Some Properties of Magic Squares

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Abstract. Some advanced mathematical properties of magic squares are discussed in this paper.

1. Introduction

A magic square of order $n$ is an $n^{th}$ order matrix such that the sum of elements in every row/column/diagonal remains the same. The common sum is known as magic constant or magic number. Cornelius Agrippa (1486 B.C. to 1535 B.C.) of China is believed to be the first to take up construction of magic squares. There it was called Loh Shu. Interest in magic squares spread from China to Japan, India and the middle East. They were introduced to Europe in Byzantine times. The first magic square of order 4 in the first century was introduced in India by a mathematician named Nagarjuna. The first magic squares of order 5 and 6 appeared in an encyclopedia in Baghdad about 983 A.D.

In India, a procedure called Vedic Method is being used to construct magic squares even before centuries. Hindu tradition assigns them to God Siva and they treated it as Bhadra Ganita. Methodological construction of magic squares based on certain mathematical principles was taken up in France in the 7th century A.D. It became very popular in the Arab countries in the 10th century A.D. One among the famous magic squares in India is Sree Rama Chakra which is of 4th order. Another magic square is found in a work of Varahamihira. Several magic squares of various orders are available to us such as Eulerian Magic Squares, Kubera Chakra, Mahadeva Sooris Magic Square, Mars Square, Nasik Magic Square, Claude Gaspar Bachet Magic square, Millenium 2000, Topsyturrv Magic Squares, Ramanujans Square, Khajuraho Squares, Wishing Caps, Albrochet Durers Melancholia 1 etc.

Apart from the recreational and mythological aspects of magic squares, it is found that they posses several advanced mathematical properties. A handful of such properties are discussed here.

2. Notations and Mathematical Preliminaries

2.1. Magic Square

A magic square of order $n$ is an $n^{th}$ order matrix $[a_{ij}]$ such that for $i = 1, 2, 3, \ldots, n$

$$\sum_{j=1}^{n} a_{ij} = k \quad (1)$$

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2.2. Magic Constant

The constant \( k \) in the above definition is known as the magic constant or magic number. Magic constant of the magic square \( A \) is denoted as \( \rho(A) \). For example, the below given magic square \( A \) is of order 3 with \( \rho(A) = 21 \).

\[
\begin{array}{ccc}
4 & 9 & 8 \\
11 & 7 & 3 \\
6 & 5 & 10 \\
\end{array}
\]

2.3. Group

A nonempty set \( G \) together with an operation \( * \) is known as a group if the following properties are satisfied:

(i) \( G \) is closed with respect to \( * \). i.e., \( a * b \in G, \forall a, b \in G \).

(ii) \( * \) is associative in \( G \). i.e., \( a * (b * c) = (a * b) * c, \forall a, b, c \in G \).

(iii) \( \exists e \in G \) such that \( e * a = a * e = a, \forall a \in G \). Here \( e \) is called the identity element in \( G \) with respect to \( * \).

(iv) \( \forall a \in G, \exists b \in G \) such that \( a * b = b * a = e \), where \( e \) is the identity element. Here \( b \) is called the inverse of \( a \) and similarly vice versa. The inverse of the element \( a \) is denoted as \( a^{-1} \).

**Note:** If \( G \) is a group with respect to \( * \). It is denoted as \( < G, * > \).

2.4. Vector Space

A nonempty set \( V \) together with two operations \( + \) and \( . \), called addition and scalar multiplication respectively, is called a Vector Space over a field \( F \) if the following conditions are satisfied:

(i) \( < V, + > \) is an abelian group. (If \( G \) is an abelian group, then \( a * b = b * a, \forall a, b \in G \).)

(ii) \( \forall a \in F \) and \( \forall \alpha \in V, a, a \in V \).

(iii) \( (a + b) \alpha = a \alpha + b \alpha, \forall a, b \in F \) and \( \forall \alpha \in V \).

(iv) \( (ab) \alpha = a (b \alpha), \forall a, b \in F \) and \( \forall \alpha \in V \).

(v) \( 1 \alpha = \alpha, \forall \alpha \in V \) and 1 is the unity element of the field \( F \).

**Note:** Here the scalar multiplication is an external composition in \( V \) over \( F \). The elements in \( F \) are called scalars and the elements in \( V \) are called vectors. Normally, \( a \alpha \) is simply written as \( a \alpha \).

