Pure Fuzzy Ideals in Ternary Semigroups

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Abstract. In this paper we initiate the study of pure fuzzy ideals in ternary semigroups. We introduce the notions of purely fuzzy maximal and purely fuzzy prime ideals of a ternary semigroup $T$. We also characterize those ternary semigroups for which each fuzzy ideal is weakly pure.

1. Introduction

Ternary semigroups introduced by Lehmer in 1932 in [4]. Sioson developed ideal theory of ternary semigroups in his paper [10]. Many researcher worked on ternary semigroups (see [7–9]). The fuzzy set was initiated by Zadeh [11]. This theory has generated great enthusiasm among intellectuals specially researchers, working in multilateral aspects of mathematics. Many concepts of Mathematics are applied to such sets and various qualities of these concepts in the contexts of fuzzy sets are established. Many examples like Rosenfeld [6] utilized this idea to delineate some essential primary concepts of groupoids. Further example of such a generalization is Kuroki’s classical semigroup [3].

Some papers by Zadeh which are published afresh in [12] along with his deeply established contributions to the domain of fuzzy sets and their modelings near to reasoning. To solve complicated social and biological problems new tools ought to be devised. Such a mathematics will be competent to handle indeterminacies taking decisions and designing large systems and networks which are complicated, irregular and distributive. Let us refer to [13] for an important article by Zadeh depicting a general theory of uncertainty. One of the profusely applied areas where fuzzy sets have been used abundantly is in executing decision making pertinent to management [14]. The monograph by Mordeson et. al which is a systematic presentation of fuzzy semigroups came to the forefront in [5], where there are theoretical results on fuzzy semigroups and their employment in fuzzy coding, fuzzy finite state machines and fuzzy languages. Fuzziness enjoys a homely place in the domain of formal languages.

The notions of pure fuzzy, purely fuzzy maximal and purely fuzzy prime ideals of a semigroup were introduced by Ahsan, et. al. in [1]. The notions of pure ideals of a ternary semigroup $T$ and verification of a ternary semigroup $T$ is right weakly regular if and only if every ideal of $T$ is right pure is introduced by Bashir and Shabir in [2].
In this paper, we introduce the concept of pure fuzzy, weakly pure fuzzy ideals of a ternary semigroup. We identify those ternary semigroups for which each fuzzy ideal is idempotent. We also characterize those ternary semigroups for which each fuzzy two-sided ideal is right weakly pure fuzzy.

2. Preliminaries

A ternary semigroup is a non-empty set $T$ together with an associative ternary operation $T$, i.e. $(x_1, x_2, x_3)T = (x_1T, x_2T, x_3T)$ for all $x_i \in T$, $1 \leq i \leq 5$. An element $0$ of a ternary semigroup $T$ is called a zero element of $T$ if $0ab = a0b = ab0 = 0$ for all $a, b \in T$. If a ternary semigroup $T$ has no zero element, then it is easy to adjoin a zero element $0$ to the set by defining $0ab = a0b = ab0 = 0$ and $000 = 0$ for all $a, b \in T$. We shall use the notation with the following meanings;

$$\mathcal{T}_p = \begin{cases} T, & \text{if } T \text{ has a zero element;} \\ T \cup \{0\}, & \text{otherwise.} \end{cases}$$

If $A, B, C$ are any three subsets of a ternary semigroup $T$, the product $ABC$ is defined as $ABC = \{abc : a \in A, b \in B, c \in C\}$. A non-empty subset $A$ of a ternary semigroup $T$ is called a left (right, lateral) ideal of $T$ if $TAT \subseteq A$ ($AT \subseteq A$). If $A$ is a left, right and lateral ideal of $T$, then it is called an ideal of $T$. A proper right pure two-sided ideal of $T$ is called purely prime if $T_1T_2 \subseteq I$ implies $T_1 \subseteq I$ or $T_2 \subseteq I$ for any right pure two-sided ideals $T_1$ and $T_2$ of $T$. A two-sided ideal $A$ of a ternary semigroup $T$ is called left (resp. right) weakly pure if $A \cap B = AAB$ (resp. $A \cap B = BAA$) for all two-sided ideals $B$ of $T$. An element $x$ of a ternary semigroup $T$ is called a regular element of $T$ if there exists an element $a$ in $T$ such that $x = axa$, that is $x \in xT$. A ternary semigroup $T$ is called regular if each element of $T$ is regular. A ternary semigroup $T$ is said to be right weakly regular if for each $x \in T$, $x \in (xT)^3$.

