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RESEARCH ARTICLE

Coupled Fixed Point Theorems in Dislocated Quasi-metric Spaces

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In this paper, we prove a coupled coincidence fixed point theorem in dislocated quasi-metric spaces. Also, we give an example to validate our main theorem and some corollaries of the main result.

Keywords: Dislocated quasi-metric spaces; fixed point; coupled coincidence point.

AMS Subject Classification: 47H10, 54H25.

1. Introduction

The following definition of dislocated metric space and it's fundamental properties was given by Hitzler and Seda [1].

Definition 1.1 Let X be a set and let $d : X \times X \longrightarrow [0, \infty)$ be a function, called a distance function. Consider the following conditions:

(M-i) For all $x \in X$, $d(x, x) = 0$,
(M-ii) For all x, $y \in X$, if $d(x, y) = d(y, x) = 0$, then $x = y$,
(M-iii) For all x, $y \in X$, $d(x, y) = d(y, x)$,
(M-iv) For all x, y, $z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$,
(M-iv') For all x, y, $z \in X$, $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

If d satisfies conditions (M-i) to (M-iv), then it is called a metric. If it satisfies conditions (M-i), (M-ii) and (M-iv), it is called a quasi-metric. If it satisfies (M-ii), (M-iii) and (M-iv), we will call it a dislocated metric (or simply d-metric). If it satisfies conditions (M-ii) and (M-iv), it is called a dislocated quasi-metric (or simply dq-metric). If a metric d satisfies the strong triangle inequality (M-iv′), then it is called an ultrametric.

The study of partial metric spaces and generalized ultrametric spaces have applications in theoretical computer science had beeen studied by Matthews [2]. Hitzler and Seda [1] introduced the concept of dislocated metric space as a generalization of metrics where self-distances need not be zero. They also proved a generalized version of Banach contraction mapping principle which was applied to obtain fixed point semantics for logic programs. Recently, many authors [3–14] studied different properties and fixed point results of dislocated (fuzzy or) metric spaces. Lakshmikantham and Ćirić [15] introduced coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces.

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In the present paper, we define the notion of a coupled coincidence fixed point and introduce a coupled coincidence fixed point theorem in dislocated quasi-metric spaces. Also, we give an example to validate our main theorem and some corollaries of the main result.

2. Fixed Point Results

Definition 2.1 A sequence (x_n) in dq-metric space (X,d) is called Cauchy if for all $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $m,n \geq n_0$, $d(x_m,x_n) < \varepsilon$ or $d(x_n,x_m) < \varepsilon$. Replacing $d(x_m,x_n) < \varepsilon$ and $d(x_n,x_m) < \varepsilon$ in this definition by $\max\{d(x_m,x_n), d(x_n,x_m)\} < \varepsilon$, the sequence (x_n) in dq-metric space (X,d) is called 'bi' Cauchy.

Definition 2.2 A sequence (x_n) dislocated quasi-converges (for short dq-converges) to x if

$$\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0.$$

In this case x is called dq-limit of (x_n).

Definition 2.3 A dq-metric space (X,d) is called complete if every Cauchy sequences in it is dq-convergent.

Definition 2.4 Let (X,d_1) and (Y,d_2) be a dq-metric spaces and let $f : X \to Y$ be a function. Then f is continuous if for each sequence (x_n) which is d_1q-convergent to x_0 in X, the sequence $(f(x_n))$ is d_2q-convergent to $f(x_0)$ in Y.

Definition 2.5 Let (X,d) be a dq-metric spaces. A map $f : X \to X$ is called contraction if there exists $0 \leq \lambda < 1$ such that

$$d(f(x),f(y)) \leq \lambda d(x,y)$$

for all $x,y \in X$.

Lemma 2.6 Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0.

Definition 2.7 An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 2.8 An element $(x,y) \in X \times X$ is called a coupled coincidence point of a mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = gx$ and $F(y,x) = gy$.

Definition 2.9 Let X be a nonempty set. Then we say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $gF(x,y) = F(gx,gy)$.

Lemma 2.10 Let (X,d) be a dq-metric spaces. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that;

$$d(F(x,y),F(u,v)) \leq \lambda [d(gx,gu) + d(gy,gv)] \tag{1}$$

for all $x,y,u,v \in X$. Assume that (x,y) is a coupled coincidence point of the mappings F and g. If $\lambda \in \left[0,\frac{1}{2}\right)$, then

$$F(x,y) = gx = gy = F(y,x).$$

Proof Since (x,y) is a coupled coincidence point of the mappings F and g, we have $F(x,y) = gx$ and $F(y,x) = gy$. Assume that $gx \neq gy$. Then by (1), we get

$$d(gx,gy) = d(F(x,y),F(y,x)) \leq \lambda [d(gx,gy) + d(gy,gx)].$$

Also by (1), we have

$$d(gy,gx)=d\left(F(y,x),F(x,y)\right)\leq\lambda\left[d(gy,gx)+d(gx,gy)\right].$$

