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RESEARCH ARTICLE

Operation on Regular Spaces

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In this paper, we introduce the concept of R_γ-open sets as a strong of γ-open sets in a topological space (X, τ). Using this set, we introduce R_γ-T_0, R_γ-$T_{\frac{1}{2}}$, R_γ-T_1, R_γ-T_2, R_γ-T_3, R_γ-T_4, R_γ-D_0, R_γ-D_1 and R_γ-D_2 spaces and study some of its properties. Finally we introduce $R_{(\gamma,\gamma')}$-continuous mappings and give some properties of such mappings.

Keywords: R_γ-open set; R_γ-g.closed set.

AMS Subject Classification: 54A40.

1. Introduction and Preliminaries

Kasahara [1] defined the concept of an operation on topological spaces and introduce the concept of α closed graphs of an operation. Ogata [2] called the operation α (respectively α closed set) as γ-operation (respectively γ-closed set) and introduced the notion of τ_γ which is the collection of all γ-open sets in a topological space. Also he introduced the concept of γ-T_i (i = 0, $\frac{1}{2}$, 1, 2) and characterized γ-T_i using the notion of γ-closed and γ-open sets.

In this paper, we introduce the concept of R_γ-open sets by using an operation γ on $RO(X, \tau)$ and we introduce the concept of R_γ-generalized closed sets and R_γ-$T_{\frac{1}{2}}$ spaces and characterize R_γ-$T_{\frac{1}{2}}$ spaces using the notion of R_γ-closed or R_γ-open sets. Also, we show that some basic properties of $R_\gamma T_i$, $R_\gamma D_i$ for $i = 0, 1, 2$ spaces and we introduce $R_{(\gamma,\gamma')}$-continuous mappings and study some of its properties. Let (X, τ) be a topological space and A be a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a topological space (X, τ) is said to be regular open [3] if $Int(Cl(A)) = A$. The complement of a regular open set is said to be regular closed [3]. The intersection of all regular closed sets containing A is called the regular closure of A and is denoted by $rCl(A)$. The family of all regular open (resp. regular closed) sets in a topological space (X, τ) is denoted by $RO(X, \tau)$ (resp. $RC(X, \tau)$).

An operation γ [2] on a topology τ is a mapping from τ in to power set $P(X)$ of X such that $V \subseteq \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of γ at V. A subset A of X with an operation γ on τ is called γ-open [2] if for each $x \in A$, there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$. Clearly $\tau_\gamma \subseteq \tau$. Complements of γ-open sets are called γ-closed. The γ-closure [2] of a subset A of X with an operation γ on τ is denoted by τ_γ-$Cl(A)$ and is defined to be the intersection of all γ-closed sets containing A. A topological X with an operation γ on τ is said to be γ-regular [2] if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that $\gamma(U)$ contained in V. It is also to be noted that $\tau = \tau_\gamma$ if and only if X is a γ-regular space [2].

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2. R_γ-open sets

Definition 2.1 Let $\gamma : RO(X,\tau) \to P(X)$ be a mapping satisfying the following property, $V \subseteq \gamma(V)$ for each $V \in RO(X,\tau)$. We call the mapping γ an operation on $RO(X,\tau)$.

Definition 2.2 Let (X,τ) be a topological space and $\gamma : RO(X,\tau) \to P(X)$ an operation on $RO(X,\tau)$. A nonempty set A of X is called a R_γ-open set of (X,τ) if for each point $x \in A$, there exists a regular open set U containing x such that $\gamma(U) \subseteq A$. The complement of a R_γ-open set is called R_γ-closed in (X,τ). We suppose that the empty set is R_γ-open for any operation $\gamma : RO(X,\tau) \to P(X)$. We denote the set of all R_γ-open (resp. R_γ-closed) sets of (X,τ) by $RO(X,\tau)_\gamma$ (resp. $RC(X,\tau)_\gamma$).

Remark 2.3 A subset A is a R_{id}-open set of (X,τ) if and only if A is regular open in (X,τ). The operation $id : RO(X,\tau) \to P(X)$ is defined by $id(V) = V$ for any set $V \in RO(X,\tau)$; this operation is called the identity operation on $RO(X,\tau)$. Therefore, we have that $RO(X,\tau)_{id} = RO(X,\tau)$.

Remark 2.4 It is clear from the definition that every R_γ-open subset of a space X is regular open, but the converse need not be true in general as shown in the following example.

Example 2.5 Consider $X = \{a,b,c\}$ with the discrete topology on X. Define an operation γ on $RO(X,\tau)$ by

$$\gamma(A) = \begin{cases} A & \text{if } b \in A, \\ X & \text{if } b \notin A. \end{cases}$$

Then $RO(X,\tau)_\gamma = \{\phi, \{b\}, \{a,b\}, \{b,c\}, X\}$ and $\{a\} \in RO(X,\tau)$, but $\{a\} \notin RO(X,\tau)_\gamma$.

Theorem 2.6 If A is a R_γ-open set in $(X.\tau)$, then A is a γ-open set.

Proof Follows from that every regular open set is open. ■

The converse of the above theorem need not be true in general as it is shown below.

Example 2.7 Consider $X = \{a,b,c\}$ with the topology $\tau = \{\phi, \{a\}, X\}$. Define an operation γ on $RO(X,\tau)$ by $\gamma(A) = A$. Then $\{a\}$ is a γ-open set but not a R_γ-open set.

Remark 2.8 Every R_γ-open set is open.

Theorem 2.9 Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of R_γ-open sets in a topological space (X,τ), then $\cup_{\alpha\in J} A_\alpha$ is R_γ-open.

Proof Let $x \in \cup_{\alpha\in J} A_\alpha$, then $x \in A_\alpha$ for some $\alpha \in J$. Since A_α is a R_γ-open set, implies that there exists a regular open set U containing x such that $\gamma(U) \subseteq A_\alpha \subseteq \cup_{\alpha\in J} A_\alpha$. Therefore $\cup_{\alpha\in J} A_\alpha$ is a R_γ-open set of (X,τ). ■

If A and B are two R_γ-open sets in (X,τ), then the following example shows that $A \cap B$ need not be R_γ-open.

Example 2.10 Consider $X = \{a,b,c\}$ with the discrete topology on X. Define an operation γ on $RO(X,\tau)$ by

$$\gamma(A) = \begin{cases} \{a,b\} & \text{if } A = \{a\} \text{ or } \{b\}, \\ A & \text{otherwise.} \end{cases}$$

Then $A = \{a,b\}$ and $B = \{a,c\}$ are R_γ-open sets but $A \cap B = \{a\}$ is not a R_γ-open set.

From the above example we notice that the family of all R_γ-open subsets of a space X is a supratopology and need not be a topology in general.

Proposition 2.11 The set A is R_γ-open in the space (X,τ) if and only if for each $x \in A$, there exists a R_γ-open set B such that $x \in B \subseteq A$.

Proof Suppose that A is R_γ-open set in the space (X,τ). Then for each $x \in A$, put $B = A$ is a R_γ-open set such that $x \in B \subseteq A$.

