Matrix-Mappings in A Separating Duality on A Non-Archimedean Valued Field

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**Abstract.** We define the notion of matrix-mappings in separated duality \(\langle X, Y \rangle\) of vector spaces over a non-archimedean valued field \(K\), and we characterize these matrix-mappings. Then, we introduce a topology in the spaces of matrix-mappings and we give some applications in the algebras of matrix-mappings over a \(p\)-adic field \(\mathbb{Q}_p\).

1. Introduction


Robinson [14] and Melvin-Melvin [11] are generalized the concept of summing to the infinite matrices formed of operators and they, separately, given the version of the two theorems for sequence spaces of a Banach space. Then Ramanujan [12] has generalized the two theorems for spaces of sequences on a Frechet space, he defined a topology on the space of infinite matrices of operators between Frechet spaces which transforms convergent sequences into convergent sequences. He shown that, with this topology, said space is a Frechet space. The Kojima-Schur theorem has been generalized by Junde, Dohan and Minhyung for the sequence spaces over a barrelled spaces [10, Theorem 1, pp. 286]. The version of these two theorems in case of sequence spaces over a non archimedean valuated field \(K\) is gave by Rangachari and Srinivasan [13].

In [5], we are interested in matricial operators which preserve schauder basis in \(p\)-adic analysis. For more about Schauder basis see [9]. In [2], Ameziane Hassani and Babahmed are interested in the theory of non-archimedean summation, they introduced a topology on the space of infinite matrices of operators between Frechet spaces on a non-archimedean \((\mathfrak{u},\mathfrak{a})\) valued field transforming null sequences into convergent sequences, and they have shown that this space is a metrizable complete topological group. Now in [6], we gave a generalization of Kojima-Schur and Toeplitz-Silverman theorems for the sequence spaces over a non archimedean locally \(K\)-convex BS-spaces thanks to a family of quasi-seminorms that we are defined.
We define the following sequence spaces over $X$. Let $\|\cdot\|_2$ be a norm on $X$. Preliminary set of conservative matrix-mappings for null sequences and show that it is a Banach space. We characterize the matrix-mappings that we will define, and then we introduce a topology on the set of conservative matrix-mappings for null sequences. Then we are interested in algebras of matrix-mappings on the $\omega$-valued field $K$ and we give some applications where the sequences are in a $p$-adic field $Q_p$.

2. Preliminary

Throughout this paper, $K$ is a non-archimedean $(n.a)$ non trivially valued complete field with valuation $|\cdot|: (X, \|\cdot\|_X)$ a $n.a$ Banach space and $Y$ a topological vector space over $K$ (or $K$-vector space) that are in separated duality $(X, Y)$. The duality theory for locally $K$-convex spaces can be found more extensively in [3], [15] and [16].

$(\omega(X), \tau_\omega(X))$ is the linear space of all sequences in $X$ endowed with the product topology $\tau_\omega(X)$; this space is noted $\omega(K)$ (or $\omega$, for short) in case when $X = K$. A sequence space over $X$ is a subspace of $\omega(X)$.

We define the following sequence spaces over $X$,

$$
c_0(X) = \{ (x_k)_k \in \omega(X) : (x_k)_k \text{ converges to zero } \}
$$

$$
c(X) = \{ (x_k)_k \in \omega(X) : (x_k)_k \text{ converges in } X \},
$$

$$
\varphi(X) = \{ (x_k)_k \in \omega(X) : \text{ there exists } k_0 \in \mathbb{N} : x_k = 0 \text{ for all } k \geq k_0 \},
$$

$$
m(X) = \{ (x_k)_k \in \omega(X) : (x_k)_k \text{ is bounded in } X \}.
$$

For more information about topologies on non-archimedean sequence spaces see [7] and [8]. Let $a \in X$ and $k \geq 1$, we put $\delta_k(a) = (0, \ldots, 0, a, 0 \ldots)$ where $a$ is in the $k$-th place and $\delta(a) = (a, a, \ldots)$. For all $y \in Y$, we put $\|y\| = \sup \{|(x, y)/\|x\|_X\| \leq 1\}$. If $\|y\| < +\infty$ for all $y \in Y$, $\|\cdot\|$ is a $n.a$ norm on $Y$; moreover, if $N_{\|\cdot\|_X} = \{ \|x\|_X / x \in X \} \subset |K|$, we have $\|y\| = \sup \{ \|x\|_X / x \neq 0 \}$, where $|K| = \{ |a| / a \in K \}$. If $\|\cdot\|$ is a $n.a$ norm on $Y$, $(Y, \|\cdot\|)$ is a $n.a$ Banach space. Henceforth, we assume that $N_{\|\cdot\|_X} \subset |K|$. Let $(y_k)_k \in \omega(Y)$, we put:

$$
\|\langle y_k, x_k \rangle\|_g = \sup \left\| \sum_{k=1}^n \langle x_k, y_k \rangle \right\|_n \geq 1, \|x_k\|_X \leq 1 (1 \leq k \leq n)
$$

and $R_n = (y_{nr}, y_{n+1r}, \ldots)$ for all $n \geq 1$. If $\|\langle y_k, x \rangle\|_g < +\infty$ for all $(y_k)_k \in \omega(Y)$, $\|\cdot\|_g$ is a $n.a$ norm on $\omega(Y)$ called $n.a$ norm of group. We have:

1. $\|y_k\|_g \leq \|R_n\|_g$ for all $k \geq n$;
2. $\|R_{n+1}\|_g \leq \|R_n\|_g$ for all $n \geq 1$;
3. $\|\sum_{k=n}^{n+p} \langle x_k, y_k \rangle \|_g \leq \|R_n\|_g \max_{n \leq k \leq n+p} \|x_k\|_X$;
4. $\|\langle y_k, x \rangle\|_g = \sup_k \|\langle y_k, x \rangle\|_g$.

