On the lattice of generalized closure operators

Kavitha T.\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, University of Calicut, Calicut University P.O., Kerala, Pin-673635, India.

Abstract

In this paper we compare the lattice of generalized closure operators and the lattice of generalized Čech closure operators.

Keywords: Generalized closure operator, generalized Čech closure operator.

2010 MSC: 54A05, 03G10.

1. Introduction

Tyagi and Choudhary introduced generalized closure operators in [5]. They studied generalized interior operator and generalized neighbourhood systems [5].

In this paper, we study the lattice of generalized form of Čech closure operators. We prove that the lattice of generalized closure operators is a complete lattice. Here we determine atoms and dual atoms of the lattice of generalized closure operators and of the lattice of generalized Čech closure operators.

2. Generalized closure operators

Let $X$ be any set. A collection $\mu$ of subsets of $X$ is said to be a generalized topology on $X$ if $\emptyset \in \mu$ and arbitrary union of elements in $\mu$ is again in $\mu$. The ordered pair $(X, \mu)$ is called a generalized topological space. The elements of $\mu$ are called $\mu$ open sets or simply open sets [1]. A subset $A$ of $X$ is said to be closed set if $X \setminus A$ is open. Note that a generalized topology is said to be strong if $X \in \mu$ [1].

An operator on $C$ on $P(X)$ which maps $g \in \mu$ to the smallest closed set containing $g$ is a closure operator on $(X, \mu)$. It satisfies the conditions $A \subseteq C(A)$, $A \subseteq B \Rightarrow C(A) \subseteq C(B)$ and $C(C(A)) = C(A)$. By relaxing the idempotent condition of $C$, a generalized closure operator is defined as follows [5].

**Definition 2.1.** [5] Let $X$ be a set. A function $C : P(X) \rightarrow P(X)$ satisfying the conditions $A \subseteq C(A)$ and $A \subseteq B \Rightarrow C(A) \subseteq C(B)$ for every $A, B \subseteq X$ is called a generalized closure operator. The ordered pair $(X, C)$ is called a generalized closure space.

Email address: kavithatrnair@gmail.com (Kavitha T.)

Received: 17 July 2017  Accepted: 23 November 2017
http://dx.doi.org/10.20454/jast.2018.1301
2090-8288 ©2018 Modern Science Publishers. All rights reserved.
A subset $A$ of $X$ is said to be closed if $C(A) = A$ and is said to be open if its complement is closed. The set of all open subsets of $X$ forms a generalized topology on $X$ called the generalized topology associated with the generalized closure operator $C$.

Let $\mu$ be a generalized topology on $X$. Then the generalized function which maps $A$ into the smallest closed set containing $A$ is a generalized closure operator on $X$. This is called the generalized closure operator associated with $\mu$. A generalized closure operator is said to be strong if $C(C(A)) = C(A)$ for every $A \subseteq X$. The generalized closure operator associated with $\mu$ is strong. 

**Example 2.2.** Let $X = \{a, b, c\}$. Define $C : P(X) \to P(X)$ as $C(\phi) = \phi$, $C(\{a\}) = \{a\}$, $C(\{b\}) = \{b, c\}$, $C(\{c\}) = \{c\}$, $C(\{a, b\}) = C(\{b, c\}) = C(\{c, a\}) = C(X) = X$. Then $C(C(\{b\})) = C(\{b, c\}) = X \neq C(\{b\})$. Thus $C$ is a generalized closure operator on $X$ which is not a strong generalized closure operator.

Let $C_1, C_2$ be two generalized closure operators on a set $X$. Then we say $C_1 \leq C_2$ if and only if $C_2(A) \subseteq C_1(A)$ for every $A \subseteq X$. Then $\leq$ is a partial order on the set of all generalized closure operators. Define $(C_1 \wedge C_2)(A) = C_1(A) \cap C_2(A)$ and $(C_1 \vee C_2)(A) = C_1(A) \cap C_2(A)$. The set of all generalized closure operators on $X$ forms a complete lattice and is denoted by $LG(X)$. The generalized closure operator defined on $X$ as $I(A) = X$ for all $A \subseteq X$ is the smallest element of $LG(X)$. The Discrete closure operator defined by $D(A) = A$ for all $A \subseteq X$ is the largest element of $LG(X)$.

**Theorem 3.2.** Let $X$ be a set. The lattice of generalized closure operators on $X$ is a complete lattice. Let $\{C_a : a \in \mathcal{A}\}$ be a nonempty family of generalized closure operators on $X$. Then the greatest lower bound $\inf_{a \in \mathcal{A}} \{C_a|a \in \mathcal{A}\}(S) = \bigcup_{a \in \mathcal{A}} \{C_a(S)\}$, for each $S \subseteq X$.

