Some New Difference Sequence Spaces Defined by A Sequence of Modulus Functions

Kuldip Raj*, Seema Jamwal*

*a School of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J&K, India.
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Abstract. In this paper, we have constructed some new difference sequence spaces defined by a sequence of modulus functions and study some topological and algebraic properties of these spaces.

1. Introduction and Preliminaries

Let $X$ be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if:

1. $p(x) \geq 0$ for all $x \in X$.
2. $p(-x) = p(x)$ for all $x \in X$.
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.
4. If $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair $(X, p)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [3, Theorem 10.4.2, pp. 183]).

A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that:

1. $f(x) = 0$ if and only if $x = 0$.
2. $f(x + y) \leq f(x) + f(y)$, for all $x, y \geq 0$.
3. $f$ is increasing.
4. $f$ is continuous from the right at 0.

It follows that $f$ must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{1}{x^p}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus function $f(x)$ is unbounded. Substantially, modulus function has been discussed in [11, 17, 24, 25] and references therein.

Let $w, l_\infty, c$ and $c_0$ denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms respectively.

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Email addresses: kuldeepraj68@rediffmail.com (Kuldip Raj), seemajamwal8@gmail.com (Seema Jamwal)
The notion of difference sequence spaces was introduced by Kizmaz [13], who studied the difference sequence spaces \( l_{o}(\Delta), \) \( c(\Delta) \) and \( c_{0}(\Delta). \) The notion was further generalized by Et and Çolak [19] by introducing the spaces \( l_{o}(\Delta^n), \) \( c(\Delta^n) \) and \( c_{0}(\Delta^n). \) Later the concept have been studied by Bektaş et al. [6] and Et et al. [20]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [5] who studied the spaces \( l_{o}(\Lambda), \) \( c(\Lambda) \) and \( c_{0}(\Lambda). \) Recently, Esi et al. [2] and Tripathy et al. [4] have introduced a new type of generalized difference operators and unified those as follows:

Let \( v, n \) be non-negative integers, then for \( Z \) a given sequence space, we have

\[
Z(\Delta^n_v) = \{ x = (x_k) \in w : (\Delta^n_v x_k) \in Z \}
\]

for \( Z = c, c_0 \) and \( l_{\infty} \) where \( \Delta^n_v x = (\Delta^n_v x_k) = (\Delta^{n-1}_v x_k - \Delta^{n-1}_v x_{k+v}) \) and \( \Delta^n_0 x_k = x_k \) for all \( k \in \mathbb{N}, \) which is equivalent to the following binomial representation

\[
\Delta^n_v x_k = \sum_{m=0}^{n} (-1)^m \binom{n}{m} x_{k+vm}.
\]

Taking \( v = 1, \) we get the spaces \( l_{o}(\Delta^n), c(\Delta^n) \) and \( c_{0}(\Delta^n) \) studied by Et and Çolak [19]. Taking \( v = n = 1, \) we get the spaces \( l_{o}(\Delta), c(\Delta) \) and \( c_{0}(\Delta) \) introduced and studied by Kizmaz [13]. For more details about sequence spaces (see [17, 18]) and references therein.

Let \( X \) be a sequence space. Then the sequence space \( X(f) \) is defined as

\[
X(f) = \{ x = (x_k) \in w : f(x_k) \in X \}.
\]

Later Kolk [8, 9] gave an extension of \( X(f) \) by considering a sequence of modulus functions \( F = (f_k) \) that is

\[
X(f) = \{ x = (x_k) \in w : f_k(x_k) \in X \}.
\]

In [1] Gaur and Mursaleen defined the following sequence spaces:

\[
l_{o}(F, \Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in l_{o}(F) \},
\]

\[
c_{0}(F, \Lambda) = \{ x = (x_k) \in w : (\Lambda x_k) \in c_{0}(F) \}.
\]

Later Bektaş and Çolak in [7] defined the following sequence spaces:

\[
l_{o}(F, \Delta^n) = \{ x = (x_k) \in w : (\Delta^n x_k) \in l_{o}(F) \},
\]

\[
c_{0}(F, \Delta^n) = \{ x = (x_k) \in w : (\Delta^n x_k) \in c_{0}(F) \}.
\]

Recently V. A. Khan in [23] defined the following sequence spaces:

\[
l_{o}(F, p, \Delta^n) = \{ x = (x_k) \in w : (\Delta^n x_k) \in l_{o}(F, p) \},
\]

\[
c_{0}(F, p, \Delta^n) = \{ x = (x_k) \in w : (\Delta^n x_k) \in c_{0}(F, p) \}.
\]

The notion of statistical convergence was introduced by H. Fast [12]. Later on, it was studied by J. A. Fridy [14, 15] from the sequence space point of view and linked with the summability theory. The notion of \( l \)-convergence is a generalization of the statistical convergence. It was studied at initial stage by Kostyrko et al. [21]. Later on, it was studied by Salat et al. [22], Demirici [16] and many others.

