : M_{R_{n-1}} \rightarrow M_{R_n}\) and by using this, one can generate a new category of measurable semi-Lindelof Quotient Radon measure manifold \((\Pi, (G, \circ))\) where \(\Pi = \{M_1 \times M_2 \times \ldots \times M_n\}\) which is closed under the group action \((G, \circ) = \{F_1, F_2, \ldots, F_n\}\) of \(C^\infty\) measurable homeomorphisms and Radon measure structure-invariant maps.

Observation 3.17. If \(M_{R_1}, M_{R_2}, \ldots, M_{R_n}\) are measurable semi-compact Quotient Radon measure manifolds and if \(\{F_1, F_2, \ldots, F_n\}\) are measurable homeomorphism and Radon measure structure-invariant transformations then \(\{F_1, F_2, \ldots, F_n\}\) form an abelian group \(G\) on \((M, \tau, \Sigma, \mu_R)\).

Observation 3.18. If \(M_{R_1}, M_{R_2}, \ldots, M_{R_n}\) are measurable semi-Lindelof Quotient Radon measure manifolds and if \(\{F_1, F_2, \ldots, F_n\}\) are measurable homeomorphism and Radon measure structure-invariant transformations then \(\{F_1, F_2, \ldots, F_n\}\) form an abelian group \(G\) on \((M, \tau, \Sigma, \mu_R)\).

4. Conclusion

Our study on Quotient Radon measure manifold admits two group structures namely \((G, \circ)\) and \((\mathcal{G}, \circ)\) on \((M, \tau, \Sigma, \mu_R)\). This study will be extended to the Network structure \((\{M, \tau, \Sigma, \mu_R\}, (G, \circ), (\mathcal{G}, \circ))\) on Quotient Radon measure manifold in our future work.

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