**Proof.** First of all we prove that $C = \{X \to \bigcup_{a \in \mathcal{A}} \{C_a|a \in \mathcal{A}\}|S \subseteq X\}$ is a generalized closure operator on $X$. We have $S \subseteq C(S)$ for every $a \in \mathcal{A}$ and $S \subseteq X$. Thus $S \subseteq C(S)$ for every $S \subseteq X$. Suppose $S_1 \subseteq S_2$. Then $C_a(S_1) \subseteq C_a(S_2)$ for each $a \in A$. Hence $\bigcup_{a \in \mathcal{A}} \{C_a(S_1)\} \subseteq \bigcup_{a \in \mathcal{A}} \{C_a(S_2)\}$. Thus $C(S_1) \subseteq C(S_2)$. Thus $C$ is a generalized closure operation. Let $C'$ be a generalized closure operator on $X$ such that $C' \leq C_a$ for every $a \in \mathcal{A}$. Then $C_a(S) \subseteq C'(S)$ for every $a \in \mathcal{A}$. This implies that $\bigcup_{a \in \mathcal{A}} \{C_a(S)\} \subseteq C'(S)$. Hence $C, \leq C$ and therefore $\inf_{a \in \mathcal{A}} \{C_a|a \in \mathcal{A}\}(S) = \bigcup_{a \in \mathcal{A}} \{C_a(S)\}$, for each $S \subseteq X$. That is every subset of $LG(X)$ has a meet. Thus $LG(X)$ is a complete lattice.

**Remark 2.4.** $(\bigvee_{a \in \mathcal{A}} \{C_a(S)\}) = \bigcap_{a \in \mathcal{A}} \{C_a(S)\}$ for each $S \subseteq X$. The map $C = \{X \to \bigcap_{a \in \mathcal{A}} \{C_a|a \in \mathcal{A}\}|S \subseteq X\}$ is a generalized closure operation on $X$. For, we have $A \subseteq C_a(S)$ for each $a \in \mathcal{A}$ and each $S \subseteq X$. Now suppose $S_1 \subseteq S_2$. Then $C_a(S_1) \subseteq C_a(S_2)$ for each $a \in \mathcal{A}$. Then $\bigcap_{a \in \mathcal{A}} \{C_a(S_1)\} \subseteq \bigcap_{a \in \mathcal{A}} \{C_a(S_2)\}$. Thus $C(S_1) \subseteq C(S_2)$. Hence $C$ is a closure operation on $X$.

Since $(\bigvee_{a \in \mathcal{A}} \{C_a(S)\}) = \bigcap_{a \in \mathcal{A}} \{C_a(S)\}$ and $(\bigwedge_{a \in \mathcal{A}} \{C_a(S)\}) = \bigcup_{a \in \mathcal{A}} \{C_a(S)\}$ for each $S \subseteq X$, we conclude that the lattice $LG(X)$ is distributive hence modular.

### 3. Infra and ultra generalized closure operators

Ramachanran determined Infra and ultra Čech closure operators in the lattice of Čech closure operators [4]. Here we determine the same in the lattice of generalized closure operators.

**Definition 3.1.** Let $X$ be any set and $x \in X$. Define $C_x : P(X) \to P(X)$ as $C_x(\phi) = X - \{x\}$ and $C_x(A) = X$ for every $A \subseteq X$. Then $C_x$ is a generalized closure operator on $X$ at each $x \in X$.

Atoms in the lattice of generalized closed sets is the generalized closure operator defined on $X$ by

**Theorem 3.2.** A generalized closure operator on $X$ is an infra generalized closure operator if and only if it is of the form $C_x$ for some $x \in X$. 
4. Generalized Čech closure operators

In this section we study generalized Čech closure operators.

Definition 4.1. [2] A function \( Cl : P(X) \to P(X) \) is said to be a generalized Čech closure operator if it satisfies the following conditions, \( Cl(\emptyset) = \emptyset, A \subseteq Cl(A) \) and if \( A \subseteq B \), then \( Cl(A) \subseteq Cl(B) \) for every \( A, B \subseteq X \).

Every Čech closure operator is a generalized Čech closure operator. Converse is not true.

Example 4.2. Let \( X = \{1, 2, 3\} \). Define \( Cl : P(X) \to P(X) \) as \( Cl(\emptyset) = \emptyset, Cl(\{1\}) = \{1\}, Cl(\{2\}) = \{2\}, Cl(\{3\}) = \{2, 3\}, Cl(\{2, 3\}) = \{2, 3\} \) and \( Cl(\{1, 2\}) = Cl(\{1, 3\}) = Cl(X) = X \). Thus \( Cl \) is a generalized Čech closure operator which is not a Čech closure operator.

A subset \( A \subseteq X \) is said to be closed if \( Cl(A) = A \) and is said to be open if its complement is closed. The set of all open sets in a generalized Čech closure space \((X, Cl)\) forms a strong generalized topology and is called the generalized topology associated with the generalized Čech closure operator \( Cl \). Now consider the converse situation. Let \((X, \mu)\) be a strong generalized topology on \( X \). Then the map which maps \( A \) to the smallest closed set containing \( A \) is a generalized Čech closure operator and is called the closure operator associated with the strong generalized topology \( \mu \) on \( X \).

With any generalized Čech closure operator, there is associated an interior operation denoted by \( int \) and is defined as below.