We denote by $M(Y)$ the set of all infinite matrices $A = (y_{nk})_{n,k}$ such that $y_{nk} \in Y$ for all $n, k \geq 1$. If $D \subset \omega(X)$, the $\beta$-dual of $D$ is the subspace of $\omega(Y)$ which is define by

$$
D^\beta = \left\{ (y_n)_n \in \omega(Y) : \lim_n \langle x_n, y_n \rangle = 0 \text{ for all } (x_n)_n \in D \right\}.
$$

$D$ is called perfect if $D^\beta = D$. If $D$ is perfect then $\varphi(X) \subset D$. We define $B^\beta$ if $B \subset \omega(Y)$ on the same way.

$\varphi(X)^\beta = \omega(Y)$ and $\omega(X)^\beta = \varphi(Y)$. For all $D \subset \omega(X)$ (or $\omega(Y)$), $D \subset D^\beta$ and $D^\beta$ is perfect.
Proposition 2.1. Let \((y_k)_k \in \omega(Y)\).

1. \((y_k)_k \in c_0(X) \iff \sup_k \|y_k\| < +\infty.\)
2. \((y_k)_k \in c(X) \iff \sup_k \|y_k\| < +\infty; \text{ for all } a \in X, (a, y_k)_k \in c(K).\)
3. \((y_k)_k \in m(X) \iff \lim_n \|R_n\|_y = 0.\)

Proof. See [1]. □

We have the criterion for permutation the order of the limits of a double sequence in \(K\).

Lemma 2.2. Let \((a_{nk})_{n,k}\) a double sequence in \(K\) such that:

1. \(\lim_k a_{nk} = a_n\) for all \(n \geq 1;\)
2. \(\lim_n a_{nk} = a_k\) uniformly on \(k.\)

Then \(\lim_k \lim_n a_{nk}\) and \(\lim_n \lim_k a_{nk}\) exist and are equal.

3. Matrix-Mappings

Let \(A = (y_{nk})_{n,k} \in \mathcal{M}(Y)\) and \(x = (x_k)_k \in \omega(X)\); if for all \(n \geq 1 \sum_k \langle x_k, y_{nk} \rangle\) converges in \(K\), we denote

\[
\langle x, A \rangle = \left(\sum_k \langle x_k, y_{nk} \rangle\right)_n
\]

the \(A\)-transform of \(x\) in \(\omega(K)\).

We say that \(A\) is a matrix-mapping of \(Y\) if there exists a space \(D\) of sequences on \(X\) and \(G\) a space of sequences on \(K\) such that for any \(x \in D\), \(\langle x, A \rangle \in G.\)

Definition 3.1. Let \(A\) a matrix-mapping on \(Y\) we say that \(A\) is:

1. conservative for the null sequences, if for all \(x \in c_0(X), \langle x, A \rangle \in c(K);\)
2. conservative, if for all \(x \in c(X), \langle x, A \rangle \in c(K);\)
3. regular for the null sequences, if for all \(x \in c_0(X), \langle x, A \rangle \in c_0(K);\)
4. coercive, if for all \(x \in m(X), \langle x, A \rangle \in c(K);\)
5. null coercive, if for all \(x \in m(X), \langle x, A \rangle \in c_0(K);\)
6. null-conservative, if for all \(x \in c(X), \langle x, A \rangle \in c_0(K).\)

Now we will characterize these matrix-mappings. We assume that \(Y\) is \(\sigma(Y,X)\)-complete.

Theorem 3.2. Let \(A\) a matrix-mapping on \(Y; A\) is conservative for the null sequences if and only if:

1. For all \(k \geq 1\), there exists \(y_k \in Y\) such that \(\lim_n y_{nk} = y_k\) in \((Y, \sigma(Y,X));\)
2. \(\sup_{n,k} \|y_{nk}\| < +\infty.\)

In this case we have: for all \((x_k)_k \in c_0(X),\)

\[
\lim_n \sum_k \langle x_k, y_{nk} \rangle = \sum_k \langle x_k, y_k \rangle.
\]
Proof. If \(A\) is conservative for the null sequences; for all \(a \in X\), \(\delta_k(a) \in c_0(X)\), therefore \((a, y_{nk})_n \in c(K)\), and then \((y_{nk})_n\) is a Cauchy sequence in \((Y, \sigma(Y, X))\), hence there exists \(y_k \in Y\) such that \((y_{nk})_n\) converges to \(y_k\) in \((Y, \sigma(Y, X))\). For all \(n \geq 1\), put

\[
T_n : c_0(X) \to K, (x_k)_k \to \sum_{k=1}^{n} \langle x_k, y_{nk} \rangle.
\]

\(T_n\) is continuous for all \(n \geq 1\) and \((T_n)_n\) is pointwise bounded on \(c_0(X)\), therefore \((T_n)_n\) is equicontinuous on \(c_0(X)\) (Banach-Steinhaus Theorem). There exists \(\rho > 0\) such that \(|T_n(x)| \leq \rho \|x\|_{\infty}\) for all \(x \in c_0(X)\) and for all \(n \geq 1\); therefore

\[
\sup_{n,k} \|y_{nk}\| \leq \rho < +\infty.
\]

