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RESEARCH ARTICLE

λ_c-Open Sets and λ_c-Separation Axioms in Topological Spaces

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The aim of this paper is to introduce a new class of sets called λ_c- open sets and to investigate some of their relationships and properties. Further, by using this set, the notion of λ_c-T_i spaces ($i = 0, 1/2, 1, 2$) and λ_c-R_j spaces ($j = 0, 1$) are introduced and some of their properties are investigated.

Keywords: s-operation; λ_c-Open Set; λ_c-T_i, $i = 0, 1, 2$; λ_c-R_j, $j = 0, 1$.

AMS Subject Classification: 54A40.

1. Introduction

The study of semi open sets and semi continuity in topological spaces was initiated by Levine [1]. Analogous to generalized closed sets which was introduced by Levine [2], Bhattacharya and Lahiri [3], introduced the concept of semi generalized closed sets in topological spaces. Kasahara [4], defined the concept of an operation on topological spaces and introduced the concept of closed graphs of a function. Ahmad and Hussain [5], continued studying the properties of operations on topological spaces introduced by Kasahara [4]. Ogata [6] introduced the concept of γ-T_i $(i = 0, 1/2, 1, 2)$ and characterized γ-T_i by the notion of γ-closed sets or γ-open sets. Maheshwari and Prasad [7] defined other new types of separation axioms and Dorsett [8], defined semi-R_1, and semi-R_0 spaces.

In this paper, we introduce and study a new class of semi open sets called λ_c-open sets in topological spaces. By using the notion of λ_c-closed and λ_c-open sets, we introduce the concept of λ_c-T_i $(i = 0, 1/2, 1, 2)$ and λ_c-R_j $(j = 0, 1)$ spaces. several properties and characterizations of these spaces are obtained.

2. Preliminaries

Throughout, X denote a topological space with out any separation axiom. Let A be a subset of X, the closure (interior) of A are denoted by $Cl(A)$ $(Int(A))$ respectively. A subset A of a topological space (X, τ) is said to be semi open [1] if $A \subseteq Cl(Int(A))$. The complement of a semi open set is said to be semi closed [1]. The family of all semi open (resp. semi closed) sets in a topological space (X, τ) is denoted by $SO(X, \tau)$ or $SO(X)$ (resp. $SC(X, \tau)$ or $SC(X)$).

Definition 2.1 [9] Let (X, τ) be a topological space and let A be a subset of X, then:

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(1) The semi interior of A ($sInt(A)$) is the union of all semi open sets of X contained in A.
(2) A point $x \in X$ is said to be a semi limit point of A if every semi open set containing x contains a point of A different from x, and the set of all semi limit points of A is called the semi derived set of A denoted by $sd(A)$.
(3) The intersection of all semi-closed sets of X containing A is called the semi-closure of A and is denoted by $sCl(A)$.

Lemma 2.2 [9] For each point $x \in X$, $x \in sCl(A)$ if and only if $V \cap A \neq \phi$ for every $V \in SO(X)$ containing x.

Definition 2.3 [3] A subset A of a space (X, τ) is called a semi-generalized closed set (sg-closed), if $A \subseteq U$ and U is semi-open implies that $sCl(A) \subseteq U$.

Definition 2.4 A topological space (X, τ) is said to be:

(1) semi-T_0 [7] if for any distinct pair of points in X, there is an semi-open set containing one of the points but not the other.
(2) semi-T_1 [7] if for any distinct pair of points x and y in X, there is a semi-open U in X containing x but not y and a semi-open set V in X containing y but not x.
(3) semi-T_2 [7] if for any distinct pair of points x and y in X, there exist semi-open sets U and V in X containing x and y, respectively, such that $U \cap V = \phi$.
(4) semi-$T_{1/2}$ [3] if every sg-closed set is semi closed.
(5) semi-R_0 [8] if for each $O \in SO(X)$ and $x \in O$, $sCl(\{x\}) \subseteq O$.
(6) semi-R_1 [8] if for each pair $x, y \in X$ such that $sCl(\{x\}) \neq sCl(\{y\})$, there exist disjoint semi-open sets U and V such that $sCl(\{x\}) \subseteq U$ and $sCl(\{y\}) \subseteq V$.

Definition 2.5 [6] Let (X, τ) be any topological space. A mapping $\lambda : \tau \to P(X)$, ($P(X)$ stands for all subsets of X), is called an operation on τ if $V \subseteq \lambda(V)$ for each non-empty open set V and $\lambda(\phi) = \phi$.

Definition 2.6 [9] Let (X, τ) be a topological space and let A be a subset of X, then:

(1) The λ-interior of A (${}_{\lambda}Int(A)$) is the union of all λ-open sets of X contained in A.
(2) A point $x \in X$ is said to be a λ-limit point of A if every λ-open set containing x contains a point of A different from x, and the set of all λ-limit points of A is called the λ-derived set of A denoted by ${}_{\lambda}d(A)$.
(3) The intersection of all λ-closed sets of X containing A is called the λ-closure of A and is denoted by ${}_{\lambda}Cl(A)$.