Conversely, suppose that for each $x \in A$, there exists a R_γ-open set B such that $x \in B \subseteq A$, thus $A = \cup B_x$ where $B_x \in RO(X,\tau)_\gamma$ for each x. Therefore, A is a R_γ-open set. ■

Definition 2.12 An operation γ on $RO(X,\tau)$ is said to be regular regular if for every regular open sets U and V of each $x \in X$, there exists a regular open set W of x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

Definition 2.13 An operation γ on $RO(X,\tau)$ is said to be regular open if for every regular open set U of each $x \in X$, there exists a R_γ-open set V such that $x \in V$ and $V \subseteq \gamma(U)$.

In the following proposition the intersection of two R_γ-open sets is also a R_γ-open set.

Proposition 2.14 Let γ be a regular regular operation on $RO(X,\tau)$. If A and B are R_γ-open sets in X, then $A \cap B$ is also a R_γ-open set.

Proof Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since A and B are R_γ-open sets, there exist regular open sets U and V such that $x \in U$ and $\gamma(U) \subseteq A$, $x \in V$ and $\gamma(V) \subseteq B$. Since γ is a regular regular operation, then there exists a regular open set W of x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V) \subseteq A \cap B$. This implies that $A \cap B$ is R_γ-open set. ■

Remark 2.15 By the above propositon, if γ is a regular regular operation on $RO(X,\tau)$. Then $RO(X,\tau)_\gamma$ form a topology on X.

Definition 2.16 A point $x \in X$ is in rCl_γ-closure of a set $A \subseteq X$, if $\gamma(U) \cap A \neq \phi$ for each regular open set U containing x. The rCl_γ-closure of A is denoted by $rCl_\gamma(A)$.

Definition 2.17 Let A be a subset of a topological space (X,τ) and γ an operation on $RO(X,\tau)$. The union of all R_γ-open sets contained in A is called the R_γ-interior of A and denoted by $R_\gamma Int(A)$.

Definition 2.18 Let A be a subset of (X,τ), and $\gamma : RO(X,\tau) \to P(X)$ be an operation on $RO(X,\tau)$. Then the R_γ-closure of A is denoted by $R_\gamma Cl(A)$ and defined as follows, $R_\gamma Cl(A) = \bigcap\{F : F$ is R_γ-closed and $A \subseteq F\}$.

The proof of the following theorem is obvious and hence omitted.

Theorem 2.19 Let (X,τ) be a topological space and γ be an operation on $RO(X,\tau)$. For any subsets A, B of X, we have the following properties:

(1) $A \subseteq R_\gamma Cl(A)$.
(2) $R_\gamma Cl(A)$ is R_γ-closed set in X.
(3) A is R_γ-closed set if and only if $A = R_\gamma Cl(A)$.
(4) $R_\gamma Cl(\phi) = \phi$ and $R_\gamma Cl(X) = X$.
(5) If $A \subseteq B$, then $R_\gamma Cl(A) \subseteq R_\gamma Cl(B)$.
(6) $R_\gamma Cl(A \cup B) \supseteq R_\gamma Cl(A) \cup R_\gamma Cl(B)$.
(7) $R_\gamma Cl(A \cap B) \subseteq R_\gamma Cl(A) \cap R_\gamma Cl(B)$.

Theorem 2.20 For a point $x \in X$, $x \in R_\gamma Cl(A)$ if and only if for every R_γ-open set V of X containing x such that $A \cap V \neq \phi$.

Proof Let $x \in R_\gamma Cl(A)$ and suppose that $V \cap A = \phi$ for some R_γ-open set V which contains x. Then $(X \setminus V)$ is R_γ-closed and $A \subseteq (X \setminus V)$, thus $R_\gamma Cl(A) \subseteq (X \setminus V)$. But this implies that $x \in (X \setminus V)$, a contradiction. Therefore $V \cap A \neq \phi$

Conversely, Let $A \subseteq X$ and $x \in X$ such that for each R_γ-open set U which contains x, $U \cap A \neq \phi$. If $x \notin R_\gamma Cl(A)$, there is a R_γ-closed set F such that $A \subseteq F$ and $x \notin F$. Then $(X \setminus F)$ is a R_γ-open set with $x \in (X \setminus F)$, and thus $(X \setminus F) \cap A \neq \phi$, which is a contradiction. ■

The proof of the following theorems are obvious and hence omitted.

Theorem 2.21 Let A be any subset of a topological space (X, τ) and γ be an operation on $RO(X, \tau)$. Then the following relation holds.
$A \subseteq \tau_\gamma\text{-}Cl(A) \subseteq rCl_\gamma(A) \subseteq R_\gamma Cl(A)$.

Theorem 2.22 Let A be a subset of a topological space (X, τ) and γ be an operation on $RO(X, \tau)$. Then, the following conditions are equivalent:

(1) A is R_γ-open.
(2) $rCl_\gamma(X \setminus A) = X \setminus A$.
(3) $R_\gamma Cl(X \setminus A) = X \setminus A$.
(4) $X \setminus A$ is R_γ-closed.

Theorem 2.23 Let $\gamma : RO(X, \tau) \to P(X)$ be an operation on $RO(X, \tau)$ and A be a subset of X, then

(1) A subset $rCl_\gamma(A)$ is a regular closed set in (X, τ).
(2) If γ is regular open, then $rCl_\gamma(A) = R_\gamma Cl(A)$, and $rCl_\gamma(rCl_\gamma(A)) = rCl_\gamma(A)$, and $rCl_\gamma(A)$ is R_γ-closed.

Proof

(1) To prove that $rCl_\gamma(A)$ is regular closed. Let $x \in rCl(rCl_\gamma(A))$, then $U \cap rCl_\gamma(A) \neq \phi$ for every regular open set U of x. Let $y \in U \cap rCl_\gamma(A)$, $y \in U$ and $y \in rCl_\gamma(A)$. Since U is regular open set containing y, implies $\gamma(U) \cap A \neq \phi$. Therefore $x \in rCl_\gamma(A)$. Hence $rCl(rCl_\gamma(A)) \subseteq rCl_\gamma(A)$. This implies $rCl_\gamma(A)$ is a regular closed set.
(2) By Theorem 2.21, we have $rCl_\gamma(A) \subseteq R_\gamma Cl(A)$. Now to prove that $R_\gamma Cl(A) \subseteq rCl_\gamma(A)$. Let $x \notin rCl_\gamma(A)$, then there exists a regular open set U such that $\gamma(U) \cap A = \phi$. Since γ is regular open, there exists a R_γ-open set V such that $x \in V \subseteq \gamma(U)$. Therefore $V \cap A = \phi$. This implies $x \notin R_\gamma Cl(A)$. Hence $R_\gamma Cl(A) \subseteq rCl_\gamma(A)$. Therefore $rCl_\gamma(A) = R_\gamma Cl(A)$. Now, $rCl_\gamma(rCl_\gamma(A)) = R_\gamma Cl(R_\gamma Cl(A)) = R_\gamma Cl(A) = rCl_\gamma(A)$.