Let \( \mathbb{N} \) be a non empty set. Then a family of sets \( I \subseteq 2^\mathbb{N} \) (Power set of \( \mathbb{N} \)) is said to be an ideal if \( I \) is additive, i.e., \( A, B \in I \Rightarrow A \cup B \in I \) and \( A \in I, B \subset A \Rightarrow B \in I. \) A non empty family of sets \( \mathcal{E}(I) \subseteq 2^\mathbb{N} \) is said to be filter on \( \mathbb{N} \) if and only if \( \emptyset \notin \mathcal{E}(I) \) for \( A, B \in \mathcal{E}(I), \) we have \( A \cap B \in \mathcal{E}(I) \) and for each \( A \in \mathcal{E}(I) \) and \( A \subseteq B \) implies \( B \in \mathcal{E}(I). \)

An ideal \( I \subseteq 2^\mathbb{N} \) is called non trivial if \( I \neq 2^\mathbb{N}. \) A non trivial ideal \( I \subseteq 2^\mathbb{N} \) is called admissible if \( \{x : x \in \mathbb{N} \} \subseteq I. \) A non-trivial ideal is maximal if there cannot exist any non trivial ideal \( J \neq I \) containing \( I \) as a subset. For each ideal \( I, \) there exist a filter \( \mathcal{E}(I) \) corresponding to \( I, \) i.e., \( \mathcal{E}(I) = \{ K \subseteq N : K^c \in I \}, \) where \( K^c = N \setminus K. \)
Definition 1.2. A sequence \((x_k) \in w\) is said to be \(I\)-convergent to a number \(L\) if for every \(\epsilon > 0\), the set \(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I\). In this case we write \(I - \lim x_k = L\).

Definition 1.3. A sequence \((x_k) \in w\) is said to be \(I\)-null if \(L = 0\). In this case we write \(I - \lim x_k = 0\).

Definition 1.4. A sequence \((x_k) \in w\) is said to be \(I\)-cauchy if for every \(\epsilon > 0\), there exist a number \(m = m(\epsilon)\) such that \(\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I\).

Definition 1.5. \([1, 7]\) The condition \([1, 7]\) in the present paper we defined the following lemmas.

Lemma 1.6. \([1, 7]\) The condition \([1, 7]\) hold if and only if there exist a point \(t_0 > 0\) such that \(\sup_k f_k(t_0) < \infty\).

Lemma 1.7. \([22]\) Let \(K \in L(I)\) and \(M \subseteq N\). If \(M \neq 1\) then \(M \cap K \neq 1\).

Lemma 1.8. \([21]\) If \(I \subseteq 2^N\) and \(M \subseteq N\). If \(M \neq 1\) then \(M \cap K \neq 1\).

Let \(F = (f_k)\) be a sequence of modulus functions, \(p = (p_k)\) be a bounded sequence of positive real numbers and \(u = (u_k)\) be a sequence of strictly positive real numbers. In the present paper we defined the following sequence spaces:

\[
c_0^I(F, p, u, \Delta_n^u) = \{(x_k) \in w : I - \lim_k f_k(|u_k \Delta_n^u x_k|^p) = 0\} \in I,
\]

\[
l_0^I(F, p, u, \Delta_n^u) = \{(x_k) \in w : I - \sup_k f_k(|u_k \Delta_n^u x_k|^p) < \infty\} \in I.
\]

If \(F = f_k(x) = x\), for all \(k\), we have

\[
c_0^I(p, u, \Delta_n^u) = \{(x_k) \in w : I - \lim_k (u_k \Delta_n^u x_k)^p = 0\} \in I,
\]

and

\[
l_0^I(p, u, \Delta_n^u) = \{(x_k) \in w : I - \sup_k (u_k \Delta_n^u x_k)^p < \infty\} \in I.
\]

