A characterization of $PSU(3, 4)$ by $nse$

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(Received: 3 May 2013; Accepted: 2 June 2013)

Abstract. Let $G$ be a group and $\omega(G)$ be the set of element orders of $G$, $k \in \omega(G)$ and $s_k$ be the number of elements of order $k$ in $G$, $nse(G) = \{s_k | k \in \omega(G)\}$. $PSL(3, 2)$ and $PSL(3, 4)$ are unique determined by $nse(G)$. In this paper, we prove that if $G$ is a group such that $nse(G) = nse(PSU(3, 4))$, then $G \cong PSU(3, 4)$.

1. Introduction

In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows (see [1]).

Thompsons Problem. Let $T(G) = \{(n, s_n)| n \in \omega(G) \text{ and } s_n \in nse(G)\}$, where $s_n$ is the number of elements with order $n$. Suppose that $T(G) = T(H)$. If $G$ is a finite solvable group, is it true that $H$ is also necessarily solvable?

It is easy to see that if $G$ and $H$ are of the same order type, then

$$nse(G) = nse(H) \text{ and } |G| = |H|.$$ 

Let $G$ be a group and $M$ some simple $K_i$-group, $i = 3, 4$, then $G \cong M$ if and only if $|G| = |M|$ and $nse(G) = nse(M)$ (see [2, 3]). And also the group $A_{12}$ is characterizable by order and $nse$ (see [4]). Recently, all sporadic simple groups are characterizable by $nse$ and order (see [5]).

Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the Thompson Problem, in other words, it remains only $nse(G)$, whether can it characterize finite simple groups? Up to now, some groups especial for $PSL(2, q)$, where $q \in \{7, 8, 9, 11, 13\}$, can be characterized by only the set $nse(G)$ (see [6, 7]).

The author has proved that the group $PSL(3, 4)$ is characterizable by $nse$ (see [8]).

In this paper, it is shown that the group $PSU(3, 4)$ also can be characterized by $nse(PSU(3, 4))$.

$PSL(n, q)$ (in short $L_n(q)$) denotes the projective special linear group of degree $n$ over finite fields of order $q$. $PSU(n, q)$ (in short $U_n(q)$) denotes the projective special unitary group of degree $n$ over finite fields of order $q$. For the other notation and notions, please refer to [9].
2. Some Lemmas

Lemma 2.1. [10] Let $G$ be a finite group and $m$ be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m | L_m(G)$.

Lemma 2.2. [11] Let $G$ be a finite group and $p \in \pi(G)$ be odd. Suppose that $P$ is a Sylow $p$-subgroup of $G$ and $n = p^r \cdot m$ with $(p, m) = 1$. If $P$ is not cyclic and $s > 1$, then the number of elements of order $n$ is always a multiple of $p^s$.

Lemma 2.3. [7] Let $G$ be a group containing more than two elements. If the maximal numbers of elements of the same order in $G$ is finite, then $G$ is finite and $|G| \leq s(s^2 - 1)$.

Lemma 2.4. [12, Theorem 9.3.1] Let $G$ be a finite solvable group and $|G| = mn$, where $m = p_1^{n_1} \cdots p_s^{n_s}$, $(m, n) = 1$. Let $\pi = \{p_1, \ldots, p_s\}$ and $h_m$ be the number of Hall $\pi$-subgroups of $G$. Then $h_m = q_1^{h_1} \cdots q_s^{h_s}$ satisfies the following conditions for all $i \in \{1, 2, \ldots, s\}$:

1. $q_i^{h_i} \equiv 1 (\text{mod } p_i)$ for some $p_i$.
2. The order of some chief factor of $G$ is divided by $q_i^{h_i}$.

To prove $G \cong \text{PSU}(3, 4)$, we need the structure of simple $K_4$-groups.

Lemma 2.5. [13] Let $G$ be a simple $K_4$-group. Then $G$ is isomorphic to one of the following groups:

1. $A_7, A_8, A_9$ or $A_{10}$.
2. $M_{11}, M_{12}$ or $J_2$.
3. One of the following:
   (a) $L_2(r)$, where $r$ is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geq 1$, $b \geq 1$, $c \geq 1$, and $v$ is a prime greater than 3.
   (b) $L_2(2^m)$, where $2^m - 1 = u, 2^m + 1 = 3^b$ with $m \geq 2, u, t$ are primes, $t > 3, b \geq 1$.
   (c) $L_2(3^m)$, where $3^m + 1 = 4t, 3^m - 1 = 2u$ or $3^m + 1 = 4t^b, 3^m - 1 = 2u$, with $m \geq 2, u, t$ are odd primes, $b \geq 1, c \geq 1$.
4. One of the following 28 simple groups: $L_2(16), L_2(25), L_2(49), L_2(81), L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), L_4(3), S_4, S_5(4), S_6(5), S_7(7), S_8(9), S_9(8), S_{10}(2), O_{10}^+(2), G_2(3), U_3(4), U_3(5), U_3(7), U_3(8), U_4(3), U_4(2), Sz(8), Sz(32), 2D_4(2)$ or $^2F_4(2)$.