Conversely, let \(x = (x_k)_k \in c_0(X)\) and \(\rho > 0\) such that \(\sup_{n,k} \|y_{nk}\| \leq \rho\). For all \(k \geq 1\),

\[
|\langle x_k, y_k \rangle| = \|x_k\| \|y_k\| \leq \rho \|x_k\| X \to 0,
\]

therefore \(\sum_k \langle x_k, y_k \rangle\) converges in \(K\). Let \(\varepsilon > 0\), there exists \(k_0 \geq 1\) such that:

\[
\left\{ \begin{array}{l}
\|x_k\| X \leq \frac{\varepsilon}{\rho} \text{ for all } k \geq k_0; \\
\sum_k \langle x_k, y_k \rangle \leq \varepsilon.
\end{array} \right.
\]

Let \(n_0 \geq 1\) such that \(\sum_{k < n_0} \langle x_k, y_{nk} - y_k \rangle \leq \varepsilon\) for all \(n \geq n_0\). For all \(n \geq n_0\), we have:

\[
\left| \sum_k \langle x_k, y_{nk} - y_k \rangle \right| \leq \max \left\{ \left| \sum_{k < n_0} \langle x_k, y_{nk} - y_k \rangle \right|, \left| \sum_{k \geq n_0} \langle x_k, y_{nk} \rangle \right|, \left| \sum_{k \geq n_0} \langle x_k, y_k \rangle \right| \right\} \leq \varepsilon.
\]

Therefore

\[
\lim_{n} \sum_k \langle x_k, y_{nk} \rangle = \sum_k \langle x_k, y_k \rangle.
\]

\(\square\)

**Corollary 3.3.** Let \(A = (y_{nk})_{n,k}\) a matrix-mapping on \(Y\); \(A\) is regular for the null sequence if, and only if:

1. For all \(k \geq 1\), \(\lim_{n} y_{nk} = 0\) in \((Y, \sigma(Y, X))\);
2. \(\sup_{n,k} \|y_{nk}\| < +\infty\).

**Theorem 3.4.** (Kojima-Schur) Let \(A = (y_{nk})_{n,k}\) a matrix-mapping on \(Y\); \(A\) is conservative if, and only if:

1. For all \(k \geq 1\), there exists \(y_k \in Y\) such that \(\lim_{n} y_{nk} = y_k\) in \((Y, \sigma(Y, X))\);
2. \(\sup_{n,k} \|y_{nk}\| < +\infty\);
3. For all \(n \geq 1\), \(\lim_{n} y_{nk} = 0\) in \((Y, \sigma(Y, X))\);
4. For all \(a \in X\), \(\lim_{n} \sum_k \langle a, y_{nk} \rangle \) exists in \(K\).

Under these conditions we have: for all \(x = (x_k)_k \in c(X)\),

\[
\lim_{n} \langle x, A \rangle = \lim_{n} \sum_k \langle x, y_{nk} \rangle + \sum_k \langle x_k - \lim x, y_k \rangle.
\]

\(\square\)
Proof. If $A$ is conservative, it is conservative for null sequences. Let $a \in X$, $x = \delta(a) \in c(X)$, therefore \( \lim_{k \to \infty} \sum_k \langle a, y_{nk} \rangle \) exists; moreover, for any $n \geq 1$, $\langle a, y_{nk} \rangle \xrightarrow{k \to \infty} 0$, and then $a_{nk} = 0$ in $(Y, c(Y, X))$.

Conversely, let $x = (x_k)_k \in c(X)$; put $a = \lim x_k$ and $z = (x_k - a)_k$. $z \in c_0(X)$ and $A$ is conservative for the null sequences (Theorem 3.2), therefore $(z, A) \in c(K)$. $(\delta(a), A) \in c(K)$, thus $(x, A) \in c(K)$ and we have \( \lim_{k \to \infty} \sum_k \langle x_k - a, y_{nk} \rangle = \sum_k \langle x_k - a, y_k \rangle \); that is to say $\lim (x, A) = \lim_{n \to \infty} \sum_k \langle x_k, y_{nk} \rangle + \sum_k \langle x_k - \lim x, y_k \rangle$. □

Corollary 3.5. Let $A = (y_{nk})_h$ a matrix-mapping on $Y$; $A$ is null-conservative if, and only if:

1. For all $k \geq 1$, $\lim_{n \to \infty} y_{nk} = 0$ in $(Y, c(Y, X))$;
2. $\sup_{n,k} \|y_{nk}\| = +\infty$;
3. For all $n \geq 1$, $\lim_{k \to \infty} y_{nk} = 0$ in $(Y, c(Y, X))$;
4. For all $a \in X$, $\lim_{n \to \infty} \sum_k \langle a, y_{nk} \rangle = 0$.

Theorem 3.6. Let $A = (y_{nk})_{h,k}$ a matrix-mapping on $Y$, $A$ is coercive if, and only if:

1. For all $k \geq 1$, there exists $y_k \in Y$ such that $\lim_{n \to \infty} y_{nk} = y_k$ in $(Y, c(Y, X))$;
2. For all $n \geq 1$, $\lim_{p \to \infty} \|R_{np}\| = 0$, where $R_{np} = (y_{np}, y_{np+1}, ...)$;
3. $\lim_{p \to \infty} \left\{ \sup_{n} \left\| R_{np} - R_p \right\| \right\} = 0$, where $R_p = (y_p, y_{p+1}, ...)$. 