■

Definition 2.24 A subset A of the space (X, τ) is said to be R_γ-generalized closed (briefly. R_γ-g.closed) if $R_\gamma Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a R_γ-open set in (X, τ). The complement of a R_γ-g.closed set is called a R_γ-g.open set.

It is clear that every R_γ-closed subset of X is also a R_γ-g.closed set. The following example shows that a R_γ-g.closed set need not be R_γ-closed.

Example 2.25 Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Define an operation γ on $RO(X, \tau)$ by

$$\gamma(A) = \begin{cases} A & \text{if } A = \{b\} \text{ or } \{a, c\}, \\ X & \text{otherwise.} \end{cases}$$

Now, if we let $A = \{a\}$, since the only R_γ-open supersets of A are $\{a, c\}$ and X, then A is R_γ-g.closed. But it is easy to see that A is not R_γ-closed.

Theorem 2.26 A subset A of (X, τ) is R_γ-g.closed if and only if $R_\gamma Cl(\{x\})$
$\cap A \neq \phi$, holds for every $x \in R_\gamma Cl(A)$.

Proof Let U be a R_γ-open set such that $A \subseteq U$ and let $x \in R_\gamma Cl(A)$. By assumption, there exists a $z \in R_\gamma Cl(\{x\})$ and $z \in A \subseteq U$. It follows from Theorem 2.20, that $U \cap \{x\} \neq \phi$, hence $x \in U$, this implies $R_\gamma Cl(A) \subseteq U$. Therefore A is R_γ-g.closed.

Conversely, suppose that $x \in R_\gamma Cl(A)$ such that $R_\gamma Cl(\{x\}) \cap A = \phi$. Since, $R_\gamma Cl(\{x\})$ is R_γ-closed, therefore $X \setminus R_\gamma Cl(\{x\})$ is a R_γ-open set in X. Since $A \subseteq X \setminus (R_\gamma Cl(\{x\}))$ and A is R_γ-g.closed implies that $R_\gamma Cl(A) \subseteq X \setminus R_\gamma Cl(\{x\})$ holds, and hence $x \notin R_\gamma Cl(A)$. This is a contradiction. Therefore $R_\gamma Cl(\{x\}) \cap A \neq \phi$. ■

Theorem 2.27 A set A of a space X is R_γ-g.closed if and only if $R_\gamma Cl(A) \setminus A$ does not contain any non-empty R_γ-closed set.

Proof Necessity: Suppose that A is R_γ-g.closed set in X. We prove the result by contradiction. Let F be a R_γ-closed set such that $F \subseteq R_\gamma Cl(A) \setminus A$ and $F \neq \phi$. Then $F \subseteq X \setminus A$ which implies $A \subseteq X \setminus F$. Since A is R_γ-g.closed and $X \setminus F$ is R_γ-open, therefore $R_\gamma Cl(A) \subseteq X \setminus F$, that is $F \subseteq X \setminus R_\gamma Cl(A)$. Hence $F \subseteq R_\gamma Cl(A) \cap (X \setminus R_\gamma Cl(A)) = \phi$. This shows that, $F = \phi$ which is a contradiction. Hence $R_\gamma Cl(A) \setminus A$ does not contains any non-empty R_γ-closed set in X.

Sufficiency: Let $A \subseteq U$, where U is R_γ-open in (X, τ). If $R_\gamma Cl(A)$ is not contained in U, then $R_\gamma Cl(A) \cap X \setminus U \neq \phi$. Now, since $R_\gamma Cl(A) \cap X \setminus U \subseteq R_\gamma Cl(A) \setminus A$ and $R_\gamma Cl(A) \cap X \setminus U$ is a non-empty R_γ-closed set, then we obtain a contradiction and therefore A is R_γ-g.closed. ■

Theorem 2.28 If A is a R_γ-g.closed set of a space X, then the following are equivalent:

(1) A is R_γ-closed.
(2) $R_\gamma Cl(A) \setminus A$ is R_γ-closed.

Proof (1) $\Rightarrow$ (2): If A is a R_γ-g.closed set which is also R_γ-closed, then by Theorem 2.27, $R_\gamma Cl(A) \setminus A = \phi$ which is R_γ-closed.

(2) $\Rightarrow$ (1): Let $R_\gamma Cl(A) \setminus A$ be R_γ-closed set and A be R_γ-g.closed. Then by Theorem 2.27, $R_\gamma Cl(A) \setminus A$ does not contain any non-empty R_γ-closed subset. Since $R_\gamma Cl(A) \setminus A$ is R_γ-closed and $R_\gamma Cl(A) \setminus A = \phi$, this shows that A is R_γ-closed. ■

Theorem 2.29 For a space (X, τ), the following are equivalent:

(1) Every subset of X is R_γ-g.closed.
(2) $RO(X, \tau)_\gamma = RC(X, \tau)_\gamma$.

Proof (1) $\Rightarrow$ (2): Let $U \in RO(X, \tau)_\gamma$. Then by hypothesis, U is R_γ-g.closed which implies that $R_\gamma Cl(U) \subseteq U$, so, $R_\gamma Cl(U) = U$, therefore $U \in RC(X, \tau)_\gamma$. Also let $V \in RC(X, \tau)_\gamma$. Then $X \setminus V \in RO(X, \tau)_\gamma$, hence by hypothesis $X \setminus V$ is R_γ-g.closed and then $X \setminus V \in RC(X, \tau)_\gamma$, thus $V \in RO(X, \tau)_\gamma$ according above we have $RO(X, \tau)_\gamma = RC(X, \tau)_\gamma$.

(2) $\Rightarrow$ (1): If A is a subset of a space X such that $A \subseteq U$ where $U \in RO(X, \tau)_\gamma$, then $U \in RC(X, \tau)_\gamma$ and therefore $R_\gamma Cl(U) \subseteq U$ which shows that A is R_γ-g.closed. ■

Proposition 2.30 If A is R_γ-open and R_γ-g.closed then A is R_γ-closed.

Proof Suppose that A is R_γ-open, R_γ-g.closed and $A \subseteq A$, we have $R_\gamma Cl(A) \subseteq A$, also $A \subseteq R_\gamma Cl(A)$, therefore $R_\gamma Cl(A) = A$. That is A is R_γ-closed. ■

Theorem 2.31 If a subset A of X is R_γ-g.closed and $A \subseteq B \subseteq R_\gamma Cl(A)$, then B is a R_γ-g.closed set in X.

Proof Let A be R_γ-g.closed set such that $A \subseteq B \subseteq R_\gamma Cl(A)$. Let U be a R_γ-open set of X such that $B \subseteq U$. Since A is R_γ-g.closed, we have $R_\gamma Cl(A) \subseteq U$. Now $R_\gamma Cl(A) \subseteq R_\gamma Cl(B) \subseteq R_\gamma Cl[R_\gamma Cl(A)] = R_\gamma Cl(A) \subseteq U$. That is $R_\gamma Cl(B) \subseteq U$, where U is R_γ-open. Therefore B is a R_γ-g.closed set in X. ■

Proposition 2.32 Let γ be an operation on $RO(X, \tau)$. Then for each $x \in X$, $\{x\}$ is R_γ-closed or $X \setminus \{x\}$ is R_γ-g.closed in (X, τ).