A fuzzy subset $f$ of a ternary semigroup $T$ is a function from $T$ to the unit interval $[0, 1]$. A fuzzy subset $f$ is non-empty if $f$ is not the constant map which assumes the value $0$. We denote an empty fuzzy subset of $T$ with $\varphi$ and $\varphi(x) = 0$ for all $x \in T$. For each subset $A$ of $T$, define $f_A : T \rightarrow [0, 1]$ by

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

$f_A$ is called the characteristic function of $A$. The characteristic function $f_T$ of $T$ is a function which gives $f_T(x) = 1$ for all $x \in T$. In this case we say that the ternary semigroup $T$ is the fuzzy subset of itself.

A fuzzy subset $f$ of a ternary semigroup $T$ is called fuzzy left (resp. right, lateral) ideal of $T$ if $f(abc) \geq f(c)$ ($f(abc) \geq f(a)$) for all $a, b, c \in T$. Let $f$ and $g$ be two fuzzy subsets of $T$, we define the relation $\leq$ between $f$ and $g$, the union, intersection and product of $f$ and $g$, respectively, as $f \leq g$ if $f(x) \leq g(x)$ for all $x \in T$.

Let $F(T)$ denote the collection of all fuzzy subsets of a ternary semigroup $T$. Let $f$, $g$, $h \in F(T)$. Then the product

$$(f \circ g \circ h)(a) = \begin{cases} \bigvee_{a = xyz} (f(x) \wedge g(y) \wedge h(z)), & \text{if there exist } x, y, z \in T \text{ such that } a = xyz; \\ 0, & \text{otherwise.} \end{cases}$$
It is easy to verify that the set \((F(T), \circ)\) is a ternary semigroup. Any intersection of fuzzy ideals of \(T\) is a fuzzy ideal of \(T\). Also union of any family of fuzzy ideals of \(T\) is a fuzzy ideal of \(T\). If \(A\) is left (right, lateral, two-sided) ideal of a ternary semigroup \(T\) if and only if \(f_A\), the characteristic function of \(A\), is a fuzzy left (right, lateral, two-sided) ideal of \(T\).

3. Pure fuzzy ideals

**Definition 3.1.** A two-sided fuzzy ideal \(\lambda\) of a ternary semigroup \(T\) is called a pure fuzzy ideal of \(T\) if \(\mu \land \lambda = \mu \circ \lambda \circ \lambda\) for all right fuzzy ideals \(\mu\) of \(T\).

**Proposition 3.2.** Let \(I\) be a two-sided ideal of a ternary semigroup \(T\). Then the following statements are equivalent:

1. \(I\) is right pure in \(T\),
2. The characteristic function of \(I\), denoted by \(\delta_I\) is pure fuzzy ideal of \(T\).

**Proof.** (1) \(\Rightarrow\) (2): Suppose that \(I\) is right pure in \(T\). Since \(I\) is an ideal of \(T\), \(\delta_I\) is obviously a fuzzy ideal of \(T\). To prove \(\delta_I\) is pure fuzzy, we show that for any fuzzy right ideal \(\xi\) of \(T\), \(\xi \land \delta_I = \xi \circ \delta_I \circ \delta_I\).

Let \(a \in T\). Then

\[
(\xi \circ \delta_I \circ \delta_I)(a) = \bigvee_{a=xyz} \{\xi(x) \land \delta_I(y) \land \delta_I(z)\}
\]

\[
\leq \bigvee_{a=xyz} \{\xi(xyz) \land \delta_I(xyz) \land \delta_I(xyz)\}
\]

\[
= \bigvee_{a=xyz} \{\xi(a) \land \delta_I(a) \land \delta_I(a)\}
\]

\[
= (\xi \land \delta_I \land \delta_I)(a).
\]

This implies that

\[
(\xi \circ \delta_I \circ \delta_I)(a) \leq (\xi \land \delta_I)(a)
\]

\[
\xi \circ \delta_I \circ \delta_I \leq \xi \land \delta_I.
\]

Now, \((\xi \land \delta_I \land \delta_I)(a) = (\xi(a) \land \delta_I(a) \land \delta_I(a)) = 0\) if \(a \notin I\). Thus \((\xi \land \delta_I \land \delta_I)(a) = 0 \leq (\xi \circ \delta_I \circ \delta_I)(a)\).