In these conditions, we have: for all $(x_k)_k \in m(X)$, $\lim_{n \to \infty} \sum_k \langle x_k, y_{nk} \rangle = \sum_k \langle x_k, y_k \rangle$.

Proof. If $A$ is coercive, it is conservative for the null sequences we therefore have (1), (2) result of the Proposition 2.1. It remains to show (3).

Let $B = \{x \in m(X) / \|x\|_c \leq 1 \}$. On $B$ we define the $n.a$ distance:

$$d(x, y) = \sup_k \frac{1}{2k+1} \|x_k - y_k\|_X$$

for all $x = (x_k)_k$, $y = (y_k)_k \in B$.

For all $n, m \geq 1$, we put:

$$f_{mn} : (B, d) \to K$$

$$x_k \mapsto \sum_k \langle x_k, y_{mk} - y_{nk} \rangle$$

$$\left| \sum_k \langle x_k - y_k, y_{mk} - y_{nk} \rangle \right| \leq \max \left\{ d(x, y)2^p \max_{1 \leq k \leq p} \|y_{mk} - y_{nk}\|^p, \|R_{np}\|_g, \|R_{np}\|_g \right\}.$$ 

Therefore $f_{mn}$ is continuous for all $n, m \geq 1$. Let $\varepsilon > 0$, put:

$$F_{mn} = \{ x \in B / |f_{mn}(x)| \leq \varepsilon \};$$

$$E_p = \bigcap_{m,n \geq p} F_{mn}.$$ 

For all $p \geq 1$, $E_p$ is closed in $B$. 

Let \( x = (x_k)_k \in B, \langle x, A \rangle \in c(K) \), therefore there exists \( q \geq 1 \) such that \( \sum_k \langle x_k, y_{nk} - y_{nk} \rangle \leq \varepsilon \) for all \( m, n \geq q \), from which \( x \in Eq_q \), and hence \( B = \bigcup_p E_{p} \). (B, d) is complete, then from theorem of Baire-Hausdorff \(([17], \text{pp.11}), \) there exists \( q_0 \geq 1, \ a \in B \) and \( \rho > 0 \) such that \( B(x, \rho) = \{x \in B/d(x, a) \leq \rho \} \subset Eq_0 \). Let \( i \geq 1 \) such that \( \sup_{k \geq i} \frac{1}{2^{k+1}} \leq \rho \); and let \( x = (x_k)_k \in B \) and \( j \geq i \); put:

\[
\begin{cases}
    z_k = a_k & \text{if } k < i; \\
    z_k = x_k & \text{if } i \leq k \leq i + j; \\
    z_k = 0 & \text{if } k > i + j
\end{cases}
\]

\[
d(z, a) = \max\left\{\max_{1 \leq k \leq i+j} \frac{1}{2^{k+1}} \|x_k - a_k\|_X, \sup_{k \geq i+j} \frac{1}{2^{k+1}} \|a_k\|_X\right\} \leq \max\left\{\max_{1 \leq k \leq i+j} \frac{1}{2^{k+1}} \|x_k\|_X, \sup_{k \geq i+j} \frac{1}{2^{k+1}} \|a_k\|_X\right\}
\leq \sup_{k \geq i} \frac{1}{2^{k+1}} \leq \rho.
\]

Therefore \( z \in Eq_0 \), and then \( \sum_{k} |\langle z_k, y_{nk} - y_{nk} \rangle| \leq \varepsilon \) for all \( m, n \geq q_0 \) that is to say

\[
\sum_{k=i}^{i+j} |\langle z_k, y_{nk} - y_{nk} \rangle| + \sum_{k=1}^{i-1} |\langle x_k, y_{nk} - y_{nk} \rangle| \leq \varepsilon
\]

for all \( m, n \geq q_0 \). Let \( M_0 \in N \) such that \( \sum_{k=i}^{i+j} |\langle z_k, y_{nk} - y_{nk} \rangle| \leq \varepsilon \) for all \( m, n \geq M_0 \). Put \( N = \max\{M_0, q_0\} \); we have \( |\sum_{k=1}^{i+j} |\langle x_k, y_{nk} - y_{nk} \rangle| \leq \varepsilon \) for all \( m, n \geq N \). Letting \( m \to +\infty \) and taking the sup on \( x \in B \) and \( j \geq i \), we have for all \( n \geq N \) \( \|R_m - R\|_g \leq \varepsilon \) and then we have:

\[
\tag{\ast} \|R_{np} - R_p\|_g \leq \varepsilon \text{ for all } n \geq N \text{ and for all } p \geq i.
\]

For all \( n = 1, \ldots, N \lim_p \|R_{np}\|_g = 0 \), therefore there exists \( p_0 \geq i \) such that \( \|R_{np}\|_g \leq \varepsilon \) for all \( p \geq p_0 \) and for all \( n = 1, \ldots, N \).

\[
\|R_{np} - R_p\|_g \leq \max\left\{\|R_{np}\|_g, \|R_{np}\|_g, \|R_{np} - R_p\|_g\right\} \leq \varepsilon
\]

for all \( p \geq p_0 \); and then, according to (\ast\), we have \( \sup_n \|R_{np} - R_p\|_g \leq \varepsilon \) for all \( p \geq p_0 \).