Lemma 2.6. Let $G$ be a simple $K_4$-group and $5^2 | |G| | 2^6.3.5.13$. Then $G \cong \text{PSU}(3, 4), \text{PSL}(2, 25)$.

Proof. From Lemma 2.5(1)(2), order consideration rules out this case. So we consider Lemma 2.5(3). We will deal with this with the following cases:

Case 1. $G \cong \text{PSL}(2, r)$, where $r \in \{3, 5, 13\}$.

   Let $r = 3$, then $|\pi(q^2 - 1)| = 1$, which contradicts $|\pi(q^2 - 1)| = 3$.
   Let $r = 5$ then $|\pi(q^2 - 1)| = 2$, which contradicts $|\pi(q^2 - 1)| = 3$.
   Let $r = 13$, then $|\pi(q^2 - 1)| = 3$, which contradicts $7 | |G|$, a contradiction.

Case 2. $G \cong \text{PSL}(2, 2^u)$, where $u \in \{3, 5, 13\}$.

   Let $u = 3$, then $m = 2$ and so $5 = 3t^b$. But the equation has no solution in $N$, a contradiction.
   Let $u = 5, 13$, then $2^m - 1 = u$. But the equation has no solution in $N$.

Case 3. $G \cong \text{PSL}(2, 3^m)$. We will consider the case by the following two subcases.
3. Main theorem and its proof

Let $G$ be a group such that $\text{nse}(G) = \text{nse}(	ext{PSU}(3, 4))$, and $s_n$ be the number of elements of order $n$. By Lemma 2.3 we have $G$ is finite. We note that $s_n = k\phi(n)$, where $k$ is the number of cyclic subgroups of order $n$. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 2.1 and the above discussion, we have

$$
\begin{cases}
\phi(m) | s_m \\
m | \sum_{d|m} s_d
\end{cases}
$$

(1)

**Theorem 3.1.** Let $G$ be a group with $\text{nse}(G) = \text{nse}(	ext{PSU}(3, 4)) = \{1, 195, 3900, 4160, 5824, 12480, 16640, 19200\}$, where $\text{PSU}(3, 4)$ is the projective special unitary group of degree 3 over field of order 4. Then $G \cong \text{PSU}(3, 4)$.

**Proof.** We prove the Theorem by first proving that $\pi(G) \subseteq \{2, 3, 5, 13\}$, second showing that $|G| = |\text{PSU}(3, 4)|$, and so $G \cong \text{PSU}(3, 4)$. By (1), $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$. If $m > 2$, then $\phi(m)$ is even, then $s_2 = 195$, $2 \in \pi(G)$. By (1) $s_3 = 4160, 16640, s_5 = 5824, s_7 = 12480, 19200$, and $s_{13} = 19200$.

If $2^a \in \omega(G)$, then $\phi(2^a) = 2^{a-1} | s_{2^a}$ and so $0 \leq a \leq 9$.

If $3^a \in \omega(G)$, then $1 \leq a \leq 2$.

If $5^a \in \omega(G)$, then $1 \leq a \leq 3$.

If $7^a \in \omega(G)$, then $1 \leq a \leq 2$. If $a=2$, then by (1), $s_7 \notin \text{nse}(G)$. So $a=1$.

If $13^a \in \omega(G)$, then $1 \leq a \leq 2$. If $a=2$, then by (1), $s_{13} \notin \text{nse}(G)$. So $a=1$.

Therefore we have that $\{2\} \subseteq \pi(G) \subseteq \{2, 3, 5, 7, 13\}$. Since $\exp(P_2) = 2$, ..., 64, 512, then by Lemma 2.1, $|P_2| \mid 1 + s_2 + s_2 + \ldots + s_{2^9}$ and so $|P_2| \mid 2^9$.

If $3 \in \pi(G)$ and $\exp(P_3) = 3, 9, 27$, then by Lemma 2.1, $|P_3| \mid 1 + s_3 + s_9 + s_{27}$ and so $|P_3| \mid 3^5$.