Therefore

$$d(gx,gy)+d(gy,gx)\leq 2\lambda\left[d(gx,gy)+d(gy,gx)\right].$$

Since $2\lambda<1$, we get

$$d(gx,gy)+d(gy,gx)<d(gx,gy)+d(gy,gx)$$

which is a contradiction. So $gx=gy$ and hence

$$F\left(x,y\right)=gx=gy=F\left(y,x\right).$$

■

Theorem 2.11 Let (X,d) be a dq-metric spaces. Let $F:X\times X\to X$ and $g:X\to X$ be two continuous mappings such that

$$d\left(F(x,y),F(u,v)\right)\leq\lambda\left[d(gx,gu)+d(gy,gv)\right] \tag{2}$$

for all $x,y,u,v\in X$. Assume that F and g satisfy the following conditions:

(i) $F(X\times X)\subseteq g(X)$.
(ii) $g(X)$ is complete dq-metric,
(iii) g commutes with F.

If $\lambda\in(0,1)$, then there is a unique x in X such that $gx=F\left(x,x\right)=x$.

Proof Let $x_0,y_0\in X$. By condition (i), Since $F(X\times X)\subseteq g(X)$, we can choose $x_1,y_1\in X$ such that $gx_1=F\left(x_0,y_0\right)$ and $gy_1=F\left(y_0,x_0\right)$. Again since $F(X\times X)\subseteq g(X)$, we can choose $x_2,y_2\in X$ such that $gx_2=F\left(x_1,y_1\right)$ and $gy_2=F\left(y_1,x_1\right)$. Continuing this process we can construct two sequences (x_n) and (y_n) in X such that $gx_{n+1}=F\left(x_n,y_n\right)$ and $gy_{n+1}=F\left(y_n,x_n\right)$. For $n\in N$, by (2), we have

$$d(gx_n,gx_{n+1})=d\left(F(x_{n-1},y_{n-1}),F(x_n,y_n)\right)\leq\lambda\left[d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)\right].$$

From

$$d(gx_{n-1},gx_n)=d\left(F(x_{n-2},y_{n-2}),F(x_{n-1},y_{n-1})\right)\leq\lambda\left[d(gx_{n-2},gx_{n-1})+d(gy_{n-2},gy_{n-1})\right]$$

and

$$d(gy_{n-1},gy_n)=d\left(F(y_{n-2},x_{n-2}),F(y_{n-1},x_{n-1})\right)\leq\lambda\left[d(gy_{n-2},gy_{n-1})+d(gx_{n-2},gx_{n-1})\right]$$

we have

$$d(gx_{n-1},gx_n)+d(gy_{n-1},gy_n)\leq 2\lambda\left[d(gx_{n-2},gx_{n-1})+d(gy_{n-2},gy_{n-1})\right]$$

holds for all $n \in N$. Thus we get that

$$\begin{aligned} d(gx_n, gx_{n+1}) &\le \lambda \left[d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\right] \\ &\le 2\lambda^2 \left[d(gx_{n-2}, gx_{n-1}) + d(gy_{n-2}, gy_{n-1})\right] \\ &\vdots \\ &\le \frac{1}{2}(2\lambda)^n \left[d(gx_0, gx_1) + d(gy_0, gy_1)\right]. \end{aligned}$$

Thus for each $n \in N$ we have

$$d(gx_n, gx_{n+1}) \le \frac{1}{2}(2\lambda)^n \left[d(gx_0, gx_1) + d(gy_0, gy_1)\right]. \tag{3}$$

Let $m, n \in N$ with $m > n$. By axiom (M-iv) of the definition of dislocated quasi metric spaces, we have;

$$d(gx_n, gx_m) \le d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m)$$

since $2\lambda < 1$, by (3), we get that

$$\begin{aligned} d(gx_n, gx_m) &\le \frac{1}{2}\left(\sum_{i=n}^{m-1}(2\lambda)^i\right)\left[d(gx_0, gx_1) + d(gy_0, gy_1)\right] \\ &\le \frac{(2\lambda)^n}{2(1-2\lambda)}\left[d(gx_0, gx_1) + d(gy_0, gy_1)\right]. \end{aligned}$$

Letting $m, n \to \infty$, we have

$$\lim_{m,n\to\infty} d(x_n, gx_m) = 0.$$

Thus (gx_n) is a Cauchy sequence in $g(X)$. Similarly, we may show that (gy_n) is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete dq-metric, we get that (gx_n) and (gy_n) are dq-convergent to some $x \in X$ and $y \in X$, respectively. Since F and g are continuous, and also, F and g are commute, we have

$$ggx_{n+1} = g(F(x_n, y_n)) = F(gx_n, gy_n),$$

and

$$ggy_{n+1} = g(F(y_n, x_n)) = F(gy_n, gx_n).$$

Thus we get that

$$\begin{aligned} gx = g(\lim_{n\to\infty} gx_n) &= \lim_{n\to\infty} g(F(x_{n-1}, y_{n-1})) = \lim_{n\to\infty} F(gx_{n-1}, gy_{n-1}) \\ &= F(\lim_{n\to\infty} gx_{n-1}, \lim_{n\to\infty} gy_{n-1}) = F(x, y). \end{aligned}$$