Conversely, let \( x = (x_k)_k \in m(X); \sum_k |\langle x_k, y_{nk} \rangle| \) converges in \( K \) for all \( n \geq 1 \) (Proposition 2.1).

\[
\|R_p\|_g \leq \max\left\{\|R_{1p} - R_p\|_g, \|R_{1p}\|_g\right\} \xrightarrow{p \to \infty} 0,
\]

therefore \( \sum_k |\langle x_k, y_k \rangle| \) converges in \( K \). Let us show that \( \lim_n \sum_k |\langle x_k, y_{nk} \rangle| = \sum_k |\langle x_k, y_k \rangle| \).
Definition 3.9. Let \( A \) be a matrix-mapping on \( Y \); \( A \) is null-coercive if, and only if:

\[
\lim_{\|x\|_{\infty}} \|A(x)\|_{g} = 0.
\]

Then we verify that:

\[
\sum_{k} \langle x_k, y_{nk} - y_k \rangle \leq \max \left\{ \sum_{k < p_0} \langle x_k, y_{nk} - y_k \rangle, \sum_{k \geq p_0} \langle x_k, y_{nk} - y_k \rangle \right\} \leq \max \left\{ \epsilon, \|x\|_{\infty} \|R_{mp_0} - R_{p_0}\| \right\} \leq \epsilon.
\]

Corollary 3.7. Let \( A = (y_{nk})_{n,k} \) a matrix-mapping on \( Y \); \( A \) is null-coercive if, and only if:

1. For all \( k \geq 1 \), \( \lim_{n} y_{nk} = 0 \) in \( (Y, \sigma(Y, X)) \);
2. For all \( n \geq 1 \), \( \lim_{p} \|R_{mp}\|_{g} = 0 \);
3. \( \lim_{p} \left\{ \sup_{n} \|R_{mp} - R_{p}\|_{g} \right\} = 0 \).

Example 3.8. Let \( X \) be a n.a Banach space over a \( p \)-adic field \( \mathbb{Q}_p \) such that \( N_X \subset N_K \), where \( p \) is a prime integer and \( N_X = \{\|x\| : x \in X\} \). Considering the following matrix-mappings on \( \mathbb{Q}_p \):

\[
A = \begin{pmatrix}
    1 & 1 & 1 & \ldots \\
    1 & 1 & 1 & \ldots \\
    1 & 1 & 1 & \ldots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad
B = \begin{pmatrix}
    1 & 0 & 0 & \ldots \\
    p & p^2 & 0 & \ldots \\
    p & p^2 & p^3 & 0 & \ldots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
    1 & 1 & 1 & \ldots \\
    p & p & p & \ldots \\
    p^2 & p^2 & p^2 & \ldots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad
D = \begin{pmatrix}
    1 & 0 & 0 & \ldots \\
    p & p & 0 & \ldots \\
    p^2 & p^2 & p^2 & 0 & \ldots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

\[
E = \begin{pmatrix}
    1 & 0 & 0 & \ldots \\
    -p^2 & p^2 & 1 & \ldots \\
    -p^3 & p^3 & -p^3 & \ldots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad
F = \begin{pmatrix}
    p & p^2 & p^3 & \ldots \\
    p & p^2 & p^3 & \ldots \\
    p & p^2 & p^3 & \ldots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad
G = \begin{pmatrix}
    p & p^2 & p^3 & \ldots \\
    p & p^2 & p^3 & \ldots \\
    p & p^2 & p^3 & \ldots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Then we verify that:

\( A \) is conservative for the null sequences but not conservative,
\( B \) is conservative but not null conservative,
\( C \) is regular for the null sequences but not null conservative,
\( D \) is null conservative,
\( E \) is regular,
\( F \) is coercive but not null coercive,
\( G \) is null coercive.

Definition 3.9. Let \( A = (y_{nk})_{n,k} \) a matrix-mapping on \( Y \) and \( f \in X^* \); we say that \( A \) is \( f \)–regular if \( A \) is conservative and for all \( x \in c(X) \) \( \lim (x, A) = f(\lim x) \).

We have the following theorem of Toeplitz-Silverman type:
Theorem 3.10. (Toeplitz-Silverman) Let $A = (y_{nk})_{n,k}$ a matrix-mapping on $Y$ and $f$ a linear form on $X$; $A$ is $f$-regular if, and only if:

1. For all $k \geq 1$, $\lim_{n} y_{nk} = 0$ in $(Y, \sigma(Y, X))$;
2. $\sup_{n,k} \|y_{nk}\| < +\infty$;
3. For all $a \in X$, $\lim_{n} \sum_{k} \langle a, y_{nk} \rangle = f(a)$;
4. For all $n \geq 1$, $\lim_{k} y_{nk} = 0$ in $(Y, \sigma(Y, X))$.

Proof. If $A$ is $f$-regular, it is regular for null sequences, and so we have (1) and (2) (Corollary 3.3). For all $a \in X$, $\delta(a) \in c(X)$ therefore, on the one hand, $\lim_{n} \langle a, y_{nk} \rangle = 0$ for all $n \geq 1$, and then $\lim_{k} y_{nk} = 0$ in $(Y, \sigma(Y, X))$, and on the other, $\lim_{n} \sum_{k} \langle a, y_{nk} \rangle = f(a)$.