If \((p_k) = 1, \) for all \(k \in \mathbb{N}\), then

\[
c_0^I(F, u, \Delta_n^u) = \{(x_k) \in w : I - \lim_k f_k(|u_k \Delta_n^u x_k|) = 0\} \in I,
\]

and

\[
l_0^I(F, u, \Delta_n^u) = \{(x_k) \in w : I - \sup_k f_k(|u_k \Delta_n^u x_k|) < \infty\} \in I.
\]

If \((p_k) = 1, \) for all \(k \in \mathbb{N}\) and \((u_k) = 1, \) for all \(k, \) we have

\[
c_0^I(F, \Delta_n^u) = \{(x_k) \in w : I - \lim_k f_k(|\Delta_n^u x_k|) = 0\} \in I,
\]

and

\[
l_0^I(F, \Delta_n^u) = \{(x_k) \in w : I - \sup_k f_k(|\Delta_n^u x_k|) < \infty\} \in I.
\]
If \( F = f_k(x) = x \), for all \( k \) and \((p_k) = 1\), for all \( k \in \mathbb{N} \), then
\[
c_0^1(u, \Delta^u_p) = \{(x_k) \in w : I - \lim(|u_k\Delta^u_p x_k|) = 0\} \in I,
\]
and
\[
l_\infty^1(u, \Delta^u_p) = \{(x_k) \in w : I - \sup_\Delta^u_p x_k < \infty\} \in I.
\]
If \( F = f_k(x) = x \), \((p_k) = 1\) and \((u_k) = 1\), for all \( k \), we have
\[
c_0^1(\Delta^u_p) = \{(x_k) \in w : I - \lim(|\Delta^u_p x_k|) = 0\} \in I,
\]
and
\[
l_\infty^1(\Delta^u_p) = \{(x_k) \in w : I - \sup_\Delta^u_p x_k < \infty\} \in I.
\]
The following inequality will be used throughout the paper. Let \( p = (p_k) \) be a sequence of positive real numbers with \( 0 < p_k \leq \sup_k p_k = H \), and let \( D = \max\{1, 2^{H-1}\} \). Then, for the factorable sequences \((a_k)\) and \((b_k)\) in the complex plane, we have
\[
|a_k + b_k|^p \leq D(|a_k|^p + |b_k|^p).
\]
The main purpose of this paper is to study some new difference sequence spaces in more general settings defined by a sequence of modulus functions. We also make an effort to study some algebraic, topological properties and interesting inclusion relations between the above defined sequence spaces.

2. Main results

**Theorem 2.1.** Let \( F = (f_k) \) be a sequence of modulus functions, then \( c_0^1(F, p, u, \Delta^u_p) \) and \( l_\infty^1(F, p, u, \Delta^u_p) \) are linear spaces.

**Proof.** Let \( x = (x_k) \) and \( y = (y_k) \in c_0^1(F, p, u, \Delta^u_p) \) and for \( \alpha, \beta \in \mathbb{C} \). Then there exist integers \( M_\alpha \) and \( M_\beta \) such that \(|\alpha| \leq M_\alpha \) and \(|\beta| \leq M_\beta \). Since \( F = (f_k) \) is a sequence of modulus functions so using the inequality (1), we have
\[
\begin{align*}
f_k(|u_k\Delta^u_p (\alpha x_k + \beta y_k)|^p) & \leq D(M_\alpha)^{f_k(|u_k\Delta^u_p x_k|^p)} + D(M_\beta)^{f_k(|u_k\Delta^u_p y_k|^p)} \\
& \rightarrow 0 \text{ as } k \rightarrow \infty.
\end{align*}
\]
Therefore \( \alpha x + \beta y \in c_0^1(F, p, u, \Delta^u_p) \). Hence \( c_0^1(F, p, u, \Delta^u_p) \) is a linear space. Similarly we can prove that \( l_\infty^1(F, p, u, \Delta^u_p) \) is also a linear space. \( \Box \)

**Theorem 2.2.** Let \( F = (f_k) \) be a sequence of modulus functions, then: \( c_0^1(F, p, u, \Delta^u_p) \) and \( l_\infty^1(F, p, u, \Delta^u_p) \) are paranormed spaces with paranorm
\[
g(x) = \sup_\Delta^u_p x_k \Delta^u_p x_k |^p \]
where \( H = \sup_k p_k < \infty \) and \( M = \max(1, H) \).
Proof. Clearly $g(x) = g(-x)$ for all $x \in c_0^\alpha(F, p, u, \Delta^n_u)$. It is trivial that $u_k \Delta^n_u x_k = 0$ for $x = 0$. Since $\frac{p}{\alpha} \leq 1$, using Minkowski’s inequality, we have

$$\|f_k([u_k \Delta^n_u x_k + u_k \Delta^n_u y_k]^p)]^\frac{1}{p} \leq \|f_k([u_k \Delta^n_u x_k]^p)]^\frac{1}{p} + \|f_k([u_k \Delta^n_u y_k]^p)]^\frac{1}{p},$$