Consider the case when \(a \in I\). Since \(I\) is right pure so for each \(a \in I\) there exist \(b, c \in I\) such that \(a = abc\). Since \(b, c \in I\), \(\delta_I(b) = 1\), \(\delta_I(c) = 1\). Therefore,

\[
(\xi \land \delta_I \land \delta_I)(a) = (\xi(a) \land \delta_I(a) \land \delta_I(a)) = (\xi(a) \land \delta_I(b) \land \delta_I(c))
\]

\[
\leq \bigvee_{a=abc} (\xi(a) \land \delta_I(b) \land \delta_I(c)) = (\xi \circ \delta_I \circ \delta_I)(a)
\]

\[
\xi \land \delta_I \land \delta_I \leq \xi \circ \delta_I \circ \delta_I.
\]

Thus \(\xi \land \delta_I \land \delta_I = \xi \circ \delta_I \circ \delta_I\). Conversely, suppose that \(\delta_I\) is pure fuzzy ideal in \(T\). We show that \(I\) is right pure in \(T\). That is for each right ideal \(J\) of \(T\), \(J \cap I = JIII\) Since \(J\) is a right ideal of \(T\), the characteristic function \(\delta_J\) of \(J\) is a fuzzy right ideal of \(T\). Since \(\delta_I\) is pure fuzzy, we have \(\delta_J \land \delta_I = \delta_J \circ \delta_I \circ \delta_I\). This implies that \(\delta_{JIII} = \delta_{JIII}\). It follows that \(J \cap I = JIII\). Hence \(I\) is right pure. □

**Proposition 3.3.** The following assertions for a ternary semigroup \(T\) are true:

1. The fuzzy ideals \(\varphi\) and \(f_T\) of \(T\), defined respectively as

\[
\varphi(x) = \begin{cases} 
0, & \text{if } x \neq 0; \\
1, & \text{if } x = 0.
\end{cases}
\]

and \(f_T(x) = 1\) for all \(x \in T\), are pure fuzzy ideals of \(T\).
(2) If \( \{ \lambda_i : i \in I \} \) is a family of pure fuzzy ideals of \( T \), then so is \( \bigvee_{i \in I} \lambda_i \).

(3) If \( \lambda_1 \) and \( \lambda_2 \) are pure fuzzy ideals of \( T \), then so is \( \lambda_1 \wedge \lambda_2 \).

**Proof.** (1) We show that for any fuzzy right ideal \( \xi \) of \( T \) we have \( \xi \wedge \varphi = \xi \circ \varphi \circ \varphi \) and \( \xi \wedge \varphi = \xi \circ \varphi \circ \varphi \). For \( a \in T \), we have

\[
(\xi \circ \varphi \circ \varphi)(a) = \bigvee_{a = xyz} (\xi(x) \wedge \varphi(y) \wedge \varphi(z)) \leq \bigvee_{a = xyz} (\xi(xyz) \wedge \varphi(xyz) \wedge \varphi(xyz))
\]

\[
= \bigvee_{a = xyz} (\xi(a) \wedge \varphi(a) \wedge \varphi(a)) = \xi(a) \wedge \varphi(a) \wedge \varphi(a)
\]

This implies that \( (\xi \circ \varphi \circ \varphi) \leq \xi \wedge \varphi \). If \( a \neq 0 \), then

\[
(\xi \wedge \varphi)(a) = (\xi \wedge \varphi \wedge \varphi)(a) \Rightarrow (\xi(a) \wedge \varphi(a) \wedge \varphi(a))
\]

\[
= (\xi(a) \wedge 0 \wedge 0) = 0 \leq (\xi \circ \varphi \circ \varphi)(a)
\]