Proof Suppose that $\{x\}$ is not R_γ-closed, then $X \setminus \{x\}$ is not R_γ-open. Let U be any R_γ-open set such that $X \setminus \{x\} \subseteq U$, implies $U = X$. Therefore $R_\gamma Cl(X \setminus \{x\}) \subseteq U$. Hence $X \setminus \{x\}$ is R_γ-g.closed. ■

3. R_γ-Separation axioms

Definition 3.1 A space (X, τ) is said to be R_γ-$T_{\frac{1}{2}}$ if every R_γ-g.closed set is R_γ-closed.

Theorem 3.2 The following statements are equivalent for a topological space (X, τ) with an operation γ on $RO(X, \tau)$:

(1) (X, τ) is R_γ-$T_{\frac{1}{2}}$.
(2) Each singleton $\{x\}$ of X is either R_γ-closed or R_γ-open.

Proof (1) $\Rightarrow$ (2): Suppose $\{x\}$ is not R_γ-closed. Then by Proposition 2.32, $X \setminus \{x\}$ is R_γ-g.closed. Now since (X, τ) is R_γ-$T_{\frac{1}{2}}$, $X \setminus \{x\}$ is R_γ-closed, i.e. $\{x\}$ is R_γ-open.

(2) $\Rightarrow$ (1): Let A be any R_γ-g.closed set in (X, τ) and $x \in R_\gamma Cl(A)$. By (2) we have $\{x\}$ is R_γ-closed or R_γ-open. If $\{x\}$ is R_γ-closed then $x \notin A$ will imply $x \in R_\gamma Cl(A) \setminus A$, which is not possible by Theorem 2.27. Hence $x \in A$. Therefore, $R_\gamma Cl(A) = A$, i.e. A is R_γ-closed. So, (X, τ) is R_γ-$T_{\frac{1}{2}}$. On the other hand, if $\{x\}$ is R_γ-open then as $x \in R_\gamma Cl(A)$, $\{x\} \cap A \neq \phi$. Hence $x \in A$. So A is R_γ-closed. ■

Definition 3.3 A subset A of a topological space (X, τ) is called a $R_\gamma D$-set if there are two $U, V \in RO(X, \tau)_\gamma$ such that $U \neq X$ and $A = U \setminus V$.

It is true that every R_γ-open set U different from X is a $R_\gamma D$-set if $A = U$ and $V = \phi$. So, we can observe the following.

Remark 3.4 Every proper R_γ-open set is a $R_\gamma D$-set.

Definition 3.5 A topological space (X, τ) with an operation γ on $RO(X, \tau)$ is said to be:

(1) $R_\gamma D_0$ if for any pair of distinct points x and y of X there exists a $R_\gamma D$-set of X containing x but not y or a $R_\gamma D$-set of X containing y but not x.
(2) $R_\gamma D_1$ if for any pair of distinct points x and y of X there exists a $R_\gamma D$-set of X containing x but not y and a $R_\gamma D$-set of X containing y but not x.
(3) $R_\gamma D_2$ if for any pair of distinct points x and y of X there exist disjoint $R_\gamma D$-sets G and E of X containing x and y, respectively.

Definition 3.6 A topological space (X, τ) with an operation γ on $RO(X, \tau)$ is said to be:

(1) $R_\gamma T_0$ if for any pair of distinct points x and y of X there exists a R_γ-open set U in X containing x but not y or a R_γ-open set V in X containing y but not x.
(2) $R_\gamma T_1$ if for any pair of distinct points x and y of X there exists a R_γ-open set U in X containing x but not y and a R_γ-open set V in X containing y but not x.
(3) $R_\gamma T_2$ if for any pair of distinct points x and y of X there exist disjoint R_γ-open sets U and V in X containing x and y, respectively.

Remark 3.7 For a topological space (X, τ) with an operation γ on $RO(X, \tau)$, the following properties hold:

(1) If (X, τ) is $R_\gamma T_i$, then it is $R_\gamma T_{i-1}$, for $i = 1, 2$.
(2) If (X, τ) is $R_\gamma T_i$, then it is $R_\gamma D_i$, for $i = 0, 1, 2$.
(3) If (X, τ) is $R_\gamma D_i$, then it is $R_\gamma D_{i-1}$, for $i = 1, 2$.

Theorem 3.8 A topological space (X, τ) is $R_\gamma D_1$ if and only if it is $R_\gamma D_2$.

Proof Sufficiency: Follows from Remark 3.7.

Necessity: Let $x, y \in X$, $x \neq y$. Then there exist $R_\gamma D$-sets G_1, G_2 in X such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 \setminus U_2$ and $G_2 = U_3 \setminus U_4$, where U_1, U_2, U_3 and U_4 are R_γ-open sets in X. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(i) $x \notin U_3$. By $y \notin G_1$ we have two subcases:
  (a) $y \notin U_1$. From $x \in U_1 \setminus U_2$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$, and by $y \in U_3 \setminus U_4$ we have $y \in U_3 \setminus (U_1 \cup U_4)$. Therefore $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4)) = \phi$.
  (b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \setminus U_2$, and $y \in U_2$. Therefore $(U_1 \setminus U_2) \cap U_2 = \phi$.

(ii) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \setminus U_4$ and $x \in U_4$. Hence $(U_3 \setminus U_4) \cap U_4 = \phi$. Therefore, X is $R_\gamma D_2$.

■

Theorem 3.9 A topological space (X, τ) with an operation γ on $RO(X, \tau)$ is $R_\gamma T_0$ if and only if for each pair of distinct points x, y of X, $R_\gamma Cl(\{x\}) \neq R_\gamma Cl(\{y\})$.

Proof Clear. ■

Theorem 3.10 A topological space (X, τ) with an operation γ on $RO(X, \tau)$ is $R_\gamma T_1$ if and only if the singletons are R_γ-closed sets.

Proof Let (X, τ) be $R_\gamma T_1$ and x any point of X. Suppose $y \in X \setminus \{x\}$, then $x \neq y$ and so there exists a R_γ-open set U such that $y \in U$ but $x \notin U$. Consequently $y \in U \subseteq X \setminus \{x\}$ i.e., $X \setminus \{x\} = \cup\{U : y \in X \setminus \{x\}\}$ which is R_γ-open.

Conversely, suppose $\{p\}$ is R_γ-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X \setminus \{x\}$. Hence $X \setminus \{x\}$ is a R_γ-open set contains y but not x. Similarly $X \setminus \{y\}$ is a R_γ-open set contains x but not y. Accordingly X is a $R_\gamma T_1$ space. ■

Proposition 3.11 The following statements are equivalent for a topological space (X, τ) with an operation γ on $RO(X, \tau)$:

(1) X is $R_\gamma T_2$.
(2) Let $x \in X$. For each $y \neq x$, there exists a R_γ-open set U containing x such that $y \notin R_\gamma Cl(U)$.
(3) For each $x \in X$, $\cap\{R_\gamma Cl(U) : U \in RO(X, \tau)_\gamma \text{ and } x \in U\} = \{x\}$.