Hence $g(x + y) \leq g(x) + g(y)$. Finally to check the continuity of scalar multiplication, let us take a complex number $\lambda$ by definition, we have

$$g(\lambda x) = \sup_k f_k([u_k \Delta^n_u \lambda x_k]^p)]^\frac{1}{p} \leq \sup_k f_k([u_k \Delta^n_u x_k]^p)]^\frac{1}{p} \leq K_1 \sup_k f_k([u_k \Delta^n_u x_k]^p)]^\frac{1}{p},$$

where $K_1$ is a positive integer such that $|\lambda| \leq K_1$. Let $\lambda \to 0$ for any fixed $x$ with $g(x) = 0$. By definition for $|\lambda| < 1$, we have

$$\sup_k f_k([u_k \Delta^n_u \lambda x_k]^p)]^\frac{1}{p} < \epsilon \text{ for } n > N(\epsilon).$$

(2) Also, for $1 \leq n \leq N$, taking $\lambda$ small enough, since $f_k$ is continuous, we have

$$\sup_k f_k([u_k \Delta^n_u \lambda x_k]^p)]^\frac{1}{p} < \epsilon.$$

(3) (2) and (3) implies that $g(\lambda x) \to 0$ as $\lambda \to 0$. This completes the proof. For more details see [24, 25].

Theorem 2.3. Let $F = (f_k)$ be a sequence of modulus functions and $\alpha = \lim_{t \to \infty} \frac{f_k(t)}{t}$, $\frac{p}{\alpha} > 0$. Then $l_{\infty}(F, p, u, \Delta^n_u) \subset l_{\infty}(p, u, \Delta^n_u)$.

Proof. Let $\alpha > 0$. By definition of $\alpha$, we have $f_k(t) \geq \alpha t$, for all $t \geq 0$. Since $\alpha > 0$, we have $t \leq \frac{1}{\alpha} f_k(t)$ for all $t \geq 0$. Let $x = (x_k) \in l_{\infty}(F, p, u, \Delta^n_u)$. Thus we have

$$l - \sup_k (u_k \Delta^n_u x_k)^{\frac{p}{\alpha}} \leq l - \frac{1}{\alpha} \sup_k f_k([u_k \Delta^n_u x_k]^p)] \leq \infty,$$

Which implies that $x = (x_k) \in l_{\infty}(p, u, \Delta^n_u)$. This completes the proof. For more details see [10].

Theorem 2.4. Let $F = (f_k)$ be a sequence of modulus functions, then

$$l_{\infty}(p, u, \Delta^n_u) \subset l_{\infty}(F, p, u, \Delta^n_u).$$

Proof. Let $x = (x_k) \in l_{\infty}(p, u, \Delta^n_u)$, then we have $l - \sup_k (u_k \Delta^n_u x_k)^{\frac{p}{\alpha}} < \infty$. Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $f_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Thus we have

$$l - \sup_k f_k([u_k \Delta^n_u x_k]^p)] = l - \sup_{k, \{x_k\} \leq \delta} f_k([u_k \Delta^n_u x_k]^p)] + l - \sup_{k, \{x_k\} > \delta} f_k([u_k \Delta^n_u x_k]^p)].$$

Since $F = (f_k)$ is a sequence of modulus functions, we have

$$l - \sup_{k, \{x_k\} \leq \delta} f_k([u_k \Delta^n_u x_k]^p)] \leq \epsilon.$$

(4) For $|\Delta^n_u x_k| > \delta$, and the fact that

$$|\Delta^n_u x_k| < \frac{|\Delta^n_u x_k|}{\delta} < \left[1 + \frac{|\Delta^n_u x_k|}{\delta}\right].$$
Proof.
Let $x \in I_k = \{x \in \mathbb{R} : k \leq x < k+1\}$ for $k \in \mathbb{Z}$. Thus

$$\frac{f_k(|u_k x|)}{|u_k x|} > \frac{2f_k(1)}{\delta}.$$  

Thus

$$\sup_{k, |x_k| > 0} f_k(|u_k x|) \leq \frac{2f_k(1)}{\delta} \sup_k |u_k x|.$$  

From equation (4) and (5) we have

$$l - \sup_k f_k(|u_k x|) \leq \epsilon + \frac{2f_k(1)}{\delta} \sup_k |u_k x|.$$  

Since $x = (x_k) \in I_\infty^p(F, p, u, \Delta_v^u)$. Hence we have $x = (x_k) \in I_\infty^p(F, p, u, \Delta_v^u)$ and this completes the proof.  