If \( a = 0 \), then

\[
(\xi \wedge \varphi)(0) = (\xi(0) \wedge \varphi(0)) \leq \bigvee_{0 = xyz} (\xi(x) \wedge \varphi(y) \wedge \varphi(z))
\]

\[
= ((\xi \circ \varphi \circ \varphi))(0).
\]

This implies that \( (\xi \wedge \varphi) \leq (\xi \circ \varphi \circ \varphi) \). Hence, \( \xi \wedge \varphi = \xi \circ \varphi \circ \varphi \).

Now consider,

\[
(\xi \circ f_r \circ f_l)(a) = \bigvee_{a = xyz} (\xi(x) \wedge f_r(y) \wedge f_l(z)) \leq \bigvee_{a = xyz} (\xi(xyz) \wedge f_r(xyz) \wedge f_l(xyz))
\]

\[
= \bigvee_{a = xyz} (\xi(a) \wedge f_r(a) \wedge f_l(a)) = \xi(a) \wedge f_r(a) \wedge f_l(a) = (\xi \wedge f_r \wedge f_l)(a)
\]

This implies that \( \xi \circ f_r \circ f_l \leq \xi \wedge f_r \). Again

\[
(\xi \wedge f_r)(a) = (\xi \wedge f_r \wedge f_r)(a) = (\xi(a) \wedge f_r(a) \wedge f_r(a))
\]

\[
\leq \bigvee_{a = xyz} (\xi(x) \wedge f_r(y) \wedge f_r(z)) = (\xi \circ f_r \circ f_r)(a).
\]

Hence, \( \xi \wedge f_r = \xi \circ f_r \circ f_r \).

(2) Let \( \{ \lambda_i : i \in I \} \) is a family of pure fuzzy ideals of \( T \). We have to show that \( \bigvee_{i \in I} \lambda_i \) is also a pure fuzzy ideal of \( T \). That is, we have to show that for any fuzzy right ideal \( \mu \) of \( T \), \( \mu \wedge (\bigvee_{i \in I} \lambda_i) = \mu \circ (\bigvee_{i \in I} \lambda_i) \circ (\bigvee_{i \in I} \lambda_i) \). Now for each \( a \in T \), we have

\[
\left[ \mu \circ (\bigvee_{i \in I} \lambda_i) \circ (\bigvee_{i \in I} \lambda_i) \right](a) = \bigvee_{a = xyz} \left[ \mu(x) \wedge (\bigvee_{i \in I} \lambda_i)(y) \wedge (\bigvee_{i \in I} \lambda_i)(z) \right] \leq \bigvee_{a = xyz} \left[ \mu(xyz) \wedge (\bigvee_{i \in I} \lambda_i)(xyz) \wedge (\bigvee_{i \in I} \lambda_i)(xyz) \right]
\]

\[
= \bigvee_{a = xyz} \left[ \mu(a) \wedge (\bigvee_{i \in I} \lambda_i)(a) \wedge (\bigvee_{i \in I} \lambda_i)(a) \right] = \mu(a) \wedge (\bigvee_{i \in I} \lambda_i)(a) \wedge (\bigvee_{i \in I} \lambda_i)(a)
\]

\[
= \left[ \mu \wedge (\bigvee_{i \in I} \lambda_i) \right](a).
\]
This implies that
\[ \mu \circ (\bigvee_{i \in I} \lambda_i) \circ (\bigvee_{i \in I} \lambda_i) \leq \mu \wedge (\bigvee_{i \in I} \lambda_i). \]

Again,
\[ \begin{align*}
\left[ \mu \wedge (\bigvee_{i \in I} \lambda_i) \right](a) &= \left[ \mu \wedge (\bigvee_{i \in I} \lambda_i) \wedge (\bigvee_{i \in I} \lambda_i) \right](a) \\
&= \bigvee_{i \in I} \left[ \mu(a) \wedge \lambda_i(a) \wedge \lambda_i(a) \right] = \bigvee_{i \in I} \left[ (\mu \circ \lambda_i \circ \lambda_i)(a) \right]
\end{align*} \]

