On M-cleavable Near Rings

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Abstract. Let \( R \) be a right near ring with identity and \( M \) be a near ring \( R \)-module. In this paper we introduce the concepts \( M \)-even, \( M \)-odd and \( M \)-cleavable such that even, odd and cleavable are special cases of this concepts respectively. If \( R \) is zero symmetric, \( M \) is faithful and \( n \) is a positive integer we show that if \( A \) belongs to the generalized matrix near ring \( \text{Mat}_n(R,M) \) and suppose that every elementary matrix appears in any expression represents \( A \) is even then \( A \) is even. Also the same is true for odd if \( R,M \) are abelian. In the end we define \( R \) is an abelian strong \( \sigma \)-generated and we show if \( R \) is \( M \)-cleavable, then \( \text{Mat}_n(R,M) \) is cleavable.

1. Introduction

\((R,+,\cdot)\) is a right near ring if (1) \((R,+)\) is a group (not necessary abelian). (2) \((R,\cdot)\) is a semigroup. (3) \((s+t)r=sr+tr\ \forall r,s,t \in R\). \( R \) is zero symmetric if \( R = R_0 = \{ r \in R : r0 = 0 \} \). Recall that, in \( R \), (1) \( r \in R \) is called even if \( r(-s) = rs, \forall r,s \in R \). (2) \( r \in R \) is called odd if \( r(-s) = -rs, \forall r,s \in R \). (3) \( R \) is called cleavable if each element \( r \in R \) is the sum of an even and an odd element. \( M \) is a left \( R \)-module if \( (M,+) \) is a group need not be abelian and there exist a map \( \cdot : R \times M \rightarrow M \) satisfies: (1) \((r+s)m = rm + sm, \forall r,s \in R, \forall m \in M \). (2) \((rm)m = r(sm)m, \forall r,s \in R, \forall m \in M \). \( M \) is faithful means that if \( r.m = 0, \forall m \in M \) then \( r = 0 \). \( M \) is unital if \( 1.m = m, \forall m \in M \). More information are in [3].

In this paper, we introduce the definitions of \( M \)-even, \( M \)-odd and \( M \)-cleavable which are extensions of the definitions of even, odd and cleavable respectively. We denote \( M^n \) the direct sum of \( n \) copies of \((M,+)\) is also a faithful left \( R \)-module. Now we define special functions in \( M_0(M^n) \) will be denoted by \( f_{ij}^r, r \in R \) and \( 1 \leq i,j \leq n \)

\[
f_{ij}^r : M^n \rightarrow M^n \text{ such that } r \in R \text{ and } 1 \leq i,j \leq n,
\]

\[
f_{ij}^r(a_1, ..., a_k) = (0, ..., 0, r a_j, 0, ..., 0) \text{ where } ra_j \text{ in the } i-\text{th position},
\]

\[
(a_1, ..., a_k) ∈ M^n \text{ and } f_{ij}^r = l_i f^r \pi_j \text{ where } l_i : M \rightarrow M^n
\]

the \( i - \)th injection, \( \pi_j : M^n \rightarrow M \) is the \( j - \)th projection and \( f^r : M \rightarrow M
\]

such that \( f^r(s) = rs \forall s \in M \).

So \( f_{ij}^r \) is the function from \( M^n \) to \( M^n \) that takes a \( n \)-tuple with entries from \( M \), multiples the \( j \)-th entry \( a_j \) by \( r \) using the module action of \( R \) on \( M \), puts the result \( ra_j \) into the \( i \)-th position and puts 0 in the other positions.

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We may sometimes write \( f_{ij} \) as \([r; i, j]\). In [2], K. C. Smith introduced the near-ring of \( n \times n \) generalized matrix near ring over \( R \) using the faithful \( R \)-module \( M \) which is the subnear ring \( \text{Mat}_n(R; M) \) of \( M_0(M^n) \) generated by \( f_{ij}^r : r \in R \) and \( 1 \leq i, j \leq n \). The identity matrix \( f_{11}^1 + f_{22}^1 + \ldots + f_{nn}^1 \). A matrix of the form \( \sum_{i=1}^{n} f_{ii}^r \) is called a diagonal matrix.