Proof (1) $\Rightarrow$ (2): Since X is $R_\gamma T_2$, there exist disjoint R_γ-open sets U and V containing x and y respectively. So, $U \subseteq X \setminus V$. Therefore, $R_\gamma Cl(U) \subseteq X \setminus V$. So $y \notin R_\gamma Cl(U)$.

(2) $\Rightarrow$ (3): If possible for some $y \neq x$, we have $y \in R_\gamma Cl(U)$ for every R_γ-open set U containing x, which then contradicts (2).

(3) $\Rightarrow$ (1): Let $x, y \in X$ and $x \neq y$. Then there exists a R_γ-open set U containing x such that $y \notin R_\gamma Cl(U)$. Let $V = X \setminus R_\gamma Cl(U)$, then $y \in V$ and $x \in U$ and also $U \cap V = \phi$. ■

Definition 3.12 A topological space (X, τ) with an operation γ on $RO(X, \tau)$ is called R_γ-regular if for each R_γ-closed set F of X not containing x, there exist disjoint R_γ-open sets U and V such that $x \in U$ and $F \subseteq V$. A R_γ-regular $R_\gamma T_1$ space is called a $R_\gamma T_3$ space.

Remark 3.13 Every $R_\gamma T_3$ space is $R_\gamma T_2$.

Proposition 3.14 A subset A of X is R_γ-g.open if and only if $F \subseteq R_\gamma Int(A)$ whenever $F \subseteq A$ and F is R_γ-closed in (X, τ).

Proof Obvious. ■

We give several characterizations of R_γ-regular spaces.

Proposition 3.15 The following are equivalent for a topological space (X, τ) with an operation γ on $RO(X, \tau)$:

(1) X is R_γ-regular.
(2) For each $x \in X$ and each R_γ-open set U containing x, there exists a R_γ-open set V containing x such that $x \in V \subseteq R_\gamma Cl(V) \subseteq U$.
(3) For each R_γ-closed set F of X, $\cap\{R_\gamma Cl(V)\colon F \subseteq V,\ V \in RO(X, \tau)_\gamma\} = F$.
(4) For each A subset of X and each $U \in RO(X, \tau)_\gamma$ with $A \cap U \neq \phi$, there exists a $V \in RO(X, \tau)_\gamma$ such that $A \cap V \neq \phi$ and $R_\gamma Cl(V) \subseteq U$.
(5) For each nonempty subset A of X and each R_γ-closed subset F of X with $A \cap F = \phi$, there exists $V, W \in RO(X, \tau)_\gamma$ such that $A \cap V \neq \phi$, $F \subseteq W$ and $W \cap V = \phi$.

(6) For each R_γ-closed set F and $x \in F$, there exists $U \in RO(X,\tau)_\gamma$ and a R_γ-g.open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.
(7) For each $A \subseteq X$ and each R_γ-closed set F with $A \cap F = \phi$, there exists $U \in RO(X,\tau)_\gamma$ and a R_γ-g.open set V such that $A \cap U \neq \phi$, $F \subseteq V$ and $U \cap V = \phi$.
(8) For each R_γ-closed set F of X, $F = \cap\{R_\gamma Cl(V)$: $F \subseteq V$, V is R_γ-g.open$\}$.

Proof (1) $\Rightarrow$ (2): Let $x \notin X \setminus U$, where U is any R_γ-open set containing x. Then there exists $G, V \in RO(X,\tau)_\gamma$ such that $(X \setminus U) \subseteq G$, $x \in V$ and $G \cap V = \phi$. Therefore $V \subseteq (X \setminus G)$ and so $x \in V \subseteq R_\gamma Cl(V) \subseteq (X \setminus G) \subseteq U$.

(2) $\Rightarrow$ (3): Let $X \setminus F$ be any R_γ-open set containing x. Then by (2) there exists a R_γ-open set U containing x such that $x \in U \subseteq R_\gamma Cl(U) \subseteq (X \setminus F)$. So, $F \subseteq X \setminus R_\gamma Cl(U) = V$, $V \in RO(X,\tau)_\gamma$ and $V \cap U = \phi$. Then by Theorem 2.20, $x \notin R_\gamma Cl(V)$. Thus $F \supseteq \cap\{R_\gamma Cl(V)$: $F \subseteq V, V \in RO(X,\tau)_\gamma\}$.

(3) $\Rightarrow$ (4): Let $U \in RO(X,\tau)_\gamma$ with $x \in U \cap A$. Then $x \notin (X \setminus U)$ and hence by (3) there exists a R_γ-open set W such that $X \setminus U \subseteq W$ and $x \notin R_\gamma Cl(W)$. We put $V = X \setminus R_\gamma Cl(W)$, which is a R_γ-open set containing x and hence $V \cap A \neq \phi$. Now $V \subseteq (X \setminus W)$ and so $R_\gamma Cl(V) \subseteq (X \setminus W) \subseteq U$.

(4) $\Rightarrow$ (5): Let F be a set as in the hypothesis of (5). Then $(X \setminus F)$ is R_γ-open and $(X \setminus F) \cap A \neq \phi$. Then there exists $V \in RO(X,\tau)_\gamma$ such that $A \cap V \neq \phi$ and $R_\gamma Cl(V) \subseteq (X \setminus F)$. If we put $W = X \setminus R_\gamma Cl(V)$, then $F \subseteq W$ and $W \cap V = \phi$.

(5) $\Rightarrow$ (1): Let F be a R_γ-closed set not containing x. Then by (5), there exist $W, V \in RO(X,\tau)_\gamma$ such that $F \subseteq W$ and $x \in V$ and $W \cap V = \phi$.

(1) $\Rightarrow$ (6): Obvious.

(6) $\Rightarrow$ (7): For $a \in A$, $a \notin F$ and hence by (6) there exists $U \in RO(X,\tau)_\gamma$ and a R_γ-g.open set V such that $a \in U$, $F \subseteq V$ and $U \cap V = \phi$. So, $A \cap U \neq \phi$.

(7) $\Rightarrow$ (1): Let $x \notin F$, where F is R_γ-closed. Since $\{x\} \cap F = \phi$, by (7) there exists $U \in RO(X,\tau)_\gamma$ and a R_γ-g.open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \phi$. Now put $V = R_\gamma Int(W)$. Using Proposition 3.14 of R_γ-g.open sets we get $F \subseteq V$ and $V \cap U = \phi$.

(3) $\Rightarrow$ (8): We have $F \subseteq \bigcap\{R_\gamma Cl(V)$: $F \subseteq V$ and V is R_γ-g.open $\} \subseteq \bigcap\{R_\gamma Cl(V)$: $F \subseteq V$ and V is R_γ-open $\} = F$.