Now,
\[ \begin{align*}
(\mu \circ \lambda_i \circ \lambda_i)(a) &= \bigvee_{a = xyz} \left[ \mu(x) \wedge \lambda_i(y) \wedge \lambda_i(z) \right] \leq \bigvee_{a = xyz} \left[ \mu(x) \wedge \left( \bigvee_{i \in I} \lambda_i \right)(y) \wedge \left( \bigvee_{i \in I} \lambda_i \right)(z) \right] \\
&= \left[ \mu \circ \left( \bigvee_{i \in I} \lambda_i \right) \circ \left( \bigvee_{i \in I} \lambda_i \right) \right](a).
\end{align*} \]

This implies that
\[ \left[ \mu \wedge (\bigvee_{i \in I} \lambda_i) \wedge (\bigvee_{i \in I} \lambda_i) \right](a) \leq \left[ \mu \circ (\bigvee_{i \in I} \lambda_i) \circ (\bigvee_{i \in I} \lambda_i) \right](a). \]

This implies that
\[ \left[ \mu \wedge (\bigvee_{i \in I} \lambda_i) \right] \leq \left[ \mu \circ (\bigvee_{i \in I} \lambda_i) \circ (\bigvee_{i \in I} \lambda_i) \right]. \]

Hence,
\[ \left[ \mu \wedge (\bigvee_{i \in I} \lambda_i) \right] = \left[ \mu \circ (\bigvee_{i \in I} \lambda_i) \circ (\bigvee_{i \in I} \lambda_i) \right]. \]

(3) Let \( \lambda_1 \) and \( \lambda_2 \) be pure fuzzy ideals of \( T \). Then \( \mu \wedge \lambda_1 = \mu \circ \lambda_1 \circ \lambda_1 \) and \( \mu \wedge \lambda_2 = \mu \circ \lambda_2 \circ \lambda_2 \) for all right fuzzy ideals of \( T \). We have to show that
\[ \mu \wedge (\lambda_1 \wedge \lambda_2) = \mu \circ (\lambda_1 \wedge \lambda_2) \circ (\lambda_1 \wedge \lambda_2). \]

Since \( \lambda_2 \) is a pure fuzzy ideal of \( T \). It follows that \( \lambda_1 \wedge \lambda_2 = \lambda_1 \circ \lambda_2 \circ \lambda_2 \). Therefore,
\[ \mu \circ (\lambda_1 \wedge \lambda_2) \circ (\lambda_1 \wedge \lambda_2) = \mu \circ (\lambda_1 \circ \lambda_2 \circ \lambda_2) \circ (\lambda_1 \circ \lambda_2 \circ \lambda_2). \]

Since \( \lambda_1 \circ \lambda_2 \circ \lambda_2 \) is a two-sided fuzzy ideal of \( T \), so
\[ \mu \circ (\lambda_1 \wedge \lambda_2) \circ (\lambda_1 \wedge \lambda_2) = \mu \wedge (\lambda_1 \circ \lambda_2 \circ \lambda_2) = \mu \wedge (\lambda_1 \wedge \lambda_2). \]

Thus we have \( \mu \wedge (\lambda_1 \wedge \lambda_2) = \mu \circ (\lambda_1 \wedge \lambda_2) \circ (\lambda_1 \wedge \lambda_2) \). Hence, \( (\lambda_1 \wedge \lambda_2) \) is a pure fuzzy ideal of \( T \).

**Theorem 3.4.** The following assertions are equivalent for a ternary semigroup \( T \):

(1) \( T \) is right weakly regular.

(2) Each fuzzy right ideal of \( T \) is idempotent.

(3) \( \lambda \wedge \mu = \mu \circ \lambda \circ \lambda \) for every fuzzy right ideal \( \mu \) and for every two-sided fuzzy ideal \( \lambda \) of \( T \).
(4) \( \lambda \wedge \mu = \mu \circ \lambda \circ \lambda \) for every fuzzy right ideal \( \mu \) and for every fuzzy ideal \( \lambda \) of \( T \).