If $5 \in \pi(G)$ and $\exp(P_5) = 5, 25, 125, 625$, then by Lemma 2.1, $|P_5| \mid 1 + s_5 + s_{25} + s_{125} + s_{625}$ and so $|P_5| \mid 5^4$.

If $7 \in \pi(G)$, then by Lemma 2.1, $|P_7| \mid 1 + s_7$ and so $|P_7| = 7$.

To remove the prime 7, we show that $13 \in \pi(G)$. Assume that $13 \notin \pi(G)$. Since $\exp(P_2) = 2$, ..., 512, then by Lemma 2.1, $|P_2| \mid 1 + s_2 + \ldots + s_{2^9}$.

If $3, 5, 7 \notin \pi(G)$, then $G$ is a 2-group and $62400 + 3900k_1 + 4160k_2 + 5824k_3 + 12480k_4 + 16640k_5 + 19200k_6 = 2^m$, where $k_1, \ldots, k_6$ and $m$ are non-negative integers and $0 \leq \sum_{i=1}^8 s_i \leq 0$. So the equation has no solution in $N$.

Hence 3, 7, or 13 belongs to $\pi(G)$. We divide the proof into the following cases.

Case a. Let $7 \in \pi(G)$. Then by Lemma 2.1, $|P_7| \mid 1 + s_7$ and so $|P_7| = 7$. Since $n_7 = s_7/\phi(7)$, then $3, 5|\pi(G)$, a contradiction.

Case b. Let $5 \in \pi(G)$. Since $\phi(5) = \phi(2.5)$, then $s_{2.5} = 5824$.

If $4.5 \in \omega(G)$, set $P$ and $Q$ are Sylow 5-subgroup of $G$, then $P$ and $Q$ are conjugate in $G$ and so $C_G(P)$ and $C_Q(Q)$ are also conjugate in $G$. Therefore we have $s_{4.5} = \phi(4.5).n_{5,k}$, where $k$ is the number of cyclic subgroups of
order 4 in $C_G(P_3)$. If $s_5 = 5824$, then $11648 | s_{15}$ and so $s_{15}=11648t$ for some integer $t$. Since $s_{15} \in \text{en}(G)$, the equation $s_{15} = 11648t$ has no solution in $N$, a contradiction. Therefore $4.5 \notin \omega(G)$. It follows that the Sylow 5-subgroup of $G$ acts fixed point freely on the set of elements of order 5 and so $|P_5| | s_4$. Therefore $|P_5| | 5^2$.

If $5^2 \cdot 3 \notin \omega(G)$, set $P$ and $Q$ are Sylow 3-subgroup of $G$, then $P$ and $Q$ are conjugate in $G$ and so $C_G(P)$ and $C_G(Q)$ are also conjugate in $G$. Therefore we have $s_{5^2 \cdot 3} = \phi(5^2 \cdot 3)n_3k$, where $k$ is the number of cyclic subgroups of order 25 in $C_G(P_3)$.

If $s_3 = 4160$, then $83200 | s_{5^2 \cdot 3}$ and so $s_{5^2 \cdot 3} = 83200t$ for some integer $t$. Since $s_{5^2 \cdot 3} \in \text{en}(G)$, the equation $s_{5^2 \cdot 3} = 83200t$ has no solution in $N$, a contradiction.

If $s_3 = 16640$, then $332800 | s_{5^2 \cdot 3}$ and so $s_{5^2 \cdot 3} = 332800t$ for some integer $t$. Since $s_{5^2 \cdot 3} \in \text{en}(G)$, the equation $s_{5^2 \cdot 3} = 332800t$ has no solution in $N$, a contradiction.

Therefore $5^2 \cdot 3 \notin \omega(G)$. It follows that the Sylow 3-subgroup of $G$ acts fixed point freely on the set of elements of order 25 and so $|P_3| | s_{25}$. Therefore $|P_3| = 3$.