(8) $\Rightarrow$ (1): Let F be a R_γ-closed set in X not containing x. Then by (8) there exists a R_γ-g.open set W such that $F \subseteq W$ and $x \in X \setminus R_\gamma Cl(W)$. Since F is R_γ-closed and W is R_γ-g.open, $F \subseteq R_\gamma Int(W)$. Take $V = R_\gamma Int(W)$. Then $F \subseteq V$, $x \in U = X \setminus R_\gamma Cl(V)$ and $U \cap V = \phi$. ■

Definition 3.16 A topological space (X,τ) with an operation γ on $RO(X,\tau)$, is said to be R_γ-normal if for any pair of disjoint R_γ-closed sets A, B of X, there exist disjoint R_γ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. A R_γ-normal $R_\gamma T_1$ space is called a $R_\gamma T_4$ space.

We give several characterizations of R_γ-normal spaces.

Proposition 3.17 For a topological space (X,τ) with an operation γ on $RO(X,\tau)$, the following are equivalent:

(1) X is R_γ-normal.
(2) For each pair of disjoint R_γ-closed sets A, B of X, there exist disjoint R_γ-g.open sets U and V such that $A \subseteq U$ and $B \subseteq V$.
(3) For each R_γ-closed A and any R_γ-open set V containing A, there exists a R_γ-g.open set U such that $A \subseteq U \subseteq R_\gamma Cl(U) \subseteq V$.
(4) For each R_γ-closed set A and any R_γ-g.open set B containing A, there exists a R_γ-g.open set U such that $A \subseteq U \subseteq R_\gamma Cl(U) \subseteq R_\gamma Int(B)$.
(5) For each R_γ-closed set A and any R_γ-g.open set B containing A, there exists a R_γ-open set G such that $A \subseteq G \subseteq R_\gamma Cl(G) \subseteq R_\gamma Int(B)$.
(6) For each R_γ-g.closed set A and any R_γ-open set B containing A, there exists a R_γ-open set U such that $R_\gamma Cl(A) \subseteq U \subseteq R_\gamma Cl(U) \subseteq B$.

(7) For each R_γ-g.closed set A and any R_γ-open set B containing A, there exists a R_γ-g.open set G such that $R_\gamma Cl(A) \subseteq G \subseteq R_\gamma Cl(G) \subseteq B$.

Proof (1) $\Rightarrow$ (2) : Follows from the fact that every R_γ-open set is R_γ-g.open.

(2) $\Rightarrow$ (3): Let A be a R_γ-closed set and V any R_γ-open set containing A. Since A and $(X \setminus V)$ are disjoint R_γ-closed sets, there exist R_γ-g.open sets U and W such that $A \subseteq U$, $(X \setminus V) \subseteq W$ and $U \cap W = \phi$. By Proposition 3.14, we get $(X \setminus V) \subseteq R_\gamma Int(W)$. Since $U \cap R_\gamma Int(W) = \phi$, we have $R_\gamma Cl(U) \cap R_\gamma Int(W) = \phi$, and hence $R_\gamma Cl(U) \subseteq X \setminus R_\gamma Int(W) \subseteq V$. Therefore $A \subseteq U \subseteq R_\gamma Cl(U) \subseteq V$.

(3) $\Rightarrow$ (1): Let A and B be any two disjoint R_γ-closed sets of X. Since $(X \setminus B)$ is a R_γ-open set containing A, there exists a R_γ-g.open set G such that $A \subseteq G \subseteq R_\gamma Cl(G) \subseteq (X \setminus B)$. Since G is a R_γ-g.open set, using Proposition 3.14, we have $A \subseteq R_\gamma Int(G)$. Taking $U = R_\gamma Int(G)$ and $V = X \setminus R_\gamma Cl(G)$, we have two disjoint R_γ-open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Hence X is R_γ-normal.

(5) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (3): Obvious.

(5) $\Rightarrow$ (3): Let A be any R_γ-closed set and V any R_γ-open set containing A. Since every R_γ-open set is R_γ-g.open, there exists a R_γ-open set G such that $A \subseteq G \subseteq R_\gamma Cl(G) \subseteq R_\gamma Int(V)$. Also we have a R_γ-g.open set G such that $A \subseteq G \subseteq R_\gamma Cl(G) \subseteq R_\gamma Int(V) \subseteq V$.

(6) $\Rightarrow$ (7): Obvious.

(7) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (5): Let A be a R_γ-closed set and B any R_γ-g.open set containing A. Using Proposition 3.14, of a R_γ-g.open set we get $A \subseteq R_\gamma Int(B) = V$, say. Then applying (3), we get a R_γ-g.open set U such that $A = R_\gamma Cl(A) \subseteq U \subseteq R_\gamma Cl(U) \subseteq V$. Again, using the same Proposition 3.14, we get $A \subseteq R_\gamma Int(U)$, and hence $A \subseteq R_\gamma Int(U) \subseteq U \subseteq R_\gamma Cl(U) \subseteq V$, which implies $A \subseteq R_\gamma Int(U) \subseteq R_\gamma Cl(R_\gamma Int(U)) \subseteq R_\gamma Cl(U) \subseteq V$, i.e.

$$A \subseteq G \subseteq R_\gamma Cl(G) \subseteq R_\gamma Int(B)$$

, where $G = R_\gamma Int(U)$.

(3) $\Rightarrow$ (7): Let A be a R_γ-g.closed set and B any R_γ-open set containing A. Since A is a R_γ-g.closed set, we have $R_\gamma Cl(A) \subseteq B$, therefore by (3) we can find a R_γ-g.open set U such that $R_\gamma Cl(A) \subseteq U \subseteq R_\gamma Cl(U) \subseteq B$.

(7) $\Rightarrow$ (6): Let A be a R_γ-g.closed set and B any R_γ-open set containing A, then by (7) there exists a R_γ-g.open set G such that $R_\gamma Cl(A) \subseteq G \subseteq R_\gamma Cl(G) \subseteq B$. Since G is a R_γ-g.open set, then by Proposition 3.14, we get $R_\gamma Cl(A) \subseteq R_\gamma Int(G)$. If we take $U = R_\gamma Int(G)$, the proof follows. ■

Every $R_\gamma T_4$ space is clearly a $R_\gamma T_3$ space, but it should not be surprising that R_γ-normal spaces need not be R_γ-regular.

Example 3.18 Consider $X = \{a, b, c\}$ with the discrete topology on X. Define an operation γ on $RO(X, \tau)$ by

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a\} \\ A \cup \{c\} & \text{if } A \neq \{a\} \end{cases}$$

Then X is R_γ-normal but not P_γ-regular.

Definition 3.19 A topological space (X, τ) with an operation γ on $RO(X, \tau)$, is said to be R_γ-symmetric if for x and y in X, $x \in R_\gamma Cl(\{y\})$ implies $y \in R_\gamma Cl(\{x\})$.