\( n \times n \) generalized matrix near ring over \( R \) using the faithful \( R \)-module \( M \) is, of course, a function from \( M^n \) to \( M^n \), but we shall often need representations of matrices. For this reason we use the set \( E_n(R) \) of matrix expressions, i.e. the subset of the free semigroup over the alphabet of symbols \( \{ f_{ij}^r : r \in R, 1 \leq i, j \leq n \} \cup \{ (,), + \} \) recursively defined by the following rules:

1. \( f_{ij}^r \in E_n (R) \) for \( 1 \leq i, j \leq n \) and all \( r \in R \).
2. If \( A, E \in E_n(R), \) then \( A + E \in E_n(R) \).
3. If \( A, E \in E_n(R), \) then \( (A)(E) \in E_n(R) \).

The length \( l(E) \) of an expression \( E \) is the number of \( f_{ij}^r \) in \( E \). The weight \( \omega(X) \) of a matrix \( X \) is the length of an expression of minimal length representing \( X \). The matrix represented by \( E \in E_n(R) \) is denoted by \( \mu(E) \). Every matrix is represented by at least one expression; however the same matrix may be represented by many different expressions. In spite of this we shall usually not distinguish between expressions and matrices, except when such a distinction becomes necessary to avoid ambiguity. Also, we shall omit parentheses if the meaning is clear. Those functions \( f_{ij}^r \), in the ring case, correspond to matrices with \( r \) in position \((i, j)\) and 0 elsewhere, so every \( n \times n \) matrix over a ring \( R \) is a sum of elements of the generating set \( X_n(R) = \{ f_{ij}^r : r \in R, 1 \leq i, j \leq n \} \).

2. Main Results

**Definition 2.1.** Let \( R \) be a near ring, \( M \) is a near ring \( R \)-module. \( r \in R \) is called \( M \)-even if for all \( m \in M : r(-m) = rm \), and is called \( M \)-odd if for all \( m \in M : r(-m) = -rm \).

It is clearly that \( r \in R \) is even if it is \( R \)-even and it is odd if it is \( R \)-odd.

**Lemma 2.2.** Let \( R \) be a near ring, \( M \) is a near ring \( R \)-module, then:

1. Every constant element in \( R \) is \( M \)-even.
2. The set of all \( M \)-even elements in \( R \) is a subnear ring and left invariant.
3. If \( R \) is abelian then the set of all \( M \)-odd elements in \( R \) is a subnear ring.

**Proof.** (1) Let \( r \in R \) be constant then we have

\[
rm = (r0)m = r(0m) = r0_m \quad \forall m \in M.
\]

So

\[
r(-m) = r0_m = rm \quad \forall m \in M.
\]

i.e., \( r \) is \( M \)-even.

(2) Let \( E \subseteq R \), denote the set of all \( M \)-even elements of \( R \). Let \( a, b \in E \). So,

\[
(a + b)(-m) = a(-m) + b(-m) = am + bm = (a + b)m \quad \forall m \in M.
\]
Hence, \(a + b \in E\). Also
\[
(-a)(-m) = -(a(-m)) = -(am) = (-a)m \quad \forall m \in M.
\]
Then \(-a \in E\). Further,
\[
(ab)(-m) = a(b(-m)) = a(bm) = (ab)m \quad \forall m \in M.
\]
So \(ab \in E\). Thus \(E\) is subnear ring of \(R\). Now let \(r \in R\). Then,
\[
(ra)(-m) = r(a(-m)) = r(am) = (ra)m.
\]
Hence, \(RE \subseteq E\).

(3) Let \(O \subseteq R\), denote the set of all \(M\)–even elements of \(R\). Let \(a, b \in O\). Then,
\[
(a + b)(-m) = a(-m) + b(-m) = -am - bm = (-a - b)m = -(a + b)m \quad \forall m \in M.
\]
Hence \(a + b \in O\). Also
\[
(-a)(-m) = -(a(-m)) = -(am) = (-a)m \quad \forall m \in M.
\]
Then \(-a \in O\). Further,
\[
(ab)(-m) = a(b(-m)) = a(bm) = -(ab)m \quad \forall m \in M.
\]
So \(ab \in O\). Thus \(O\) is subnear ring of \(R\). □

In the following \(R\) be a zero-symmetric near ring with identity, \(M\) be a faithful near ring \(R\)–module.

**Lemma 2.3.** If \(n > 1\), then:

(1) \(a \in R\) is \(M\)–even if and only if \(f_{ij}^a \in \text{Mat}_n(R, M)\) is even.

(2) \(a \in R\) is \(M\)–odd if and only if \(f_{ij}^a \in \text{Mat}_n(R, M)\) is odd.