2.5. Homomorphism or linear transformation of vector spaces

Let \( U \) and \( V \) be two vector spaces over the same field \( F \). Then a mapping \( f : U \rightarrow V \) is called a homomorphism or linear transformation of \( U \) into \( V \) if

\[
f(aa + bb) = af(a) + bf(b), \quad \forall a, b \in F \quad \text{and} \quad \forall \alpha, \beta \in U.
\]

**Note:** A field \( F \) is a vector space over itself. If \( f \) is a linear transformation from \( V \) into \( F \), then \( f \) is called a linear functional on \( V \).
3. Propositions and Theorems

**Proposition 3.1.** If $A$ and $B$ are two $n^{th}$ order magic squares with $\rho(A) = a$ and $\rho(B) = b$, then $\forall \alpha, \beta \in \mathbb{R}$, $C = (\alpha + \beta)(A + B)$ is a magic square with $\rho(C) = (\alpha + \beta) [\rho(A) + \rho(B)]$.

*Proof.* Given that $A$ and $B$ are two $n^{th}$ order magic squares with $\rho(A) = a$ and $\rho(B) = b$. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then, since $A$ and $B$ are matrices,

$$
C = (\alpha + \beta)(A + B)
= (\alpha + \beta)[a_{ij} + b_{ij}]
= [(\alpha + \beta)a_{ij} + (\alpha + \beta)b_{ij}].
$$

Let the sum of first row elements of $C = s$. Then,

$$
s = (\alpha + \beta)(a_{11} + b_{11}) + (\alpha + \beta)(a_{12} + b_{12}) + ... + (\alpha + \beta)(a_{1n} + b_{1n})
= (\alpha + \beta)\{a_{11} + a_{12} + ... + a_{1n}) + (b_{11} + b_{12} + ... + b_{1n})\}
= (\alpha + \beta)(a + b)
= (\alpha + \beta)[\rho(A) + \rho(B)].
$$

(Since $A$ and $B$ are magic squares, for $i = 1, 2, 3, ..., n, \sum_{j=1}^{n} a_{ij} = \rho(A)$ and $\sum_{j=1}^{n} b_{ij} = \rho(B)$).

Similarly, we can show that the sum of elements of each row/column/diagonal of $C = (\alpha + \beta)[\rho(A) + \rho(B)]$. i.e., $C = (\alpha + \beta)(A + B)$ is a magic square with $\rho(C) = (\alpha + \beta)[\rho(A) + \rho(B)]$; whenever $A$ and $B$ are magic squares of the same order. $\square$

From the above Proposition we can deduce the following results.

If $A$ and $B$ are two $n^{th}$ order magic squares then $\forall \alpha, \beta, c \in \mathbb{R}$,

(i) $A + B$ is a magic square with $\rho(A + B) = \rho(A) + \rho(B)$.

(ii) $cA$ is a magic square with $\rho(cA) = c\rho(A)$.

(iii) $C = \alpha A + \beta B$ is a magic square with $\rho(C) = \alpha \rho(A) + \beta \rho(B)$.

(iv) $(-A)$ is a magic square with $\rho(-A) = -\rho(A)$.

*Proof.* (i) In Proposition 3.1, put $\alpha = 1$ and $\beta = 0$.

(ii) In proposition 3.1, put $\alpha = c, \beta = 0$ and $B = 0$, where 0 denotes the zero matrix of order $n$. (Zero matrix is a magic square with $\rho(0) = 0$.)

(iii) It immediately follows from results (i) and (ii).

(iv) In results (ii) put $c = -1$. $\square$

**Proposition 3.2.** If $A$ and $B$ are two $n^{th}$ order magic squares, then $\forall \alpha, \beta \in \mathbb{R}$:

(i) $\alpha(A + B) = \alpha A + \alpha B$.

(ii) $(\alpha + \beta)A = \alpha A + \beta A$.

(iii) $(\alpha \beta)A = \alpha (\beta A)$.

(iv) $1A = A$, where 1 is the unity element of the field $\mathbb{R}$ of real numbers.

*Proof.* Given that $A$ and $B$ are two $n^{th}$ order magic squares and $\alpha, \beta \in \mathbb{R}$. Before going to the proof, please remember that all magic squares are square matrices.