Proposition 3.20 If (X,τ) is a topological space with an operation γ on $RO(X,\tau)$, then the following are equivalent:

(1) (X,τ) is R_γ-symmetric space.
(2) Every singleton is R_γ-g.closed, for each $x \in X$.

Proof (1) $\Rightarrow$ (2). Assume that $\{x\} \subseteq U \in RO(X,\tau)_\gamma$, but $R_\gamma Cl(\{x\}) \nsubseteq U$. Then $R_\gamma Cl(\{x\}) \cap X \setminus U \neq \phi$. Now, we take $y \in R_\gamma Cl(\{x\}) \cap X \setminus U$, then by hypothesis $x \in R_\gamma Cl(\{y\}) \subseteq X \setminus U$ and $x \notin U$, which is a contradiction. Therefore $\{x\}$ is R_γ-g.closed, for each $x \in X$.

(2) $\Rightarrow$ (1). Assume that $x \in R_\gamma Cl(\{y\})$, but $y \notin R_\gamma Cl(\{x\})$. Then $\{y\} \subseteq X \setminus R_\gamma Cl(\{x\})$ and hence $R_\gamma Cl(\{y\}) \subseteq X \setminus R_\gamma Cl(\{x\})$. Therefore $x \in X \setminus R_\gamma Cl(\{x\})$, which is a contradiction and hence $y \in R_\gamma Cl(\{x\})$. ■

Corollary 3.1 If a topological space (X,τ) with an operation γ on $RO(X,\tau)$ is a $R_\gamma T_1$ space, then it is R_γ-symmetric.

Proof In a $R_\gamma T_1$ space, every singleton is R_γ-closed (Theorem 3.10) and therefore is R_γ-g.closed. Then by Proposition 3.20, (X,τ) is R_γ-symmetric. ■

Corollary 3.2 For a topological space (X,τ) with an operation γ on $RO(X,\tau)$, the following statements are equivalent:

(1) (X,τ) is R_γ-symmetric and $R_\gamma T_0$.
(2) (X,τ) is $R_\gamma T_1$.

Proof By Corollary 3.1, it suffices to prove only (1) $\Rightarrow$ (2).

Let $x \neq y$ and by $R_\gamma T_0$, we may assume that $x \in U \subseteq X \setminus \{y\}$ for some $U \in RO(X,\tau)_\gamma$. Then $x \notin R_\gamma Cl(\{y\})$ and hence $y \notin R_\gamma Cl(\{x\})$. There exists a R_γ-open set V such that $y \in V \subseteq X \setminus \{x\}$ and thus (X,τ) is a $R_\gamma T_1$ space. ■

Proposition 3.21 If (X,τ) is a R_γ-symmetric space with an operation γ on $RO(X,\tau)$, then the following statements are equivalent:

(1) (X,τ) is a $R_\gamma T_0$ space.
(2) (X,τ) is a $R_\gamma T_{\frac{1}{2}}$ space.
(3) (X,τ) is a $R_\gamma T_1$ space.

Proof (1) $\Leftrightarrow$ (3): Obvious from Corollary 3.2.

(3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are obvious. ■

4. $R_{(\gamma,\gamma')}$-Continuous Maps

Throughout this section, let (X,τ) and (Y,σ) be two topological spaces and let $\gamma : RO(X,\tau) \to P(X)$ and $\gamma' : RO(Y,\sigma) \to P(Y)$ be the operations on $RO(X,\tau)$ and $RO(Y,\sigma)$, respectively.

Definition 4.1 A mapping $f : (X,\tau) \to (Y,\sigma)$ is said to be $R_{(\gamma,\gamma')}$-continuous if for each x of X and each $R_{\gamma'}$-open set V containing $f(x)$, there exists a R_γ-open set U such that $x \in U$ and $f(U) \subseteq V$.

Theorem 4.2 Let $f : (X,\tau) \to (Y,\sigma)$ be a $R_{(\gamma,\gamma')}$-continuous mapping. Then

(1) $f(R_\gamma Cl(A)) \subseteq R_{\gamma'} Cl(f(A))$ holds for every subset A of (X,τ).
(2) For every $R_{\gamma'}$-closed set B of (Y,σ), $f^{-1}(B)$ is R_γ-closed in (X,τ).

Proof

(1) Let $y \in f(R_\gamma Cl(A))$ and V be the $R_{\gamma'}$-open set containing y, then there exists a point $x \in X$ and a R_γ-open set U such that $f(x) = y$, $x \in U$ and $f(U) \subseteq V$. Since $x \in R_\gamma Cl(A)$, we have $U \cap A \neq \phi$, and hence $\phi \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. This implies $y \in R_{\gamma'} Cl(f(A))$.

(2) It is sufficient to prove that (1) implies (2). Let B be the $R_{\gamma'}$-closed set in (Y,σ). That is $R_{\gamma'}Cl(B)=B$. By using (1) we have $f(R_\gamma Cl(f^{-1}(B)))\subseteq R_{\gamma'}Cl(f(f^{-1}(B)))\subseteq R_{\gamma'}Cl(B)=B$ holds. Therefore $R_\gamma Cl(f^{-1}(B))\subseteq f^{-1}(B)$, and hence $f^{-1}(B)=R_\gamma Cl(f^{-1}(B))$. Hence $f^{-1}(B)$ is R_γ-closed set in (X,τ).

■

Definition 4.3 A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be $R_{(\gamma,\gamma')}$-closed if for any R_γ-closed set A of (X,τ), $f(A)$ is $R_{\gamma'}$-closed in (Y,σ).

Remark 4.4 If f is $R_{(id,\gamma')}$-closed, then $f(F)$ is $R_{\gamma'}$-closed for any regular closed set F of (X,τ).

Remark 4.5 If f is bijective mapping and $f^{-1}:(Y,\sigma)\to(X,\tau)$ is $R_{(\gamma',id)}$-continuous, then f is $R_{(id,\gamma')}$-closed.

Proof Follows from the Definitions 4.3 and Remark 4.4. ■

Theorem 4.6 Suppose $f:(X,\tau)\to(Y,\sigma)$ is $R_{(\gamma,\gamma')}$-continuous and f is $R_{(\gamma,\gamma')}$-closed, then

(1) For every R_γ-g.closed set A of (X,τ) the image $f(A)$ is $R_{\gamma'}$-g.closed.
(2) For every $R_{\gamma'}$-g.closed set B of (Y,σ) the inverse set $f^{-1}(B)$ is R_γ-g.closed.