**Theorem 2.5.** The inclusion $I_\infty^p(F, p, u, \Delta_v^u) \subseteq c_0^v(u, \Delta_v^u)$ holds if and only if

$$\lim_k f_k(t) = \infty \text{ for } t > 0.$$  

**Proof.** Let $I_\infty^p(F, p, u, \Delta_v^u) \subseteq c_0^v(u, \Delta_v^u)$ such that $\lim_k f_k(t) = \infty$ for $t > 0$ does not hold. Then there is a number $t_0 > 0$ and a sequence $(k_i)$ of positive integers such that

$$f_k(t_0) \leq M < \infty.$$  

Define the sequence $x = (x_k)$ by:

$$(x_k) = \begin{cases} t_0, & \text{if } k = k_i, \ i = 1, 2, 3 \ldots; \\ 0, & \text{otherwise}. \end{cases}$$  

Thus $x \in I_\infty^p(F, p, u, \Delta_v^u)$ by (7). But $x \not\in c_0^v(u, \Delta_v^u)$, for $v_k = p_k = 1$, for all $k \in \mathbb{N}$ so that (6) must holds.

Conversely, let (6) hold. If $x \in I_\infty^p(F, p, u, \Delta_v^u)$, then $f_k(|u_k x|) \leq M < \infty$, for all $k$. Suppose that $x \not\in c_0^v(u, \Delta_v^u)$. Then for some number $c_0 > 0$ and positive integer $k_0$, we have $|u_k x| < c_0$ for $k \geq k_0$. Therefore $f_k(c_0) \geq f_k(|u_k x|) \leq M$ for $k \geq k_0$, which contradicts (6). Hence $x \in c_0^v(u, \Delta_v^u)$.  

**Theorem 2.6.** The inclusion $I_\infty^p(u, \Delta_v^u) \subseteq c_0^v(F, p, u, \Delta_v^u)$ holds if and only if

$$\lim_k f_k(t) = 0 \text{ for } t > 0.$$  

**Proof.** Suppose that $I_\infty^p(u, \Delta_v^u) \subseteq c_0^v(F, p, u, \Delta_v^u)$ but (8) does not hold then

$$\lim_k f_k(t_0) = l \neq 0, \text{ for some } t_0 > 0.$$  

Define the sequence $x = (x_k)$ by

$$(x_k) = t_0 \sum_{v=0}^{k-1} (-1)^v \left( \frac{n + k - v - 1}{k - v} \right)$$  

for $k = 1, 2, 3, \ldots$ Then $x \not\in c_0^v(F, p, u, \Delta_v^u)$ by (9) for $v_k = p_k = 1$, for all $k \in \mathbb{N}$. Hence (8) must hold.

Conversely, let $x \in I_\infty^p(u, \Delta_v^u)$ and suppose that (8) holds. Then $|u_k| x| \leq M < \infty$ for $k = 1, 2, 3, \ldots$. Therefore $f_k(|u_k x|) \leq f_k(M)$ for $k = 1, 2, 3, \ldots$ and $\lim_k f_k(|u_k x|) \leq \lim_k f_k(M) = 0$ by (8). Hence $x \in c_0^v(F, p, u, \Delta_v^u)$.  

Theorem 2.7. Let \( F = (f_k) \) be a sequence of modulus functions, then the following statements are equivalent:

(i) \( \prod_{k=1}^n (u, v_k) \subseteq \prod_{k=1}^n (u, \Delta^n_k) \).

(ii) \( c^0_k(u, \Delta^n_k) \subseteq \prod_{k=1}^n (u, \Delta^n_k) \).

(iii) \( \sup_k f_k(t) < \infty, (t > 0) \).

Proof. (i) \( \Rightarrow \) (ii): is obvious.