3. λ_c-Open Sets

In this section, we introduce a new class of semi open sets called λ_c-open sets. Further, the notion of λ_c-derived set, λ_c-closure and λ_c-interior are introduced and their properties are discussed.

Definition 3.1 A mapping $\lambda : SO(X) \to P(X)$ is called an s-operation on $SO(X)$ if $V \subseteq \lambda(V)$ for each non-empty semi open set V and $\lambda(\phi) = \phi$.

If $\lambda : SO(X) \to P(X)$ is any s-operation, then it is clear that $\lambda(X) = X$.

Definition 3.2 Let (X, τ) be a topological space and $\lambda : SO(X) \to P(X)$ be an s-operation defined on $SO(X)$, then a subset A of X is λs-open set if for each $x \in A$ there exists a semi open set U such that $x \in U$ and $\lambda(U) \subseteq A$.

Definition 3.3 A λs-open subset A of X is called λ_c-open if for each $x \in A$ there exists a closed subset F of X such that $x \in F \subseteq A$.

The complement of a λ_c-open set is said to be λ_c-closed. The family of all λ_c-open (resp., λ_c-closed) subsets of a topological space (X, τ) is denoted by $SO_{\lambda c}(X, \tau)$ or $SO_{\lambda c}(X)$ (resp., $SC_{\lambda c}(X, \tau)$ or $SC_{\lambda c}(X)$).

Proposition 3.4 For any topological space (X, τ), we have $SO_{\lambda c}(X) \subseteq SO_{\lambda}(X) \subseteq SO(X)$.

Proof Obvious. ■

The following examples show that the equality in the above proposition may not be true in general.

Example 3.5 Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \{a, c\}$ or A is empty and $\lambda(A) = X$ otherwise. Here, we have $\{a, c\}$ is λs-open set but it is not λ_c-open.

The following example shows that τ is incomparable with $\lambda_c O(X)$.

Example 3.6 Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A \neq \{a\}$ or $\{b\}$ and $\lambda(A) = \{a, b\}$ if $A = \{a\}$ or $\{b\}$. Now, the family of open sets in (X, τ) is incomparable with $\lambda_c O(X)$.

Proposition 3.7 Let $\{A_\alpha\}_{\alpha \in I}$ be any collection of λ_c-open sets in a topological space (X, τ), then $\bigcup\limits_{\alpha \in I} A_\alpha$ is a λ_c-open set.

Proof Since A_α is λ_c-open set for all $\alpha \in I$, then A_α is a λs-open set for all $\alpha \in I$. This implies that there exists a semi open set U such that $\lambda(U) \subseteq A_{\alpha_0} \subseteq \bigcup\limits_{\alpha \in I} A_\alpha$. Therefore, $\bigcup\limits_{\alpha \in I} A_\alpha$ is a λs-open subset of (X, τ). Let $x \in \bigcup\limits_{\alpha \in I} A_\alpha$, then there exists an $\alpha_0 \in I$ such that $x \in A_{\alpha 0}$. Since A_α is a λ_c-open set for all $\alpha \in I$, then there exists a closed set F such that $x \in F \subseteq A_{\alpha 0}$ but $A_{\alpha 0} \subseteq \bigcup\limits_{\alpha \in I} A_\alpha$, then $x \in F \subseteq \bigcup\limits_{\alpha \in I} A_\alpha$. Hence, $\bigcup\limits_{\alpha \in I} A_\alpha$ is λ_c-open. ■

The following example shows that the intersection of two λ_c-open sets need not be λ_c-open.

Example 3.8 Let $X = \{a, b, c\}$ and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty or $A \neq \{a\}$ nor $\{b\}$ and $\lambda(A) = X$ otherwise. So we have $\{a, b\}$ and $\{b, c\}$ are λ_c-open sets but $\{a, b\} \cap \{b, c\} = \{b\}$ is not λ_c-open.

Proposition 3.9 The set A is λ_c-open in the space (X, τ) if and only if for each $x \in A$ there exists a λ_c-open set B such that $x \in B \subseteq A$.

Proof Suppose that A is λ_c-open in (X, τ). Then for each $x \in A$ we put $B = A$ is a λ_c-open set such that $x \in B \subseteq A$.

Conversely, Suppose that for each $x \in A$ there exists a λ_c-open set B_x such that $x \in B_x \subseteq A$. Thus $A = \bigcup B_x$, where $B_x \in SO_{\lambda c}(X)$ for each x. Therefore, by Proposition 3.7, A is λ_c-open. ■

Definition 3.10 Let (X, τ) be a topological space. An s-operation λ is said to be s-regular if for every semi open sets U and V containing $x \in X$, there exists a semi open set W containing x such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V)$.

Theorem 3.11 Let λ be an s-regular s-operation. If A and B are λ_c-open sets in X, then $A \cap B$ is also λ_c-open.