**Proof.** (1)\((\Rightarrow)\) Suppose that \(a \in R\) is \(M\)–even and let \(A \in \text{Mat}_n(R, M), a \in M^n\). If \(A\alpha = (b_1, \ldots, b_n)\), then
\[
(f_{ij}^a(-A))\alpha = f_{ij}^a(-b_1, \ldots, -b_n) = (0, 0, \ldots, a(-b_i), 0, \ldots, 0) = (0, 0, \ldots, ab_j, 0, \ldots, 0) = f_{ij}^a(A\alpha).
\]
Hence, \( f_{ij}^0(-A) = f_{ij}^0A \), for all \( A \in \text{Mat}_n(R, M) \), and so \( f_{ij}^0 \in \text{Mat}_n(R, M) \) is even.

\((\Leftarrow)\) Conversely, suppose that \( f_{ij}^0 \in \text{Mat}_n(R, M) \) is even. Let \( \delta = (x_1, \ldots, x_n) \in M^n \). Note that \( f_{ij}^0(-I) = f_{ij}^{0}I = f_{ij}^0 \), so \( (f_{ij}^0(-I))(x_1, \ldots, x_n) = f_{ij}^0(x_1, \ldots, x_n) \). In particular, take \( \delta = (0, 0, \ldots, m, 0, \ldots, 0) \in M^n \), then
\[
(f_{ij}^0)^{-1}_1 + \ldots + f_{ij}^{-1}_m)(0, 0, \ldots, m, 0, \ldots, 0) = f_{ij}^0(0, 0, \ldots, m, 0, \ldots, 0)
\]
\[(0, 0, \ldots, a(-m), 0, \ldots, 0) = (0, 0, \ldots, am, 0, \ldots, 0).
\]
Then, \( a(-m) = am \ \forall m \in M \), and so \( a \in R \) is \( M \)-even.

\((2)(\Rightarrow)\) Suppose that \( a \in R \) is \( M \)-odd and let \( A \in \text{Mat}_n(R, M) \), \( a \in M^n \). If \( A\alpha = (b_1, \ldots, b_n) \), then
\[
(f_{ij}^0(-A))\alpha = f_{ij}^0(-b_1, \ldots, -b_n)
\]
\[
= (0, 0, \ldots, a(-b_i), 0, \ldots, 0)
\]
\[
= (0, 0, \ldots, -ab_i, 0, \ldots, 0)
\]
\[
= -f_{ij}^0(A\alpha).
\]
Hence, \( f_{ij}^0(-A) = -f_{ij}^0A \), for all \( A \in \text{Mat}_n(R, M) \), and so \( f_{ij}^0 \in \text{Mat}_n(R, M) \) is odd.

\((\Leftarrow)\) Conversely, suppose that \( f_{ij}^0 \in \text{Mat}_n(R, M) \) is odd. Let \( \delta = (x_1, \ldots, x_n) \in M^n \). Note that \( f_{ij}^0(-I) = -f_{ij}^{0}I = -f_{ij}^0 \), so \( (f_{ij}^0(-I))(x_1, \ldots, x_n) = -f_{ij}^{0}(x_1, \ldots, x_n) \). In particular, take \( \delta = (0, 0, \ldots, m, 0, \ldots, 0) \in M^n \), then
\[
(f_{ij}^0)^{-1}_1 + \ldots + f_{ij}^{-1}_m)(0, 0, \ldots, m, 0, \ldots, 0) = -f_{ij}^0(0, 0, \ldots, m, 0, \ldots, 0)
\]
\[(0, 0, \ldots, a(-m), 0, \ldots, 0) = (0, 0, \ldots, -am, 0, \ldots, 0).
\]
Then, \( a(-m) = -am \ \forall m \in M \), and so \( a \in R \) is \( M \)-odd. \(\square\)

**Lemma 2.4.** Let \( A \in \text{Mat}_n(R, M) \) and suppose that every elementary matrix appears in any expression represents \( A \) is even. Then \( A \) is even.

**Proof.** We use induction on \( w(A) \). Basis of induction:

Suppose that \( w(A) = 1 \), so \( A = f_{ij}^0, a \in A, 1 \leq i, j \leq n \) then the results follows from the hypothesis.