We know that $\exp(P_3)=5$, 25. Let $\exp(P_3)=5$. Then by Lemma 2.1, $|P_3| | 1 + s_5$ and so $|P_3| | 5^2$. If $|P_3| = 5$, then since $n_5 = s_5//\phi(5)$, 7, 13 $\in \pi(G)$, a contradiction. If $|P_3| = 5^2$, then since $13,7 \notin \pi(G)$ and $n_5 = s_5//\phi(5)$, 7, 13, 13 $\in \pi(G)$, we can assume that $[2] \notin \pi(G) \subseteq [2, 3, 5]$ and so $62400 + 3900k_1 + 4160k_2 + 5824k_3 + 12480k_4 + 16640k_5 + 19200k_6 = 2^m \cdot 3^2$, where $k_1, \ldots, k_6$ and $m$ are non-negative integers and $0 \leq \sum_{i=1}^{6} s_i \leq 15$. Since $62400 \leq |G| = 2^m \cdot 3^2 \leq 62400 + 19200 \cdot 15$, then the equation has no solution for $m$.

Case c. Let $3 \in \pi(G)$. We know that $\exp(P_3)=3$, or 9.

Let $\exp(P_3)=3$. Then by Lemma 2.1, $|P_3| | 1 + s_3$ and so $|P_3| | 9$. If $|P_3| = 9$, then since $n_3 = s_3//\phi(3)$, 5 $\in \pi(G)$, a contradiction. If $|P_3| = 3$, then since $13,7,5 \notin \pi(G)$, we can assume that $G$ is a $[2, 3]$-group. Therefore $62400 + 3900k_1 + 4160k_2 + 5824k_3 + 12480k_4 + 16640k_5 + 19200k_6 = 2^m \cdot 3^2$, where $k_1, \ldots, k_6$ and $m$ are non-negative integers and $0 \leq \sum_{i=1}^{8} s_i \leq 9$. Since $62400 \leq |G| = 2^m \cdot 3^2 \leq 62400 + 19200 \cdot 9$, then the equation has no solution in $N$.

Let $\exp(P_3)=9$. Then by Lemma 2.1, $|P_3| | 1 + s_3$ and so $|P_3| = 9$. Since $n_3 = s_3//\phi(3^2)$, then 5 $\in \pi(G)$, a contradiction.

Therefore $13 \in \pi(G)$.

Let $7 \in \pi(G)$. If $7,13 \notin \omega(G)$, then by Lemma 2.1, $7,13 | 1 + s_7 + s_{13}$ and so $s_{7,13} \notin \text{en}(G)$. Therefore $7,13 \notin \omega(G)$. It follows that the Sylow 7-subgroup of $G$ acts fixed point freely on the set of elements of order 13 and so $|P_7| | s_{13}$, a contradiction. Therefore $7 \notin \pi(G)$. So $\pi(G) \subseteq [2, 3, 13]$.

In the following, we consider the proper set of $[2, 3, 5, 13]$ which contains the primes 2 and 13.

Case a. $\pi(G) = [2, 13]$.

Since $\exp(P_{13})=13$, then by Lemma 2.1, $|P_{13}| | 1 + s_{13}$ and so $|P_{13}| = 13$. Since $n_{13} = s_{13}/\phi(13)$, then 3, 5 $\in \pi(G)$, a contradiction.

Case b. $\pi(G) = [2, 3, 13]$.

Similarly as the “Case a $\pi(G) = [2, 13]$”, 5 $\in \pi(G)$, a contradiction.

Case c. $\pi(G) = [2, 5, 13]$.

Similarly as the “Case a $\pi(G) = [2, 13]$”, 3 $\in \pi(G)$, a contradiction.

Case d. $\pi(G) = [2, 3, 5, 13]$.

In the following, we first prove the possible orders of $G$ and then show that there is no group such that $|G| = 2^m \cdot 3^2 \cdot 13$, where $m=7, 8$ and $\text{en}(G) = \text{en}(\text{PSU}(3, 4))$. Finally have the desired result by [3].

Step 1. $|G| = 2^m \cdot 3^2 \cdot 13$, where $m=6, 7, 8$. 

First we show that $2.13 \notin \omega(G)$.

If $2.13 \in \omega(G)$, set $P$ and $Q$ are Sylow 13-subgroup of $G$, then $P$ and $Q$ are conjugate in $G$ and so $C_G(P)$ and $C_G(Q)$ are also conjugate in $G$. Therefore we have $s_{2.13} = \phi(2.13)\cdot n_{13}, k$, where $k$ is the number of cyclic subgroups of order 2 in $C_G(P_{13})$. If $s_{13} = 19200$, then $19200 \mid s_{2.13}$ and so $s_{2.13}=19200t$ for some integer $t$. Since $s_{2.13} \in \text{ense}(G)$, $s_{2.13}=19200$. On the other hand, by Lemma 2.1, $2.13 \mid 1 + s_2 + s_{13} + s_{2.13}$, a contradiction. Therefore $2.13 \notin \omega(G)$. It follows that the Sylow 2-subgroup of $G$ acts fixed point freely on the set of elements of order 13 and so $|P_{2}||G_{13}$. Thus $|P_{2}| = 2^6$ and $|P_{13}| = 13$.