(ii) \( \Rightarrow \) (iii): Let \( c^0_k(u, \Delta^n_k) \subseteq \prod_{k=1}^n (u, \Delta^n_k) \). Suppose that (iii) is not true. Then by Lemma 1.5

\[
\sup_k f_k(t) = \infty, \quad \text{for all} \quad t > 0
\]

and therefore there is a sequence \((k_i)\) of positive integers such that

\[
f_k \left( \frac{1}{i} \right) > i, \quad \text{for each} \quad i = 1, 2, 3 \ldots \tag{10}
\]

Define the sequence \( x = (x_k) \) as follows:

\[
(x_k) = \begin{cases} 
\frac{1}{i}, & \text{if} \ k = k_i \quad i = 1, 2, 3 \ldots; \\
0, & \text{otherwise}.
\end{cases}
\]

Then \( x \in c^0_k(u, \Delta^n_k) \) but by (10) \( x \notin \prod_{k=1}^n (u, \Delta^n_k) \), for \( v_k = p_k = 1 \), for all \( k \in \mathbb{N} \) which contradicts (ii). Hence (iii) must hold.

(iii) \( \Rightarrow \) (i): Let (iii) be satisfied and \( x \in \prod_{k=1}^n (u, \Delta^n_k) \). If we suppose that \( x \notin \prod_{k=1}^n (u, \Delta^n_k) \). Then

\[
\sup_k f_k(l_k^{\Delta^n_k x_k}) = \infty \quad \text{for} \quad u \Delta^n x_k \in \prod_{k=1}^n.
\]

If we take \( t = |u_k \Delta^n x_k| \). Then \( \sup_k f_k(t) = \infty \) which contradicts (iii). Hence \( \prod_{k=1}^n (u, \Delta^n_k) \subseteq \prod_{k=1}^n (u, \Delta^n_k) \).

Theorem 2.8. Let \( F = (f_k) \) be a sequence of modulus functions, then the following statements are equivalent:

(i) \( c^0_k(F, p, u, \Delta^n_k) \subseteq c^0_k(u, \Delta^n_k) \).

(ii) \( c^0_k(F, p, u, \Delta^n_k) \subseteq \prod_{k=1}^n (u, \Delta^n_k) \).

(iii) \( \inf_k f_k(t) > 0, (t > 0) \).

Proof. (i) \( \Rightarrow \) (ii): is obvious.

(ii) \( \Rightarrow \) (iii): Let \( c^0_k(F, p, u, \Delta^n_k) \subseteq \prod_{k=1}^n (u, \Delta^n_k) \). Suppose that (iii) is not true. Then by Lemma 1.6,

\[
\inf_k f_k(t) = 0, \quad \text{for all} \quad t > 0
\]

and therefore there is a sequence \((k_i)\) of positive integers such that

\[
f_k(k^2) < \frac{1}{i}, \quad \text{for each} \quad i = 1, 2, 3 \ldots \tag{11}
\]

Define the sequence \( x = (x_k) \) as follows:

\[
(x_k) = \begin{cases} 
k^2, & \text{if} \ k = k_i \quad i = 1, 2, 3 \ldots; \\
0, & \text{otherwise}.
\end{cases}
\]

By (11) \( x \in c^0_k(F, p, u, \Delta^n_k) \) but \( x \notin \prod_{k=1}^n (u, \Delta^n_k) \), for \( v_k = p_k = 1 \), for all \( k \in \mathbb{N} \) which contradicts (ii). Hence (iii) must hold.

(iii) \( \Rightarrow \) (i): Let (iii) be satisfied and \( x \in c^0_k(F, p, u, \Delta^n_k) \). i.e \( I - \lim_k f_k(l_k^{\Delta^n_k x_k}) = 0 \). If we suppose that \( x \notin c^0_k(u, \Delta^n_k) \). Then for some number \( \varepsilon_0 > 0 \) and positive integer \( k_0 \), we have \( |u_k \Delta^n x_k| \leq \varepsilon_0 \) for \( k > k_0 \). Therefore \( f_k(\varepsilon_0) \geq f_k(|u_k \Delta^n x_k|) \) for \( k > k_0 \) and hence \( \lim_k f_k(\varepsilon_0) > 0 \), which contradicts our assumption that \( x \notin c^0_k(u, \Delta^n_k) \). Thus \( c^0_k(F, p, u, \Delta^n_k) \subseteq c^0_k(u, \Delta^n_k) \).
References