Proof Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since A and B are λs-open sets, so there exist semi open sets U and V such that $x \in U$ and $\lambda(U) \subseteq A$, $x \in V$ and $\lambda(V) \subseteq B$. Since λ is s-regular, this implies that there exists a semi open set W of x such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V) \subseteq A \cap B$. Therefore, $A \cap B$ is λs-open set. Again for each $x \in A \cap B$, we have $x \in A$ and $x \in B$ and since A and B areλ_c-open sets, then there exist closed sets E, F such that $x \in E \subseteq A$ and $x \in F \subseteq B$. Therefore, $x \in E \cap F \subseteq A \cap B$. Since $E \cap F$ is closed, so by Definition 3.2, we obtain that $A \cap B$ is λ_c-open. ■

Definition 3.12 Let (X, τ) be a topological space and let A be subset of X, then a point $x \in X$ is called a λ_c-limit point of A if every λ_c-open set containing x contains a point of A different from x.

The set of all λ_c-limit points of A is called the λ_c-derived set of A denoted by $\lambda_c d(A)$.

Definition 3.13 Let A be subset of the space (X, τ), then the λ_c-closure of A ($\lambda_c Cl(A)$) is the intersection of all λ_c-closed sets containing A.

Here we introduce some properties of λ_c-closure of the sets.

Proposition 3.14 For subsets A, B of a topological space (X, τ), the following statements are true:

(1) $A \subseteq \lambda_c Cl(A)$.
(2) $\lambda_c Cl(A)$ is λ_c-closed set in X.
(3) $\lambda_c Cl(A)$ is smallest λ_c-closed set which contain A.
(4) A is λ_c-closed set if and only if $A = \lambda_c Cl(A)$.
(5) $\lambda_c Cl(\phi) = \phi$ and $\lambda_c Cl(X) = X$.
(6) If $A \subseteq B$. Then $\lambda_c Cl(A) \subseteq \lambda_c Cl(B)$.
(7) $\lambda_c Cl(A) \cup \lambda_c Cl(B) \subseteq \lambda_c Cl(A \cup B)$.
(8) $\lambda_c Cl(A \cap B) \subseteq \lambda_c Cl(A) \cap \lambda_c Cl(B)$.

Proof Obvious. ■

In general the equalities (7) and (8) of the above proposition is not true, as it is shown in the following examples:

Example 3.15 Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \phi$ or $c \in A$ and $\lambda(A) = Cl(A)$ otherwise. Now, if $A = \{a\}$ and $B = \{c\}$ then $\lambda_c Cl(A) = \{a\}$ and $\lambda_c Cl(B) = \{c\}$, but $\lambda_c Cl(A \cup B) = X$, where $A \cup B = \{a, c\}$. Hence $\lambda_c Cl(A \cup B) \neq \lambda_c Cl(A) \cup \lambda_c Cl(B)$.

Example 3.16 Let $X = \{a, b, c\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \phi$, $A = \{a, b\}$ or $\{b, c\}$ and $\lambda(A) = X$ otherwise. Now, if $A = \{a, b\}$ and $B = \{a, c\}$, then $\lambda_c Cl(A) = X$ and $\lambda_c Cl(B) = X$, but $\lambda_c Cl(A \cap B) = \{a\}$, where $A \cap B = \{a\}$,

$$\lambda_c Cl(A \cap B) \neq \lambda_c Cl(A) \cap \lambda_c Cl(B).$$

Proposition 3.17 Let A be any subset of a space X, then $\lambda_c Cl(A) = A \cup \lambda_c d(A)$.

Proof Obvious. ■

Proposition 3.18 If A is a subset of (X, τ), then $x \in \lambda_c Cl(A)$ if and only if $V \cap A \neq \phi$ for every λ_c-open set V containing x.

Proof Let $x \in \lambda_c Cl(A)$ and suppose that $V \cap A = \phi$ for some λ_c-open set V which contains x. This implies that $X \backslash V$ is λ_c-closed and $A \subseteq (X \backslash V)$, so $\lambda_c Cl(A) \subseteq (X \backslash V)$. But this implies that $x \in (X \backslash V)$ which is contradiction. Therefore, $V \cap A \neq \phi$.

Conversely, Let $A \subseteq X$ and $x \in X$ such that for each λ_c-open set V containing x, $U \cap A \neq \phi$. If $x \notin \lambda_c Cl(A)$, then there is a λ-closed set S such that $A \subseteq S$ and $x \notin S$. Hence, $(X \backslash S)$ is a λ_c-open set with $x \in (X \backslash S)$ and thus $(X \backslash S) \cap A \neq \phi$ which is a contradiction. Therefore, $x \in \lambda_c Cl(A)$. ■

Proposition 3.19 If A is any subset of a topological space (X, τ), then ${}_s Cl(A) \subseteq \lambda_c Cl(A)$.

Proof Obvious. ■

The following example shows that the equality in the above proposition is not true in general.