Conversely, $A$ is conservative (Theorem 3.4). Let $x = (x_{k})_{k} \in c(X)$; put $a = \lim_{k} x_{k} \cdot (x_{k} - a) \in c_{0}(X)$, therefore $\lim_{n} \sum_{k} \langle x_{k} - a, y_{nk} \rangle = 0$ (Theorem 3.2), and then $\lim_{n} \sum_{k} \langle x_{k} - y_{nk} \rangle = \lim_{n} \sum_{k} \langle a, y_{nk} \rangle = f(a)$. □

Definition 3.11. Let $A$ a matrix-mapping on $Y$; we say that $A$ is:

1. $c_{0}$-permanent if for all $x \in c_{0}(X)$, $(x, A) \in m(K)$;
2. $c$-permanent if for all $x \in c(X)$, $(x, A) \in m(K)$;
3. $m$-permanent if for all $x \in m(X)$, $(x, A) \in m(K)$.

$A$ is $m$-permanent $\Rightarrow A$ is $c$-permanent $\Rightarrow A$ is $c_{0}$-permanent.

Theorem 3.12. Let $A = (y_{nk})_{n,k}$ a matrix-mapping on $Y$; $A$ is $c_{0}$-permanent if, and only if $\sup_{n,k} \|y_{nk}\| < +\infty$.

Proof. Suppose that $A$ is $c_{0}$-permanent; for all $n \geq 1$ put $T_{n} : c_{0}(X) \rightarrow K$, $(x_{k})_{k} \rightarrow \sum_{k} \langle x_{k}, y_{nk} \rangle$. $T_{n} \in c_{0}(X)'$ for all $n \geq 1$ and the sequence $(T_{n})_{n}$ is pointwise bounded on $c_{0}(X)$, then it is equicontinuous on $c_{0}(X)$. Let $\rho > 0$ such that $|T_{n} - x| \leq \rho \|x\|_{c_{0}}$ for all $x \in c_{0}(X)$ and for all $n \geq 1$. Therefore $\sup_{n,k} \|y_{nk}\| \leq \rho$.

Conversely, let $x = (x_{k})_{k} \in c_{0}(X)$, for all $n \geq 1$, $\sum_{k} \langle x_{k}, y_{nk} \rangle$ converges in $K$ (Proposition 2.1). Let $\lambda \in K$ such that $\|x\|_{c_{0}} \leq |\lambda|$, we have $\sup_{n} \left\| \sum_{k} \langle x_{k}, y_{nk} \rangle \right\| \leq |\lambda| \sup_{n,k} \|y_{nk}\| < +\infty$. Therefore $(x, A) \in m(K)$. □

Corollary 3.13. Let $A = (y_{nk})_{n,k}$ a matrix-mapping on $Y$, $A$ is $c$-permanent if, and only if:

1. $\sup_{n,k} \|y_{nk}\| < +\infty$;
2. For all $n \geq 1$, $\lim_{k} y_{nk} = 0$ in $(Y, \sigma(Y, X))$.

Corollary 3.14. Let $A = (y_{nk})_{n,k}$ a matrix-mapping on $Y$, $A$ is $m$-permanent if, and only if:

1. $\sup_{n,k} \|y_{nk}\| < +\infty$;
2. For all $n \geq 1$, $\lim_{p} \|R_{np}\| = 0$. 

Example 3.15. Let $X$ be a n.a Banach space over a $p$−adic field $Q_p$ such that $N_X \subseteq N_K$, where $p$ is a prime integer. Considering the following matrix-mappings on $Q_p$:

$$H = \begin{pmatrix} -p & -p & -p & \ldots \\ -p & -p & p & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad I = \begin{pmatrix} -p & p & p & \ldots \\ p & p & p & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$H$ is $c_0(X)$−permanent but not $c(X)$−permanent and not conservative for the null sequences.
$I$ is $m(X)$−permanent then it is $c(X)$−permanent but not conservative and not coercive.

4. Topological Characterization

We denote $CN(Y), C(Y), NC(Y), C_0(Y), NC_0(Y), RN(Y), R_f(Y), C_0\mathcal{P}(Y), c\mathcal{P}(Y)$ and $m\mathcal{P}(Y)$ the spaces of matrix-mappings on $Y$ conservatives for the null sequences, conservatives, null-conservatives, coercives, null-coercives, regulars for the null sequences, $f$−regulars, $c_0$−permanents, $c$−permanents and $m$−permanents respectively.

We have the following of inclusion diagram:

$$\begin{array}{cccc}
NC_0(Y) & \rightarrow & C_0(Y) & \rightarrow & m\mathcal{P}(Y) \\
\downarrow & & \uparrow & & \downarrow \\
RN(Y) & \leftarrow & c_0\mathcal{P}(Y) & \leftarrow & C\mathcal{P}(Y) \\
\downarrow & & \uparrow & & \downarrow \\
NC(Y) & \rightarrow & CN(Y) & \rightarrow & R_f(Y) \\
\end{array}$$

Let $A \in c_0\mathcal{P}(Y)$, we put $H(A) = \sup_{n,k} \|y_{nk}\|$. $H$ is a n.a norm on $c_0\mathcal{P}(Y)$.

**Theorem 4.1.** ($c_0\mathcal{P}(Y), H$) is a n.a Banach space.

**Proof.** Let $(A^r)_r$, a Cauchy sequence in ($c_0\mathcal{P}(Y), H$), with $A^r = (y_{nk}^r)_r$ for all $r \geq 1$. We have: for all $\varepsilon > 0$, there exists $r_0 \geq 1$ such that for all $r, s \geq r_0$, $\|y_{nk}^r - y_{nk}^s\| \leq \varepsilon$ for all $n, k \geq 1$ ($\ast$). Therefore $(y_{nk}^r)_r$ is uniformly of Cauchy sequence on $n, k$ in $(Y, \sigma(Y, X))$; there exists $y_{nk} \in Y$ such that $y_{nk}^r \xrightarrow{\text{r} \to \infty} y_{nk}$ uniformly on $n, k$ in $(Y, \sigma(Y, X))$. Put $A = (y_{nk})_{n,k}$. By letting $s$ to $\infty$ in ($\ast$), we will $A^r \xrightarrow{\text{r} \to \infty} A(H)$.