Induction step: assume that the results holds for all matrices with weight less than \( m, m \geq 2 \). If \( w(A) = m \), then \( A = A' + A'' \) or \( A = CD \), where \( w(C), w(D) < m \). For any \( B \in \text{Mat}_n(R, M) \),
\[
A(-B) = (C + D)(-B)
\]
\[
= C(-B) + D(-B)
\]
\[
= CB + DB
\]
\[
= (C + D)B
\]
\[
= AB.
\]

\(\square\)
Lemma 2.5. Let $R, M$ be abelian, $A \in \text{Mat}_n(R, M)$ and suppose that every elementary matrix appears in any expression represents $A$ is odd then $A$ is odd.

Proof. We use induction on $w(A)$. Basis of induction: Suppose that $w(A) = 1$, so $A = f^a_{ij}, a \in A, 1 \leq i, j \leq n$ then the results follows from the hypothesis.

Induction step: assume that the results holds for all matrices with weight less than $m, m \geq 2$. If $w(A) = m$, then $A = C + D$ or $A = CD$, where $w(C), w(D) < m$. For any $B \in \text{Mat}_n(R, M)$,

$$A(-B) = (C + D)(-B) = C(-B) + D(-B) = -CB - DB = -(D + C)B = -(C + DB) = -AB.$$  

From Definition 2.6. $\text{Mat}_n(R, M)$ is said to be strong $\sigma$--generated if every $n \times n$ generalized matrix $A \in \text{Mat}_n(R, M)$ is a sum of elements of $X_n(R) = \{ f^a_{ij} : r \in R, 1 \leq i, j \leq n \}$. Also we say that $R$ is $M, \sigma$--generated if $\text{Mat}_n(R, M)$ is strong $\sigma$--generated for any natural number $n$.

Lemma 2.7. Let $R$ be an abelian $M, \sigma$--generated, $M$ is abelian. If $A \in \text{Mat}_n(R, M)$ is even then every elementary matrix appears in any expression represents $A$ is even.

Proof. Since $R$ is abelian $M, \sigma$--generated, then every $A \in \text{Mat}_n(R, M)$ can be written as

$$A = \sum_{1 \leq i, j \leq n} f^a_{ij} a_{ij} \in R.$$  

To show that $f^a_{ij}$ is even we show that $a_{ij}$ is $M$--even, for all $i, j$. Let $\alpha = (x_1, ..., x_n) \in M^n$. Note that $A(-I) = AI = A$, so $A(-I)\alpha = A\alpha$. In particular, take $\alpha = (x, 0, ..., 0)$. It follows that

$$\pi_k(A(-I)(x, 0, ..., 0)) = \pi_k(A(x, 0, ..., 0))$$  

and so

$$a_{k1}(-x) + a_{k2}0 + ... + a_{kn}0 = a_{k1}x + a_{k2}0 + ... + a_{kn}0.$$  

This means that $a_{k1}(-x) = a_{k1}x$, for all $k$. Also if $\alpha = (0, x, ..., 0)$ then $a_{k2}(-x) = a_{k2}x$, for all $k$. Hence, by the same arrangement, $a_{ij}(-x) = a_{ij}x$, for all $i, j, x \in M$. Thus $a_{ij}$ is $M$--even, and so $f^a_{ij}$ is even for all $i, j$.  

Definition 2.8. $R$ is said to be $M$--cleavable if each $r \in R$ is the sum of $M$--even and $M$--odd element.
Theorem 2.9. Let $R$ be an abelian $M, \sigma$–generated. If $R$ is $M$–cleavable, then $\text{Mat}_n(R, M)$ is cleavable.

Proof. Since $R$ is abelian $M, \sigma$–generated then every $A \in \text{Mat}_n(R, M)$ can be written as

$$A = \sum_{1 \leq i, j \leq n} a_{ij} \in R.$$  

Since $R$ is $M$–cleavable, then $a_{ij} = e_{ij} + o_{ij}$, where $e_{ij} \in E$ (the set of all $M$–even), $o_{ij} \in O$ (the set of all $M$–odd) for all $i, j$. It follows that

$$A = \sum_{1 \leq i, j \leq n} e_{ij} + \sum_{1 \leq i, j \leq n} o_{ij}.$$  

Since $e_{ij}$ is $M$–even, $o_{ij}$ is $M$–odd so from Lemma 2.3, $f_{ij}^{e}$ is even and $f_{ij}^{o}$ is odd, for all $i, j$. Hence

$$\sum_{1 \leq i, j \leq n} f_{ij}^{e}$$  

is even and

$$\sum_{1 \leq i, j \leq n} f_{ij}^{o}$$  

is odd. This means that $\text{Mat}_n(R, M)$ is cleavable. $\square$

References