(i) Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then $\forall \alpha, \beta \in \mathbb{R}, A + B = [a_{ij} + b_{ij}]$ and

$$
\alpha(A + B) = \alpha [a_{ij} + b_{ij}]
= [(\alpha a_{ij} + \alpha b_{ij}]
= [\alpha a_{ij}] + \alpha [b_{ij}]
= \alpha A + \alpha B, \forall \alpha \in \mathbb{R}.
$$
(ii)

\[(\alpha + \beta)A = (\alpha + \beta)[a_{ij}] = [(\alpha + \beta)a_{ij}] = [\alpha a_{ij} + \beta b_{ij}] = \alpha[a_{ij}] + \beta[b_{ij}] = \alpha A + \beta B, \quad \forall \alpha, \beta \in \mathbb{R}.\]

(iii)

\[(\alpha \beta)A = (\alpha \beta)[a_{ij}] = [(\alpha \beta)(a_{ij})] = [\alpha \beta a_{ij}] = \alpha[\beta a_{ij}] = \alpha(\beta[a_{ij}]) = \alpha(\beta A), \quad \forall \alpha, \beta \in \mathbb{R}.\]

(iv)

\[1_A = 1.[a_{ij}] = [1.a_{ij}] = [a_{ij}] = A.\]

\(\Box\)

**Theorem 3.3.** If \(V\) is the set of all \(n^{th}\) order magic squares, then \(<V, +>\) is an abelian group, where + denotes matrix addition.

**Proof.** Given that \(V\) is the set of all \(n^{th}\) order magic squares. We have to show that \(V\) is an abelian group with respect to matrix addition. Since matrix addition is associative and commutative, we need to prove only the following properties:

(i) Closure Property : Let \(A, B \in V\). Then by result (i), \(A + B \in V\).
(ii) The zero matrix \(0\) of order \(n\) is a magic square with \(\rho(0) = 0\) and so \(0 \in V\) and acts as the identity element

\[A + 0 = 0 + A = A, \quad \forall A \in V.\]

(iii) If \(A \in V\), then by result (iv), \(-A \in V\) and it can be easily verified that \(A + (-A) = (-A) + A = 0\). i.e., \(-A\) is the additive inverse of \(A\) and similarly vice versa. So \(<V, +>\) is a group. Hence, \(<V, +>\) is an abelian group.

\(\Box\)

**Theorem 3.4.** The set \(V\) of all \(n^{th}\) order magic squares is a vector space over the field \(\mathbb{R}\) of real numbers with respect to addition of matrices as addition of vectors and multiplication of a matrix by a scalar as scalar multiplication.

**Proof.** It immediately follows from Theorem 3.1 result (ii) and Proposition 3.2.

**Theorem 3.5.** The set \(\rho_s\) of all magic constants forms an abelian group under addition of real numbers.

**Proof.** Since addition of real numbers is associative and commutative, we need to prove only the following properties:
(i) Closure Property : Let \( a, b \in \rho_s \). Then, \( \exists A, B, \in \mathbb{V} \) such that \( \rho(A) = a \) and \( \rho(B) = b \). Then by result(i), \( A + B \) is magic square with \( \rho(A + B) = \rho(A) + \rho(B) = a + b \). i.e., \( a + b \) is also a magic constant and so \( a + b \in \rho_s \).

(ii) Since \( \rho(0) = 0 \in \rho_s \) and will act as the identity element under addition.

(iii) Let \( a = \rho(A) \in \rho_s \) where \( A \in \mathbb{V} \). Then by result (iv), \( -A \) is a magic square with \( \rho(-A) = -\rho(A) \). i.e., \( -a \in \rho_s \). Hence \(< \rho_s, + >\) is an abelian group.

\[\Box\]

**Proposition 3.6.** The function \( f : \mathbb{V} \to \mathbb{R} \) defined by \( f(A) = \rho(A) \), \( \forall A \in \mathbb{V} \) is a linear functional on \( \mathbb{V} \).

**Proof.** The function \( f : \mathbb{V} \to \mathbb{R} \) is a linear functional on \( \mathbb{V} \) if the following condition is satisfied

\[ f(aA + bB) = af(A) + bf(B), \quad \forall a, b \in \mathbb{R} \quad \text{and} \quad \forall A, B \in \mathbb{V}. \]

Now,

\[ f(aA + bB) = \rho(aA + bB) = a\rho(A) + b\rho(B) \longrightarrow (4) \]

\[ = af(A) + bf(B), \quad \forall a, b \in \mathbb{R}, \quad \forall A, B \in \mathbb{V}. \]

Equation (4) is possible by result(iii) and hence the proof. \( \Box \)

4. Conclusion

We have seen that the set \( \mathbb{V} \) of all \( n^{th} \) order magic squares forms a vector space over the field \( \mathbb{R} \) of real numbers and the set \( \rho_s \) of all magic constants forms an abelian group under ordinary addition of real numbers. Finally, we proved that the function \( f : \mathbb{V} \to \mathbb{R} \) defined by \( f(A) = \rho(A) \) is a linear functional on \( \mathbb{V} \).

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References