It remains to show that $A \in c_0\mathcal{P}(Y)$.

For all $n, k \geq 1$ $\|y_{nk}\| \leq \max_{n,k} \{\|y_{nk}^r - y_{nk}\|, \|y_{nk}^0\|\}$.

$$\Rightarrow \sup_{n,k} \|y_{nk}\| \leq \max \{\varepsilon, H(A^0)\} < +\infty.$$ 

\square

**Corollary 4.2.** ($c\mathcal{P}(Y), H$) is a n.a Banach space.

**Proof.** It suffices to show that $c\mathcal{P}(Y)$ is a closed subspace of ($c_0\mathcal{P}(Y), H$). Let $(A^r)_r$, a sequence in $c\mathcal{P}(Y)$ and $A \in c_0\mathcal{P}(Y)$ such that $A^r \xrightarrow{r \to \infty} A$, with $A^r = (y_{nk}^r)_r$ for all $r \geq 1$ and $A = (y_{nk})_{n,k}$.

Let $r_0 \geq 1$ such that $H(A^{r_0} - A) \leq 1$; $H(A) \leq \max \{H(A^{r_0} - A), H(A^n)\} < +\infty$. ($y_{nk}^r$), converges uniformly on $n, k$ to $y_{nk}^r$ in $(Y, \sigma(Y, X))$. Let $n \geq 1$, $\lim_{k \to \infty} y_{nk}^r = \lim_{k \to \infty} y_{nk}^r = \lim_{k \to \infty} y_{nk}^r = 0$ (Lemma 2.2). Therefore $A \in c_0\mathcal{P}(Y)$ (Corollary 3.7). \square
We also show that all other spaces listed in the diagram above are closed subspaces of \((c_0\mathcal{P}(Y), H)\) and therefore they are \(n.a\) Banach spaces.

5. Algebras of Matrix-Mappings

**Definition 5.1.** Let \(E\) an algebra over \(K\) having a unitary element \(e\); we say that \(E\) is a \(n.a\) Banach algebra if there exists a \(n.a\) norm \(\|\cdot\|\) on \(E\) such that \((E, \|\cdot\|)\) is a \(n.a\) Banach space and it verifies:

1. \(\|e\| = 1;\)
2. For all \(x, y \in E, \|x y\| \leq \|x\| \|y\|\).

Let \((X, \|\cdot\|_X)\) and \((Y, \|\cdot\|_Y)\) two \(n.a\) Banach algebras with unitary element \(e_X, e_Y\) respectively put in separating duality \((X, Y)\); if \(A = (x_{nk})_{n,k} \in \mathcal{M}(X)\) and \(B = (y_{nk})_{n,k} \in \mathcal{M}(Y)\), we put \(A.B = (\lambda_{nk})_{n,k}\), with \(\lambda_{nk} = \sum_j \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle\) for all \(n, k \geq 1\) where there is.

**Proposition 5.2.** If \(A = (x_{nk})_{n,k} \in c\mathcal{P}(X)\) and \(B = (y_{nk})_{n,k} \in c_0\mathcal{P}(Y)\), \(A.B\) exists.

**Proof.** Let \(n, k \geq 1\);

\[
\lim j \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle \leq \|y_{jk}\| \|\langle x_{nj}, e_Y \rangle\| \rightarrow 0 \quad \text{(Corollary 3.7)},
\]

therefore \(\sum_j \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle\) converges. \(\square\)

**Proposition 5.3.** If \(A = (x_{nk})_{n,k} \in \mathcal{P}(X)\) and \(B = (y_{nk})_{n,k} \in \mathcal{P}(Y)\), \(A.B \in \mathcal{P}(K)\).

**Proof.** \(A.B\) exists (Proposition 5.2).

\[
\sup_{n,k} \left| \sum_j \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle \right| \leq \left( \sup_{n,j} \left| \langle x_{nj}, e_Y \rangle \right| \left( \sup_{j,k} \|y_{jk}\| \right) \right) < +\infty.
\]

Let \(n \geq 1\);

\[
\lim j \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle \leq \|y_{jk}\| \|\langle x_{nj}, e_Y \rangle\| \rightarrow 0 \quad \text{uniformly on } k \quad \text{(Corollary 3.7)}.
\]

Let \(j \geq 1\);

\[
\lim k \sum_j \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle = \sum_j \lim k \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle = \sum_j \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle = 0.
\]

Therefore \(A.B \in \mathcal{P}(Y)\). \(\square\)

**Proposition 5.4.** If \(A = (x_{nk})_{n,k} \in m\mathcal{P}(X)\) and \(B = (y_{nk})_{n,k} \in m\mathcal{P}(Y)\), \(A.B \in m\mathcal{P}(K)\).

**Proof.** It suffices to show that \(\lim k \sum_j \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle = 0\) for all \(n \geq 1\). Let \(\rho > 0\) such that \(\sup_{j,k} \|y_{jk}\| \leq \rho\).