Hence $gx = F(x, y)$. Similarly, we may show that $gy_n = F(y, x)$. By Lemma 2.10, (x, y) is a coupled fixed point of the mappings F and g. So

$$gx = F\,(x, y) = F\,(y, x) = gy.$$

Since (gx_{n+1}) is subsequence of (gx_n) we have that (gx_{n+1}) is dq-convergent to x. Thus

$$\begin{aligned} d(gx_{n+1}, gx) &= d\left(gx_{n+1}, F(x,y)\right) = d(F(x_n, y_n), F(x,y)) \\ &\leq \lambda \left[d(gx_n, gx) + d(gy_n, gy)\right]. \end{aligned}$$

Letting $n \to \infty$ and using the fact that d is continuous on its variables, we get that

$$d(x, gx) \leq \lambda \left[d(x, gx) + d(y, gy)\right].$$

Similarly, we may show that

$$d(y, gy) \leq \lambda \left[d(x, gx) + d(y, gy)\right].$$

Thus

$$d(x, gx) + d(y, gy) \leq 2\lambda \left[d(x, gx) + d(y, gy)\right].$$

Since $2\lambda < 1$, the last inequality happens only if $d(x, gx) = 0$ and $d(y, gy) = 0$. Similarly,

$$\begin{aligned} d(gx, gx_{n+1}) &= d\left(F(x,y), gx_{n+1}\right) = d(F(x,y), F(x_n, y_n)) \\ &\leq \lambda \left[d(gx, gx_n) + d(gy, gy_n)\right]. \end{aligned}$$

Letting $n \to \infty$ and using the fact that d is continuous on its variables, we get that;

$$d(gx, x) \leq \lambda \left[d(gx, x) + d(gy, y)\right].$$

Similarly, we may show that

$$d(gy, y) \leq \lambda \left[d(gx, x) + d(gy, y)\right].$$

Thus

$$d(gx, x) + d(gy, y) \leq 2\lambda \left[d(gx, x) + d(gy, y)\right].$$

Since $2\lambda < 1$, the last inequality happens only if $d(gx, x) = 0$ and $d(gy, y) = 0$. From Definition 2.1 (M-ii), we have $x = gx$ and $y = gy$. Thus we get

$$gx = F(x, x) = x.$$

To prove the uniqueness, let $z \in X$ with $z \neq x$ such that

$$z = gz = F(z, z).$$

Then

$$\begin{aligned} d(x, z) &= d(F(x,x), F(z,z)) \leq \lambda \left[d(gx, gz) + d(gx, gz)\right] \\ &= 2\lambda d(gx, gz). \end{aligned}$$

Since $2\lambda < 1$, we get $d(x, z) < d(x, z)$, which is a contradiction. Thus F and g have a unique common fixed point. ■

Corollary 2.1 Let (X,d) be a complete dq-metric spaces. Let $F : X \times X \to X$ be a continuous mapping such that

$$d\left(F(x,y),F(u,v)\right) \leq \lambda\left[d(x,u)+d(y,v)\right]$$

for all $x,y,u,v \in X$. If $\lambda \in \left[0,\frac{1}{2}\right)$, then there is a unique x in X such that $F\left(x,x\right)=x$.

Proof Define $g : X \to X$ by $g\left(x\right)=x$. Then F and g satisfy all the hypothesis of theorem. Hence the result follows. ■

Example 2.12 Let $X=[0,1]$. Define $d : X \times X \to [0,\infty)$ by

$$d(x,y)=|x-y|+|x|$$

for all $x,y \in X$. Then (X,d) is a complete dq-metric spaces. Define a map $F : X \times X \to X$ by $F\left(x,y\right)=\frac{1}{6}xy$ for all $x,y \in X$. Also, define $g : X \to X$ by $g\left(x\right)=\frac{1}{2}x$ for all $x \in X$. Since

$$|xy-uv| \leq |x-u|+|y-v| \quad \text{and} \quad |xy| \leq |x|+|y|$$

holds for all $x,y,u,v \in X$. We have

$$\begin{aligned} d\left(F(x,y),F(u,v)\right) &= \frac{1}{6}\left|xy-uv\right|+\frac{1}{6}\left|xy\right| \leq \frac{1}{6}\left[|x-u|+|y-v|\right]+\frac{1}{6}\left[|x|+|y|\right] \\ &\leq \frac{1}{3}\left[d(gx,gu)+d(gy,gv)\right] \end{aligned}$$

holds for all $x,y,u,v \in X$. It is an easy matter to see that F and g satisfy all the hypothesis of Theorem. Thus F and g have a unique common fixed point. Here $F\left(0,0\right)=g\left(0\right)=0$.

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