Similarly $3.13 \notin \omega(G)$.

Therefore we can assume that $|G| = 2^6, 3, 5, 13$. Since $62400 = 2^6, 3, 5, 13 \leq |G| = 2^6, 3, 5, 13$. So $p=2$ and $m=6, 7, 8$.

**Step 2.** There is no group such that $|G| = 2^6, 3, 5, 13$, and $n_{s}(G)=n_{s}(PSU(3,4))$.

Let $|G| = 2^6, 3, 5, 13$.


Let $n_{13}(H) = 3^{a_1+s_1+a_2+s_2+\cdots+b_s}$ by Lemma 2.4, $3^6 \equiv 1 \pmod{13}$, $3^5 \equiv 1 \pmod{13}$, $i = 1, \ldots, r$, $j = 1, 2, \ldots, s$, where $a_1, s_1, a_2, s_2, \ldots, b_s$ are non-negative integers and $a_1 + s_1 + a_2 + s_2 + \cdots + b_s \leq 2$. Hence $n_{13}(H) = 1$. So $12 \leq s_{13}(G) \leq 1536$ and $12 \mid s_{13}(G)$, but $s_{13}(G) \notin n_{s}(G)$, a contradiction. So $G$ is insoluble.

Therefore $G$ has a normal series 1 $\triangleleft K$ $\triangleleft L$ $\triangleleft G$ such that $L/K$ is isomorphic to a simple $K_\omega$-group with $i=3, 4$ as 9 and 169 don’t divide the order of $G$.

If $L/K$ is isomorphic to a simple $K_\omega$-group, from [14], $L/K \cong A_5, A_6$. Let $L/K \cong A_5$. Then $|G/L| = 2^6, 5, 13$.

Let $A / K := C_{G/K}(L/K)$. Then $A/K \cap L/K = 1$. We see that $(G/K)/(A/K) \cong \text{Aut}(L/K)=S_5$ and so $G/K \cong S_5$. Since $A/K, L/K, A/K \times L/K \leq G/K$. Therefore $[L/K] \mid |G/K|$ and so $G/K \cong A_5$ or $S_5$, i.e., $A=2^6, 5, 13$ or $2^5, 5, 13$.

By Sylow’s theorem, $n_{13}(A)=1, 40$. Since $A \triangleleft G$, we have that $n_{13}(A) = n_{13}(G)$, and so $s_{13}(G) = 12, 480$, which contradicts $s_{13}(G) \notin \text{ense}(G)$. Similarly we can rules out the other cases “$L/K \cong A_6$”. Hence $G$ is isomorphic to a simple $K_\omega$-group, then by Lemma 2.6, $L/K \cong PSU(3,5)$ or $PSL(2,25)$. If $L/K \cong PSL(2,25)$, then order consideration rules out this case. So $G/A \cong \text{Aut}(PSU(3,4))$. Therefore $G/A \cong PSU(3,4), G/A \cong PSL(3,4)$, or $G/A \cong A_5$.

If $G/A \cong PSU(3,4)$, then order consideration $|A| = 2$. It follows that $A$ is a normal subgroup generated by a 2-central element of $G$. So there exists an element of order 2.13, which is a contradiction.

If $G/A \cong PSL(3,4)$, then order consideration $|A| = 1$. By Sylow’s theorem, $n_{s}(2.PSU(3,4)) = 1, 16, 26, 96, 156, 416, 2496$ and so the number of elements of order 5 of $G$ is at most $24, 384, 624, 2304, 3744, 9984, 59904$, but none of which belongs to $n_{s}(G)$, a contradiction.

If $G/A \cong PSU(3,4)$, then order consideration rules out this case. Let $|G| = 2^6, 3, 5, 13$. Similarly as the proof of $|G| = 2^6, 3, 5, 13$, we can rules out this case.

**Step 3.** $G \cong PSU(3,4)$.

From Steps 1 and 2, $|G| = 2^6, 3, 5, 13$, and by assumption, $n_{s}(G)=n_{s}(PSU(3,4))$, then by [3], $G \cong PSU(3,4)$.

This completes the proof of the theorem. $\square$

**4. conclusion**

The group $U_2(4)$ is also characterized by nse only.

**Acknowledgments**

The author is very grateful for the helpful suggestions of the referee.
References