Example 3.20 Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. We define an s-operation λ : SO(X) $\to$ P(X) as $\lambda(A) = A$ if $A = \phi$ or $A = \{a\}$ and $\lambda(A) = X$ otherwise. Now, if $A = \{c\}$, then ${}_s Cl(A) = \{c\}$ and $\lambda_c Cl(A) = X$.

Definition 3.21 Let (X, τ) be a topological space and let A be subset of X, then the λ_c-interior of A ($\lambda_c Int(A)$) is the union of all λ_c-open sets of X contained in A.

Proposition 3.22 For subsets A, B of a space X, the following statements hold:

(1) $\lambda_c Int(A)$ is the union of all λ_c-open sets which are contained in A.
(2) $\lambda_c Int(A)$ is a λ_c-open set in X.
(3) $\lambda_c Int(A) \subseteq A$.
(4) $\lambda_c Int(A)$ is the largest λ_c-open set contained in A.
(5) A is λ_c-open set if and only if $\lambda_c Int(A) = A$.
(6) $\lambda_c Int(\lambda_c Int(A)) = \lambda_c Int(A)$.
(7) If $A \subseteq B$, then $\lambda_c Int(A) \subseteq \lambda_c Int(B)$.
(8) $\lambda_c Int(\phi) = \phi$ and $\lambda_c Int(X) = X$.
(9) $\lambda_c Int(A) \cup \lambda_c Int(B) \subseteq \lambda_c Int(A \cup B)$.
(10) $\lambda_c Int(A \cap B) \subseteq \lambda_c Int(A) \cap \lambda_c Int(B)$.

Proof Obvious. ■

In general the equalities of (9) and (10) of the above proposition is not true, as it is shown in the following examples:

Example 3.23 Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{\text{a, c}\}, X\}$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $b \in A$ and $\lambda(A) = Cl(A)$ if $b \notin A$. Now, let $A = \{a\}$ and $B = \{c\}$, then $\lambda_c Int(A) = \phi$, and $\lambda_c Int(B) = \phi$, but $\lambda_c Int(A \cup B) = \{a, c\}$. Thus $\lambda_c Int(A \cup B) \neq \lambda_c Int(A) \cup \lambda_c Int(B)$.

Example 3.24 Let $X = \{a, b, c\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \phi$, $A = \{c\}$, $\{a, b\}$ or $\{a, c\}$ and $\lambda(A) = X$ otherwise. Now, if $A = \{a, b\}$ and $B = \{a, c\}$, then $\lambda_c Int(A) = \{a, b\}$ and $\lambda_c Int(B) = \{a, c\}$, but $\lambda_c Int(A \cap B) = \phi$. Hence, $\lambda_c Int(A \cap B) \neq \lambda_c Int(A) \cap \lambda_c Int(B)$.

Proposition 3.25 If A is a subset of a space X, then $\lambda_c Int(A) = A \backslash \lambda_c d(X \backslash A)$.

Proof Obvious. ■

Proposition 3.26 If A is any subset of a space X, then the following statements are true:

(1) $X \backslash \lambda_c Int(A) = \lambda_c Cl(X \backslash A)$.
(2) $\lambda_c Cl(A) = X \backslash \lambda_c Int(X \backslash A)$.
(3) $X \backslash \lambda_c Cl(A) = \lambda_c Int(X \backslash A)$.
(4) $\lambda_c Int(A) = X \backslash \lambda_c Cl(X \backslash A)$.

Proof Obvious. ■

Proposition 3.27 If A is a subset of a topological space (X, τ), then $\lambda_c Int(A) \subseteq {}_sInt(A)$.

Proof Obvious. ■

The equality in the above proposition need not be true in general, as shown by the following example:

Example 3.28 Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty or $A = \{b\}$ and $\lambda(A) = X$ otherwise. Now, if $A = \{b, c\}$, then $\lambda_c Int(A) = \phi$ and ${}_sInt(A) = \{b, c\}$.

Theorem 3.29 Let A, B be subsets of X. If the s-operation $\lambda : SO(X) \to P(X)$ is s-regular, then we have:

(1) $\lambda_c Cl(A \cup B) = \lambda_c Cl(A) \cup \lambda_c Cl(B)$.
(2) $\lambda_c Int(A \cap B) = \lambda_c Int(A) \cap \lambda_c Int(B)$.

Proof Obvious. ■

4. λ_c-Separation Axioms

In this section, we define new types of separation axioms called λ_c-T_i $(i = 0, 1/2, 1, 2)$ and λ_c-R_j $(j = 0, 1)$ by using the notion of λ_c-open and λ_c-closed sets. First, we begin with the following definition.

Definition 4.1 A subset A of (X, τ) is said to be generalized λ_c-closed (briefly, g-λ_c-closed) if $\lambda_c Cl(A) \subseteq U$, whenever $A \subseteq U$ and U is a λ_c-open set in (X, τ).