Definition 4.3. Let \((X, Cl)\) be a generalized Čech closure space. \( int : P(X) \to P(X) \) defined by \( int(S) = X - Cl(X - S) \). The set \( int(S) \) is called interior of \( S \) in \((X, Cl)\).
From the definition of interior operator and generalized Čech closure operator we have the following proposition.

**Proposition 4.4.** In a generalized Čech closure space we have the following:

(a) \( \text{int} X = X \).

(b) For each \( S \subseteq X \), \( \text{int} S \subseteq S \).

(c) If \( A \subseteq B \), \( \text{int} A \subseteq \text{int} B \).

**Remark 4.5.** \( \text{int}(A \cap B) \neq \text{int}(A) \cap \text{int} B \). For example Let \( X = \{a, b, c\} \). \( \text{Cl}(\{\phi\}) = \phi \), \( \text{Cl}(\{a\}) = \{a\}, \text{Cl}(\{b\}) = \{b\}, \text{Cl}(\{c\}) = \{c\}, \text{Cl}(\{a, b\}) = \text{Cl}(\{b, c\}) = \text{Cl}(\{a, c\}) = \text{Cl}(X) = X \). Then \( \text{Cl} \) is a generalized Čech closure operator on \( X \). Then \( \text{int}(\{a\}) = \text{int}(\{b\}) = \text{int}(\{c\}) = \phi \), \( \text{int}(\{a, b\}) = \{a, b\}, \text{int}(\{b, c\}) = \{b, c\} \). Then \( \text{int}(\{a, b\} \cap \{b, c\}) \neq \text{int}(\{a, b\}) \cap \{b, c\} \).

**Proposition 4.6.** Let \((X, Cl)\) be a generalized Čech closure space. Then \( S \subseteq X \) is open if and only if \( \text{int} S = S \).

**Proof.**

\[
\text{int} S = S \iff X - \text{Cl}(X - S) = S \\
\iff X - S = \text{Cl}(X - A) \\
\iff S \text{ is open.}
\]

**Definition 4.7.** A neighbourhood of a subset \( S \) of a generalized Čech closure operator is any subset \( N \) containing \( S \) in its interior. Thus \( N \) is a neighbourhood of \( S \) if and only if \( S \subseteq \text{int}(N) \). We say \( N \) is a neighbourhood of an element \( x \in X \), if \( N \) is a neighbourhood of the singleton set \( \{x\} \).

**Proposition 4.8.** Let \((X, Cl)\) be a generalized Čech closure space. A subset \( N \) of \( X \) is a neighbourhood of a subset \( S \) of \( X \) if and only if \( N \) is a neighbourhood of each point of \( X \). Also a subset \( S \) of \( X \) is open if and only if it is a neighbourhood of each of its points.

**Proof.** Proof is clear from the definition of neighbourhoods and the Proposition 4.6

**Theorem 4.9.** Let \( \mathcal{N} \) be the neighbourhood system of a subset \( S \) of a generalized Čech closure space \((X, Cl)\). Then every member of \( \mathcal{N} \) contains \( S \) and if \( X \supset N_1 \supset N_2 \in \mathcal{N} \), then \( N_2 \in \mathcal{N} \).

**Proof.** This result is obvious from the definition of a neighbourhood of a subset of \( X \).

**Theorem 4.10.** Let \((X, Cl)\) be a generalized Čech closure space. Then a point \( x \) is in the closure of a subset \( S \) of \( X \) if and only if every neighbourhood of \( x \) meets \( S \).

**Proof.** Suppose \( x \notin Cl(S) \), Then by the definition of neighbourhoods, \( X - S \) is a neighbourhood of \( S \). That is there exists a neighbourhood of \( x \) which does not meet \( S \). Thus every neighbourhood of \( x \) meets \( S \) implies that \( x \in Cl(S) \). Conversely suppose \( N \) is a neighbourhood of \( x \) which does not meet \( S \). Then \( x \in \text{int} N \) and \( S \cap N = \phi \). Thus \( x \in \text{int} N \notin \text{int} (X - S) = X - \text{Cl}(S) \). This implies that \( x \notin Cl(S) \). Hence \( x \) is in the closure of a subset \( S \) of \( X \) implies that every neighbourhood of \( x \) meets \( S \).

Arbitrary join and Arbitrary meet of a non void collection of generalized Čech closure operators are same as the arbitrary join and arbitrary meet of generalized closure operators. Thus the lattice of generalized Čech closure operators, \( \text{LGC}(X) \) is distributive and modular. Atoms in the lattice of generalized Čech closure operators are same as the atoms in the lattice of Čech closure operators.

**Theorem 4.11.** Atoms in the lattice of generalized Čech closure operators are same as the atoms in the lattice of Čech closure operators.

**Proof.** Let \( C \) be a generalized Čech closure operator such that \( I \leq C < V_{a,b} \). Then \( V_{a,b}(\{a\}) \subset C(\{a\}) \subseteq I(\{a\}) \). Then \( X - \{b\} \subset C(\{a\}) \subseteq X \). Then \( C(\{a\}) = X \). Hence \( C = I \).
5. Acknowledgement

I would like to express my sincere gratitude to my supervising teacher Dr. Ramachandran P. T. for his help and support during the preparation of this paper.

References