Let \(n \geq 1\);

\[
\lim j \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle \leq \rho \|x_{nj}\| \|\langle e_X, y_{jk} \rangle\| \rightarrow 0 \quad \text{uniformly on } k \quad \text{(Corollary 3.13)}.
\]

For all \(j \geq 1\);

\[
\lim k \sum_j \langle x_{nj}, e_Y \rangle \langle e_X, y_{jk} \rangle \leq \|x_{nj}\| \|y_{jk}\| \|\langle e_X, y_{jk} \rangle\| \rightarrow 0 \quad \text{uniformly on } k.
\]

Therefore \(A.B \in m\mathcal{P}(Y)\). \(\square\)

Using the same techniques, we show:
Corollary 5.8. Let $x$ be a sequence in $Q_p$.

Example 5.7. Let $Q$ be a sequence in $Q_p$.

Proof. If $A \in C_0(X)$ and $B \in C_0(Y)$, then $A \cdot B \in C_0(K)$.

Theorem 5.5. $(cP(K), H)$ is a Banach algebra.

Proof. $cP(K)$ is an algebra admitting the identity matrix $I$ as a unitary element. $(cP(K), H)$ is a Banach space (Corollary 4.2), and we have: $H(I) = 1$ and $H(A \cdot B) \leq H(A)H(B)$ for all $A, B \in cP(K)$.

We show that $mP(K)$, $C(K)$, $\mathcal{R}N(K)$ and $\mathcal{C}N(K)$ are Banach subalgebras of $(cP(K), H)$.

Proposition 5.6. Let $A \in cP(K)$ and $\lambda \in K$ such that $|\lambda| < \frac{1}{H(A)}$, then $I + \lambda A$ and $I - \lambda A$ are nonsingular in $cP(K)$.

Proof. $H((-\lambda)^n A^n) \leq (|\lambda| H(A))^n \rightarrow 0$, therefore the series $B = \sum_{n=0}^{\infty} (-\lambda)^n A^n$ and $C = \sum_{n=0}^{\infty} \lambda^n A^n$ converge in $cP(K)$, and we have:

$(I + \lambda A) B = B (I + \lambda A) = I$;
$(I - \lambda A) C = C (I - \lambda A) = I$.

We have the same result for $mP(K)$, $C(K)$, $\mathcal{R}N(K)$ and $\mathcal{C}N(K)$.

Example 5.7. Let $Q_p$ be the $p$–adic field, where $p$ is a prime integer; considering the following matrix-mappings on $Q_p$:

$$A = \begin{bmatrix}
p & 0 & 0 & 0 & \cdots 
p^2 & p & 0 & 0 & \cdots 
p^3 & p^2 & p & 0 & \cdots 
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad B = \begin{bmatrix}
p & 0 & 0 & 0 & \cdots 
p^2 & p & 0 & 0 & \cdots 
p^3 & p^2 & p & 0 & \cdots 
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad C = \begin{bmatrix}
p & p^2 & p^3 & \cdots 
p^2 & p^3 & p^4 & \cdots 
p^3 & p^4 & p^5 & \cdots 
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$

$A$ is conservative for the null sequences.
$B$ is conservative.
$C$ is regular for the null sequences.

Corollary 5.8. Let $x = (x_k)_k \in \omega(Q_p)$ and $\lambda \in Q_p$ such that $|\lambda| < 1$.

1. If the sequence of general term $y_n = \lambda p \sum_{k=1}^{n-1} x_k + (1 + \lambda p)x_n$ converges to 0 in $Q_p$, the sequence $(x_k)_k$ converges in $Q_p$.

2. If the sequence of general term $y_n = \lambda \sum_{k=1}^{n-1} x_k + (1 + \lambda p^n)x_n$ converges in $Q_p$, the sequence $(x_k)_k$ converges in $Q_p$.

3. If the sequence of general term $y_n = \lambda p^n \sum_{k \neq n} x_k + (1 + \lambda p^n)x_n$ converges to 0 in $Q_p$, the sequence $(x_k)_k$ converges to 0 in $Q_p$. 


Proof. (1) \((y_k)_k = (I + \lambda A)(x_k)_k; H(A) = \sup_{k \geq 1} |p|^k = \frac{1}{p},\) therefore \(|\lambda| < \frac{1}{H(A)},\) and then \((I + \lambda A)\) is nonsingular in \(CN(Q_p).\) There exists \(A' \in CN(Q_p)\) such that \(A' = (I + \lambda A)^{-1}.\) \((x_k)_k = A'(y_k)_k\) and then \((x_k)_k \in c(Q_p).\)

(2) \((y_k)_k = (I + \lambda B)(x_k)_k\) and \((I + \lambda B)\) is nonsingular in \(C(Q_p).\)

(3) \((y_k)_k = (I + \lambda C)(x_k)_k\) and \((I + \lambda C)\) is nonsingular in \(RN(Q_p).\)

6. Conclusion

In this work we gave a generalization of Kojima-Schur and Toeplitz-Silverman theorems for conservative and \(f\)-regular matrix-mappings in separated duality \((X, Y),\) where \(X\) and \(Y\) are respectively a Banach space and vector space over a non-archimedean valued field \(K.\) We are also defined at the same other matrix-mappings and established others results characterizing them. We have provided space for conservative matrix-mappings and space for conservative for the null sequences matrix-mappings with a topology and we have demonstrated that they are non-archimedean Banach spaces. Some applications to spaces of non-archimedean scalar matrix-mappings were given. However a natural question arises:

Is what one can generalize these results at spaces of matrix-mappings on locally \(K-\) convex spaces?

References