We say that a subset B of X is generalized λ_c-open (briefly, g-λ_c-open) if its complement $X \backslash B$ is generalized λ_c-closed in (X, τ).

Theorem 4.2 If a subset A of X is g-λ_c-closed and $A \subseteq B \subseteq \lambda_c Cl(A)$, then B is a g-λ_c-closed set in X.

Proof Let A be g-λ_c-closed set such that $A \subseteq B \subseteq \lambda_c Cl(A)$. Let U be a λ_c-open set of X such that $B \subseteq U$. Since A is g-λ_c-closed, we have λ_cCl(A) $\subseteq U$. Now $\lambda_c Cl(A) \subseteq \lambda_c Cl(B) \subseteq \lambda_c Cl(\lambda_c Cl(A)) = \lambda_c Cl(A) \subseteq U$. This implies that $\lambda_c Cl(B) \subseteq U$, where U is λ_c-open. Therefore, B is a g-λ_c-closed set in X. ■

In the following example, we have two g-λ_c-closed sets A and B such that $A \subseteq B$ but $B \not\subseteq \lambda_c Cl(A)$.

Example 4.3 Let $X = \{a, b, c\}$, and $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Let $\lambda : SO(X) \to P(X)$ be identity s-operation. If $A = \{a\}$ and $B = \{a, b\}$, then A and B are g-λ_c-closed sets in (X, τ). But $A \subseteq B \not\subseteq \lambda_c Cl(A)$.

Theorem 4.4 Let $\lambda : SO(X) \to P(X)$ be an s-operation, then for each singleton set $\{x\}$ is λ_c-closed or $X \backslash \{x\}$ is g-λ_c-closed in (X, τ).

Proof Suppose that $\{x\}$ is not λ_c-closed, then $X \backslash \{x\}$ is not λ_c-open. Let U be any λ_c-open set such that $X \backslash \{x\} \subseteq U$, then $U = X$. Therefore $\lambda_c Cl(X \backslash \{x\}) \subseteq U$. Hence $X \backslash \{x\}$ is g-λ_c-closed. ■

Proposition 4.5 A subset A of (X, τ) is g-λ_c-closed if and only if $\lambda_c Cl(\{x\}) \cap A \neq \phi$, for every $x \in \lambda_c Cl(A)$.

Proof Let U be aλ_c-open set such that $A \subseteq U$ and let $x \in \lambda_c Cl(A)$. By assumption, there exists a $z \in \lambda_c Cl(\{x\})$ and $z \in A \subseteq U$. It follows From Proposition 3.18, that $U \cap \{x\} \neq \phi$, hence $x \in U$, implies $\lambda_c Cl(A) \subseteq U$. Therefore A is g-λ_c-closed.

Conversely, suppose that $x \in \lambda_c Cl(A)$ such that $\lambda_c Cl(\{x\}) \cap A = \phi$. Since $A \subseteq X \backslash \lambda_c Cl(\{x\})$ and A is g-λ_c-closed implies that $\lambda_c Cl(A) \subseteq X \backslash \lambda_c Cl(\{x\})$ holds, and hence $x \notin \lambda_c Cl(A)$, which is contradiction. Therefore $\lambda_c Cl(\{x\}) \cap A \neq \phi$. ■

Theorem 4.6 If a subset A of X is g-closed set in X, then $\lambda_c Cl(A) \backslash A$ does not contain any non empty λ_c-closed set in X.

Proof Let A be a g-λ_c-closed set in X. Let F be a λ_c-closed set such that $F \subseteq \lambda_c Cl(A) \backslash A$ and $F \neq \phi$. Then $F \subseteq X \backslash A$ which implies that $A \subseteq X \backslash F$. Since A is g-λ_c-closed and $X \backslash F$ is a λ_c-open set, therefore $\lambda_c Cl(A) \subseteq X \backslash F$, so $F \subseteq X \backslash \lambda_c Cl(A)$. Hence $F \subseteq \lambda_c Cl(A) \cap X \backslash \lambda_c Cl(A) = \phi$. This shows that, $F = \phi$ which is a contradiction. Hence $\lambda_c Cl(A) \backslash A$ does not contains any non empty λ_c-closed set in X. ■

Definition 4.7 Let (X, τ) be a topological space, then (X, τ) is said to be:

(1) a λ_c-T_0 space if for each distinct points $x, y \in X$ there exists a λ_c-open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.
(2) a λ_c-$T_{1/2}$ space if every g-λ_c-closed set in (X, τ) is λ_c-closed.
(3) a λ_c-T_1 space if for each distinct points $x, y \in X$, there exists a λ_c-open set, containing and respectively such that $y \notin U$ and $x \notin V$.

(4) a λ_c-T_2 space if for each $x, y \in X$ there exists a λ_c-open sets U, V such that $x \in U$ and $y \in V$ and $U \cap V \neq \phi$.