**Proof.** (1) \( \implies \) (2): Let \( \delta \) be a fuzzy right ideal of \( T \). We prove that \( \delta^3 = \delta \circ \delta \circ \delta = \delta \). Let \( x \in T \). Then

\[
(\delta \circ \delta \circ \delta)(x) = \bigwedge_{x=abc} \{ \delta(a) \wedge \delta(b) \wedge \delta(c) \} \leq \bigwedge_{x=abc} \{ \delta(abc) \wedge \delta(b) \wedge \delta(c) \}
\]

where \( a, b, c, d, e, f \in T \) and \( T \) is right weakly regular. Thus \( x = (xab)(xcd)(xef) \). So

\[
\delta(x) \leq \bigwedge_{x=abc} (\delta(w) \wedge \delta(y) \wedge \delta(z)) = (\delta \circ \delta \circ \delta)(x).
\]

Thus \( \delta \leq \delta \circ \delta \circ \delta = \delta^3 \). Hence \( \delta = \delta^3 \).

(2) \( \implies \) (1): Let \( x \in T \). We show that \( x \in (xTT)^3 \). Let \( A = x \cup xTT \) be the right ideal generated by \( x \). Let \( \delta_A \) be the characteristic function of \( A \) and is a fuzzy right ideal of \( T \), hence by (2), \( \delta_A = \delta_A \circ \delta_A \circ \delta_A = \delta_A \). This implies that \( A = A^3 \). Since \( x \in A \), it follows that \( x \in A^3 = (x \cup xTT)^3 \). This implies that

\[
x \in (x \cup xTT)^3 = (x \cup xTT)(x \cup xTT)(x \cup xTT) = xxx \cup xxTTx \cup xTTxx \cup xTTxTTx \cup xxxTT
\]

Thus implies that \( x \in (xTT)^3 \). Hence \( T \) is right weakly regular.

(1) \( \implies \) (4): Let \( \lambda \) be a fuzzy ideal and \( \mu \) a fuzzy right ideal of \( T \). We show that \( \lambda \wedge \mu = \mu \circ \lambda \circ \lambda \). Since \( \mu \circ \lambda \circ \lambda \leq f_T \circ f_T \circ \lambda \leq \lambda \). Also \( \mu \circ \lambda \circ \lambda \leq \mu \circ f_T \circ f_T \leq \mu \). This implies that \( \mu \circ \lambda \circ \lambda \leq \mu \wedge \lambda \).

Now we show that \( \mu \wedge \lambda \leq \mu \circ \lambda \circ \lambda \). Let \( x \in T \) and \( T \) is right weakly regular, so there exist \( t_1, t_2, t_3, s_1, s_2, s_3 \in T \) such that \( x = (xt_1s_1)(xt_2s_2)(xt_3s_3) \). Thus

\[
(\mu \wedge \lambda)(x) = (\mu(x) \wedge \lambda(x) \wedge \lambda(x)) \leq (\mu(xt_1s_1) \wedge \lambda(xt_2s_2) \wedge \lambda(xt_3s_3))
\]

\[
\leq \bigwedge_{x=abc} (\mu(a) \wedge \lambda(b) \wedge \lambda(c)) = (\mu \circ \lambda \circ \lambda)(x).
\]

This implies that \( (\mu \wedge \lambda) \leq (\mu \circ \lambda \circ \lambda) \). Thus \( (\mu \wedge \lambda) = (\mu \circ \lambda \circ \lambda) \). Hence \( \lambda \) is fuzzy pure.

(4) \( \implies \) (1): We show that \( T \) is right weakly regular. Let \( x \in T \) and let \( A = \{ x \} \cup TTx \cup xTT \cup TTxTT \) be two-sided ideal generated by \( x \). Let \( \delta_A \) be the characteristic function of \( A \). Then \( \delta_A \) is fuzzy ideal of \( T \), \( \delta_A \) is pure fuzzy. Thus by the Proposition 3.2, \( A \) is right pure in \( T \). Since \( x \in A \) and \( A \) is pure in \( T \), therefore there exist \( y, z \in A \) such that \( x = xyz \). This means that

\[
x \in xAA = x[x] \cup TT \cup xTT \cup TTxTT \cup [x] \cup TTx \cup xTT \cup TTxTT
\]

\[
= xxx \cup xTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx \cup xxTTx
\]

Simple calculations show that \( x \in (xTT)^3 \). Hence \( T \) is right weakly regular ternary semigroup.