Proof

(1) Let V be any $R_{\gamma'}$-open set in (Y,σ) such that $f(A)\subseteq V$, then by Theorem 4.2 (2), $f^{-1}(V)$ is R_γ-open. Since A is R_γ-g.closed and $A\subseteq f^{-1}(V)$, we have $R_\gamma Cl(A)\subseteq f^{-1}(V)$, and hence $f(R_\gamma Cl(A))\subseteq V$. By assumption $f(R_\gamma Cl(A))$ is a $R_{\gamma'}$-closed set, therefore $R_{\gamma'}Cl(f(A))$ $\subseteq R_{\gamma'}Cl(f(R_\gamma Cl(A)))=f(R_\gamma Cl(A))\subseteq V$. This implies $f(A)$ is $R_{\gamma'}$-g.closed.
(2) Let U be any R_γ-open set such that $f^{-1}(B)\subseteq U$. Let $F=R_\gamma Cl(f^{-1}(B))$ $\cap(X\setminus U)$, then F is R_γ-closed in (X,τ). This implies $f(F)$ is $R_{\gamma'}$-closed set in (Y,σ). Since $f(F)=f(R_\gamma Cl((f^{-1}(B))\cap(X\setminus U)))\subseteq R_{\gamma'}Cl(B)\cap f(X\setminus U)\subseteq R_{\gamma'}Cl(B)\cap(Y\setminus B)$. This implies $f(F)=\phi$ and hence $F=\phi$. Therefore $R_\gamma Cl(f^{-1}(B))\subseteq U$. This implies $f^{-1}(B)$ is R_γ-g.closed.

■

Theorem 4.7 Suppose $f:(X,\tau)\to(Y,\sigma)$ is $R_{(\gamma,\gamma')}$-continuous and $R_{(\gamma,\gamma')}$-closed, then

(1) If f is injective and (Y,σ) is $R_{\gamma'}$-$T_{\frac{1}{2}}$, then (X,τ) is R_γ-$T_{\frac{1}{2}}$.
(2) If f is surjective and (X,τ) is R_γ-$T_{\frac{1}{2}}$, then (Y,σ) is $R_{\gamma'}$-$T_{\frac{1}{2}}$.

Proof

(1) Let A be a R_γ-g.closed set of (X,τ). Now to prove that A is R_γ-closed. By Theorem 4.6 (1), $f(A)$ is $R_{\gamma'}$-g.closed. Since (Y,σ) is $R_{\gamma'}$-$T_{\frac{1}{2}}$, this implies that $f(A)$ is $R_{\gamma'}$-closed. Since f is $R_{(\gamma,\gamma')}$-continuous, then by Theorem 4.2, we have $A=f^{-1}(f(A))$ is R_γ-closed. Hence (X,τ) is R_γ-$T_{\frac{1}{2}}$.
(2) Let B be a $R_{\gamma'}$-g.closed set in (Y,σ). Then $f^{-1}(B)$ is R_γ-closed, since (X,τ) is R_γ-$T_{\frac{1}{2}}$ space. It follows from the assumption that B is $R_{\gamma'}$-closed.

■

Definition 4.8 A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be $R_{(\gamma,\gamma')}$-homeomorphic, if f is bijective, $R_{(\gamma,\gamma')}$-continuous and f^{-1} is $R_{(\gamma',\gamma)}$-continuous.

Remark 4.9 If $f:(X,\tau)\to(Y,\sigma)$ is bijective and $f^{-1}:(Y,\sigma)\to(X,\tau)$ is $R_{(\gamma',\gamma)}$-continuous, then f is $R_{(\gamma,\gamma')}$-closed.

Theorem 4.10 Let $f:(X,\tau)\to(Y,\sigma)$ be $R_{(\gamma,\gamma')}$-homeomorphic. The space (X,τ) is R_γ-$T_{\frac{1}{2}}$ if and only if (Y,σ) is $R_{\gamma'}$-$T_{\frac{1}{2}}$.

Proof Necessity: Let B be a $R_{\gamma'}$-g.closed set of (Y, σ). By Theorem 4.6, $f^{-1}(B)$ is R_γ-g.closed and hence R_γ-closed. Since f is $R_{(\gamma,\gamma')}$-closed, we have $B = f(f^{-1}(B))$ is $R_{\gamma'}$-closed.

Sufficiency: Let A be a R_γ-g.closed set of (X, τ). By Theorem 4.6, $f(A)$ is $R_{\gamma'}$-g.closed and hence $R_{\gamma'}$-closed. Since f is $R_{(\gamma,\gamma')}$-continuous, then by Theorem 4.2, we have $A = f^{-1}(f(A))$ is R_γ-closed. ■

Theorem 4.11 If $f : (X, \tau) \to (Y, \sigma)$ is a $R_{(\gamma,\gamma')}$-continuous surjective mapping and E is a $R_{\gamma'}D$-set in Y, then the inverse image of E is a $R_\gamma D$-set in X.

Proof Let E be a $R_{\gamma'}D$-set in Y. Then there are $R_{\gamma'}$-open sets U_1 and U_2 in Y such that $E = U_1 \setminus U_2$ and $U_1 \neq Y$. By the $R_{(\gamma,\gamma')}$-continuous of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are R_γ-open in X. Since $U_1 \neq Y$ and f is surjective, we have $f^{-1}(U_1) \neq X$. Hence, $f^{-1}(E) = f^{-1}(U_1) \setminus f^{-1}(U_2)$ is a $R_\gamma D$-set. ■

Theorem 4.12 If (Y, σ) is $R_{\gamma'}$-D_1 and $f : (X, \tau) \to (Y, \sigma)$ is $R_{(\gamma,\gamma')}$-continuous bijective, then (X, τ) is $R_\gamma D_1$.

Proof Suppose that Y is a $R_{\gamma'}$-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is $R_{\gamma'}$-D_1, there exist $R_{\gamma'}D$-sets G_x and G_y of Y containing $f(x)$ and $f(y)$ respectively, such that $f(x) \notin G_y$ and $f(y) \notin G_x$. By Theorem 4.11, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are $R_\gamma D$-sets in X containing x and y, respectively, such that $x \notin f^{-1}(G_y)$ and $y \notin f^{-1}(G_x)$. This implies that X is a $R_\gamma D_1$ space. ■

Theorem 4.13 A topological space (X, τ) is $R_\gamma D_1$ if for each pair of distinct points $x, y \in X$, there exists a $R_{(\gamma,\gamma')}$-continuous surjective mapping $f : (X, \tau) \to (Y, \sigma)$, where Y is a $R_{\gamma'}$-D_1 space such that $f(x)$ and $f(y)$ are distinct.

Proof Let x and y be any pair of distinct points in X. By hypothesis, there exists a $R_{(\gamma,\gamma')}$-continuous, surjective mapping f of a space X onto a $R_{\gamma'}$-D_1 space Y such that $f(x) \neq f(y)$. By Theorem 3.8, there exist disjoint $R_{\gamma'}D$-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is $R_{(\gamma,\gamma')}$-continuous and surjective, by Theorem 4.11, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $R_\gamma D$-sets in X containing x and y, respectively. hence by Theorem 3.8, X is $R_\gamma D_1$ space. ■

References

[1] S. Kasahara. Operation-compact spaces. *Math. Japonica*, 24:97:105, 1979.
[2] H. Ogata. Operation on topological spaces and associated topology. *Math. Japonica*, 36(1):175:184, 1991.
[3] M. H. Stone. Applications of the theory of boolean rings to topology. *Trans. Amer. Math. Soc.*, 41:375:481, 1937.