Example 4.8 Let $X = \{a, b, c\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if $A = \{a\}$ and $\lambda(A) = X$ otherwise. Then the space X is a semi-T_0 but it is not λ_c-T_0 space. Moreover a space is semi-T_i, for $i = 0, 1/2, 1, 2$.

Theorem 4.9 A space X is λ_c-T_0 if and only if for each distinct points x and y in X, either $x \notin \lambda_c Cl(\{y\})$ or $y \notin \lambda_c Cl(\{x\})$, .

Proof Let $x \neq y$ in a λ_c-T_0 space X. Then there exists an λ_c-open set U containing one of them but not the other, without loss of generality, we assume that U contains x but not y. Then $U \cap \{y\} = \phi$, this implies that $x \notin \lambda_c Cl(\{y\})$.

Conversely, Let x and y be two distinct points of X, then by hypothesis, either $x \notin \lambda_c Cl(\{y\})$ or $y \notin \lambda_c Cl(\{x\})$. With out loss of generality, we assume that $y \notin \lambda_c Cl(\{x\})$. Then $X \backslash \lambda_c Cl(\{x\})$ is an λ_c-open subset of X containing y but not x. Therefore, X is λ_c-T_0. ■

Theorem 4.10 Let $\lambda : SO(X) \to P(X)$ be an s-operation, then the following statements are equivalent:

(1) (X, τ) is λ_c-$T_{1/2}$.
(2) Each singleton $\{x\}$ of X is either λ_c-closed or λ_c-open.

Proof (1) $\Rightarrow$ (2) : Suppose that $\{x\}$ is not λ_c-closed. Then by Theorem 4.4, $X \backslash \{x\}$ is g-λ_c-closed. Since (X, τ) is λ_c-$T_{1/2}$, then $X \backslash \{x\}$ is λ_c-closed. Hence, $\{x\}$ is λ_c-open.

(2) $\Rightarrow$ (1) : Let A be any g-λ_c-closed set in (X, τ) and $x \in \lambda_c Cl(A)$. By (2), we have $\{x\}$ is λ_c-closed or λ_c-open. If $\{x\}$ is λ_c-closed, then $x \notin A$ will imply $x \in \lambda_c Cl(A) \backslash A$ which is not true by Theorem 4.6, so $x \in A$. Therefore, $\lambda_c Cl(A) = A$, so A is λ_c- closed. Therefore, (X, τ) is λ_c-$T_{1/2}$.

On the other hand, if $\{x\}$ is λ_c-open, then as $x \in \lambda_c Cl(A)$, we have $\{x\} \cap A \neq \phi$. Hence $x \in A$, so A is λ_c-closed. ■

Corollary 4.1 Each λ_c-$T_{1/2}$ space is λ_c-T_0 space.

Proof Follows from Theorem 4.10 and Theorem 4.9. ■

Example 4.11 Let $X = \{a, b, c\}$ and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty, $A = \{a\}$ or $\{a, b\}$ and $\lambda(A) = X$ otherwise. Then (X, τ) is a λ_c-T_0 space but not λ_c-$T_{1/2}$ space because $\{a, c\}$ is g-λ_c- closed but not λ_c- closed.

Theorem 4.12 Each λ_c-T_1 space is λ_c-$T_{1/2}$ space.

Proof Follows from Theorem 4.6. ■

Example 4.13 $X = \{a, b\}$ and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty or $A = \{a\}$ and $\lambda(A) = X$ otherwise. Then (X, τ) is a λ_c-$T_{1/2}$ space but not λ_c-T_1 space.

Definition 4.14 A topological space (X, τ) is called a λ_c-symmetric space if for x and y in X, $x \in \lambda_c Cl(\{y\})$ implies that $y \in \lambda_c Cl(\{x\})$.

Theorem 4.15 Let (X, τ) be a λ_c-symmetric space, then the following are equivalent:

(1) (X, τ) is λ_c-T_0.
(2) (X, τ) is λ_c-$T_{1/2}$.
(3) (X, τ) is λ_c-T_1.

Proof It is enough to prove only the necessity of (1) $\Leftrightarrow$ (2). Let $x \neq y$ and since (X, τ) is λ_c-T_0, we may assume that $x \in U \subseteq X \backslash \{y\}$ for some $U \in SO_{\lambda c}(X)$. Then $x \notin \lambda_c Cl(\{y\})$ and hence $y \notin \lambda_c Cl(\{x\})$. Therefore, there exists $V \in SO_{\lambda c}(X)$ such that $y \in V \subseteq X \backslash \{x\}$ and (X, τ) is a λ_c-T_1 space. ■

Example 4.16 Let $X = \{a, b, c\}$ and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty, $A = \{a, b\}$, $\{a, c\}$ or $\{b, c\}$ and $\lambda(A) = X$ otherwise. Clearly (X, τ) is λ_c-T_1 space, but it is not λ_c-T_2.