(1) \( \Leftrightarrow \) (3): Obvious. \( \square \)

**Theorem 3.5.** The following assertions for a ternary semigroup \( T \) are equivalent:

1. \( T \) is right weakly regular.
2. Every two-sided fuzzy ideal \( \lambda \) of \( T \) is fuzzy pure.
3. Every fuzzy ideal \( \lambda \) of \( T \) is fuzzy pure.

**Proof.** The proof follows from the Theorem 3.4 and Proposition 3.2. \( \square \)
4. Weakly pure fuzzy ideals

Definition 4.1. A two-sided ideal \( \lambda \) of a ternary semigroup \( T \) is called left (resp. right) weakly fuzzy pure if \( \lambda \land \mu = \lambda \circ \lambda \circ \mu \) (resp. \( \lambda \land \mu = \mu \circ \lambda \circ \lambda \)) for all two-sided fuzzy ideals \( \mu \) of \( T \).

Proposition 4.2. For a ternary semigroup \( T \), the following assertions are equivalent:

1. Each two-sided fuzzy ideal of \( T \) is left weakly pure fuzzy.
2. Each two-sided fuzzy ideal of \( T \) is idempotent.
3. Each two-sided fuzzy ideal of \( T \) is right weakly pure fuzzy.

Proof. (1) \( \Rightarrow \) (2): Suppose each two-sided fuzzy ideal of \( T \) is left weakly pure fuzzy. Let \( \lambda \) be a two-sided fuzzy ideal of \( T \). Then for each two-sided fuzzy ideal \( \mu \) of \( T \), we have \( \lambda \land \mu = \lambda \circ \lambda \circ \mu \). In particular \( \lambda = \lambda \land \lambda = \lambda \circ \lambda \circ \lambda = \lambda^2 \). Hence each two-sided fuzzy ideal of \( T \) is idempotent.

(2) \( \Rightarrow \) (1): Suppose each two-sided fuzzy ideal of \( T \) is idempotent. Let \( \lambda \) be a two-sided fuzzy ideal of \( T \), then for any two-sided fuzzy ideal \( \mu \) of \( T \), we always have \( \lambda \circ \lambda \circ \mu \leq \lambda \land \mu \). On the other hand, \( \lambda \land \mu = (\lambda \land \mu) \circ (\lambda \land \mu) \circ (\lambda \land \mu) \leq \lambda \circ \lambda \circ \mu \). Thus we have \( \lambda \land \mu = \lambda \circ \lambda \circ \mu \). Hence \( \lambda \) is left weakly pure fuzzy.

(2) \( \Rightarrow \) (3): Suppose each two-sided fuzzy ideal of \( T \) is idempotent. Let \( \lambda \) be a two-sided fuzzy ideal of \( T \). Then for any two-sided fuzzy ideal \( \mu \) of \( T \), we always have \( \mu \circ \lambda \circ \lambda \leq \mu \land \lambda \). On the other hand \( \mu \land \lambda ) = (\mu \land \lambda) \circ (\mu \land \lambda) \circ (\mu \land \lambda) \leq \mu \circ \lambda \circ \lambda \). Thus we have \( \mu \circ \lambda \circ \lambda = \mu \land \lambda \). Hence \( \lambda \) is right weakly pure fuzzy.

(3) \( \Rightarrow \) (2): Suppose each two-sided fuzzy ideal of \( T \) is right weakly pure fuzzy. Let \( \lambda \) be any two-sided fuzzy ideal of \( T \). Then \( \lambda \) is right weakly pure fuzzy. Hence for each two-sided fuzzy ideal \( \mu \) of \( T \), we have \( \lambda \land \mu = \mu \circ \lambda \circ \lambda \). In particular \( \lambda = \lambda \land \lambda = \lambda \circ \lambda \circ \lambda = \lambda^3 \). Hence each two-sided fuzzy ideal of \( T \) is idempotent. \( \square \)

References