Remark 4.17 From the definitions of λ_c-T_i, semi-T_i $(i = 0, 1/2, 1, 2)$ and previous results, we get the following diagram of implications:

$$\lambda_c\text{-T}_2 \longrightarrow \lambda_c\text{-T}_1 \longrightarrow \lambda_c\text{-T}_{1/2} \longrightarrow \lambda_c\text{-}T_0$$

$$\lambda_c\text{-}T_i \longrightarrow \text{semi-}T_i \text{ for } (i = 0, 1/2, 1, 2).$$

Definition 4.18 Let $\lambda : SO(X) \to P(X)$ be an s-operation, a topological space (X, τ) is called λ_c-R_0 if $U \in SO_{\lambda c}(X)$ and $x \in U$, then $\lambda_c Cl(\{x\}) \subseteq U$.

Theorem 4.19 For any topological space X and any s-operation λ, the following are equivalent:

(1) X is λ_c-R_0.
(2) $F \in SC_{\lambda c}(X)$ and $x \notin F$ implies $F \subseteq U$ and $x \notin U$ for some $U \in SO_{\lambda c}(X)$.
(3) $F \in SC_{\lambda c}(X)$ and $x \notin F$ implies $F \cap \lambda_c Cl(\{x\}) = \phi$.
(4) For any two distinct points x, y of X, either $\lambda_c Cl(\{x\}) = \lambda_c Cl(\{y\})$ or $\lambda_c Cl(\{x\}) \cap \lambda_c Cl(\{y\}) = \phi$.

Proof (1) $\Rightarrow$ (2): $F \in SC_{\lambda c}(X)$ and $x \notin F$ implies $x \in X \backslash F \in SO_{\lambda c}(X)$ then $\lambda_c Cl(\{x\}) \subseteq X \backslash F$. By (1), if we put $U = X \backslash \lambda_c Cl(\{x\})$, then $x \notin U \in SO_{\lambda c}(X)$ and $F \subseteq U$.

(2) $\Rightarrow$ (3) : if $F \in SC_{\lambda c}(X)$ and $x \notin F$, then there exists $U \in SO_{\lambda c}(X)$ such that $x \notin U$ and $F \subseteq U$. By (2), we have $U \cap \lambda_c Cl(\{x\}) = \phi$, so $U \cap \lambda_c Cl(\{x\}) = \phi$.

(3) $\Rightarrow$ (4) : Suppose that for any two distinct points x, y of X, $\lambda_c Cl(\{x\}) \neq \lambda_c Cl(\{y\})$. Then suppose, without loss of generality, that there exists some $z \in \lambda_c Cl(\{x\})$ such that $z \notin \lambda_c Cl(\{y\})$. Thus there exists a λ_c-open set V such that $z \in V$ and $y \notin V$ but $x \in V$. Thus $x \notin \lambda_c Cl(\{y\})$. Hence by (3), $\lambda_c Cl(\{x\}) \cap \lambda_c Cl(\{y\}) = \phi$.

(4) $\Rightarrow$ (1): Let $U \in SO_{\lambda c}(X)$ and $x \in U$. Then for each $y \notin U$, $x \notin \lambda_c Cl(\{y\})$. Thus $\lambda_c Cl(\{x\}) \neq \lambda_c Cl(\{y\})$. Hence by (4), $\lambda_c Cl(\{x\}) \cap \lambda_c Cl(\{y\}) = \phi$, for each $y \in X \backslash U$. So $\lambda_c Cl(\{x\}) \cap [\cup \{\lambda_c Cl(\{y\}) : \cup\{\lambda_c Cl(\{y\}) : y \in X \backslash U\}] = \phi$. Now, $U \in SO_{\lambda c}(X)$ and $y \in X \backslash U$, then $\{y\} \subseteq \lambda_c Cl(\{y\}) \subseteq \lambda_c Cl(X \backslash U) = X \backslash U$. Thus $X \backslash U = \bigcup \{\lambda_c Cl(\{y\}) : y \in X \backslash U\}$. Hence, $\lambda_c Cl(\{x\}) \cap X \backslash U = \phi$, so $\lambda_c Cl(\{x\}) \subseteq U$. This implies that (X, τ) is λ_c-R_0. ■

Theorem 4.20 Let (X, τ) be a topological space and $\lambda : SO(X) \to P(X)$ be any s-operation, then the following are equivalent:

(1) X is λ_c-T_1.
(2) $\lambda_c Cl(\{x\}) = \{x\}$ for all $x \in X$.
(3) X is λ_c-R_0 and λ_c-T_0.

Proof (1) $\Rightarrow$ (2) : Let $y \notin \{x\}$, then there exists $U \in SO_{\lambda c}(X)$ such that $y \in U$, $x \notin U$, so $U \cap \{x\} = \phi$. Hence $y \notin \lambda_c Cl(\{x\})$ implies $\lambda_c Cl(\{x\}) \subseteq \{x\}$ also $\{x\} \subseteq \lambda_c Cl(\{x\})$ always, hence $\lambda_c Cl(\{x\}) = \{x\}$ for all $x \in X$.

(2) $\Rightarrow$ (3) : Let x, $y \in X$ with $x \neq y$. Then $\{x\}$ and $\{y\}$ are λ_c-closed and hence $X \backslash \{x\}$ is a λ_c-open set containing y but not x. This shows that X is λ_c-T_0. Again, x, $y \in X$ with $x \neq y$, then $\lambda_c Cl(\{x\}) \neq \lambda_c Cl(\{y\})$. Also, $\lambda_c Cl(\{x\}) \cap \lambda_c Cl(\{y\}) = \phi$. Thus, by Theorem 4.19, X is λ_c-R_0.

(3) $\Rightarrow$ (1) : Let x, $y \in X$ with $x \neq y$. there exists $U \in SO_{\lambda c}(X)$ such that $x \in U$ and $y \notin U$ then, $\lambda_c Cl(\{x\}) \subseteq U$ (as X is λ_c-R_0) and so $y \notin \lambda_c Cl(\{x\})$. Hence $x \in U \in SO_{\lambda c}(X)$, $y \notin U$ and $y \in X \backslash \lambda_c Cl(\{x\}) \in SO_{\lambda c}(X)$, $x \notin X \backslash \lambda_c Cl(\{x\})$. Therefore, X is a λ_c-T_1 space. ■

Definition 4.21 Let (X, τ) be a topological space $\lambda : SO(X) \to P(X)$ be an s-operation. The space X is said to be λ_c-R_1 if for x, $y \in X$ with $\lambda_c Cl(\{x\}) \neq \lambda_c Cl(\{y\})$, there exist disjoint λ_c-open sets U and V such that $\lambda_c Cl(\{x\}) \subseteq U$ and $\lambda_c Cl(\{y\}) \subseteq V$.

Theorem 4.22 If $\lambda : SO(X) \to P(X)$ is an s-operation and X is λ_c-R_1, then X is λ_c-R_0.

Proof Let $U \in SO_{\lambda c}(X)$ and $x \in U$. If $y \notin U$, then $\lambda_c Cl(\{x\}) \neq \lambda_c Cl(\{y\})$ (as$x \notin \lambda_c Cl(\{y\})$). Hence there exists $V \in SO_{\lambda c}(X)$ such that $\lambda_c Cl(\{y\}) \subseteq V$ and $x \notin V$. This gives that $y \notin \lambda_c Cl(\{x\})$, so $\lambda_c Cl(\{x\}) \subseteq U$. Hence, X is a λ_c-R_0 space. ■

By the following examples, we show the converse of above theorem is not true in general, and also we show λ_c-R_0 and semi-R_0 are independent.

Example 4.23 Let $X = \{a, b\}$ and $\tau = \{\phi, \{a\}, X\}$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty and $\lambda(A) = X$ otherwise. Clearly X is λ_c-R_0, but it is neither semi-R_0 nor semi-R_1.

Example 4.24 Let $X = \{a, b\}$, and $\tau = P(X)$. We define an s-operation $\lambda : SO(X) \to P(X)$ as $\lambda(A) = A$ if A is empty or $A = \{a\}$ and $\lambda(A) = X$ otherwise. Clearly X is semi-R_0 and semi-R_1, but it is not λ_c-R_0.

Theorem 4.25 Let (X, τ) be a topological space $\lambda : SO(X) \to P(X)$ be an s-operation. Then the following are equivalent:

(1) X is λ_c-T_2.
(2) X is λ_c-R_1 and λ_c-T_1.
(3) X is λ_c-R_1 and λ_c-T_0.

Proof (1) $\Rightarrow$ (2) : Let X be λ_c-T_2, then X is clearly λ_c-T_1. Now if $x,\ y \in X$ with $\lambda_c Cl(\{x\}) \neq \lambda_c Cl(\{y\})$ then $x \neq y$, so there exist $U,\ V \in SO_{\lambda c}(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Hence by Theorem 4.20, $\lambda_c Cl(\{x\}) = \{x\} \subseteq U$ and $\lambda_c Cl(\{y\}) = \{y\} \subseteq V$ and $U \cap V = \phi$. Therefore, X is λ_c-R_1.

(2) $\Rightarrow$ (3) : It is obvious.

(3) $\Rightarrow$ (1) : Let X be λ_c-R_1 and λ_c-T_0, then by Theorem 4.22, X is λ_c-R_0 and λ_c-T_0. Hence, by Theorem 4.20, X is λ_c-T_1 . If $x, y \in X$ with $x \neq y$, then $\lambda_c Cl(\{x\}) = \{x\} \neq \{y\} = \lambda_c Cl(\{y\})$. Since X is λ_c-R_1, so there exist $U, V \in SO_{\lambda c}(X)$ such that $\lambda_c Cl(\{x\}) = \{x\} \subseteq U, \lambda_c Cl(\{y\}) = \{y\} \subseteq V$ and $U \cap V = \phi$. Hence, X is λ_c-T_2. ■

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