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RESEARCH ARTICLE

Uniform Structures in KK-algebras

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In this paper, we consider a collection of ideals of a KK-algebra X. We use the concept of congruence relation with respect to ideals to construct a uniformity that induces a topology on X which makes this to a topological KK-algebras.

Keywords: KK-algebra; ideal; uniform space.

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1. Introduction and Preliminaries

S. Asawasamrit and A. Sudprasert [1] introduced the concept of KK-algebra as a generalization of the BCI-algebra [2] and studied some important properties. In this paper we consider a collection of ideals to define a uniformity and make the KK-algebra into a uniform topological space with the desired subset as the open sets. In this section, we recall certain definitions and theorems which are taken from [1].

Definition 1.1 An algebra $(X, *, 0)$ with a binary operation $*$ and a nullary operation 0. Then X is called KK-algebra if it satisfies for all $x, y, z \in X$:

(KK-1) $(x * y) * ((y * z) * (x * z)) = 0$.
(KK-2) $0 * x = x$.
(KK-3) $x * y = 0$ and $y * x = 0$ if and only if $x = y$.

Example 1.2 Let $*$ be defined on an abelian group G by $x * y = x^{-1}y$, where x, y in G, with e is unity element of G. Then $(G, \cdot, e)$ is a KK-algera.

Example 1.3 Let $X = \{0, 1\}$ and $*$ is defined by

$*$	0	1
0	0	1
1	1	0

Then $(X, *, 0)$ is a KK-algebra.

Theorem 1.4 Let $(X, *, 0)$ be a KK-algebra if and only if it satisfies the following conditions: for all $x, y, z \in X$,

(1) $(x * y) * ((y * z) * (x * z)) = 0$.
(2) $x * ((x * y) * y) = 0$.

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(3) $x * x = 0$.
(4) $x * y = 0$ and $y * x = 0$ if and only if $x = y$.

Define a binary relation $\leq$ on KK-algebra X by $x \leq y$ if and only if $y * x = 0$. Then $(X, \leq)$ is a poset.

Theorem 1.5 Let $(X, *, 0)$ be a KK-algebra if and only if it satisfies the following conditions: for all $x, y, z \in X$,

(1) $((y * z) * (x * z)) \leq (x * y)$.
(2) $((x * y) * y) \leq x$.
(3) $x \leq y$ if and only if $y * x = 0$.

Proposition 1.6 Let x, y, z be any elements in a KK-algebra X. Then:

(1) $x \leq y$ implies $y * z \leq x * z$.
(2) $x \leq y$ implies $z * x \leq z * y$.

Proposition 1.7 Let x, y, z be any elements in a KK-algebra X. Then $x * (y * z) = y * (x * z)$.

Corollary 1.1 Let x, y, z be any elements in a KK-algebra X. Then:

(1) $y * z \leq x$ if and only if $x * z \leq y$.
(2) $(z * x) * (z * y) \leq x * y$.
(3) $x \leq y$ implies $x * z \leq y * z$.

Proposition 1.8 Let x, y, z be any elements in a KK-algebra X. Then:

(1) $((x * y) * y) * y = x * y$.
(2) $(x * y) * 0 = (x * 0) * (y * 0)$.

Definition 1.9 A non-empty subset A of a KK-algebra X is called a closed of X on condition that $x * y \in A$ whenever $x, y \in A$.

Definition 1.10 A non-empty subset A of a KK-algebra X is called an ideal of X if it satisfies the following conditions:
(I-1) $0 \in A$.
(I-2) For any $x, y \in X$, $x * y \in A$ and $x \in A$ imply $y \in A$.

Example 1.11 Let $X = \{0, 1, 2, 3\}$ and let $*$ be defined by the table

$*$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	3	3	0	0
3	3	2	1	0

Thus, it can be easily shown that X is a $KK-$algebra. And we see that $I = \{0, 1\}$ and $J = \{0, 3\}$ are closed ideals of X.

Definition 1.12 Let I be an ideal of a KK-algebra X. Define a relation θ_I on X by $x\theta_I y$ if and only if $x * y \in I$ and $y * x \in I$.

Example 1.13 Let $X = \{0, 1, 2, 3\}$ and $*$ be defined by the table

$*$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	3	3	0	0
3	3	2	1	0

Then X is a KK-algebra. Let $I = \{0, 1\}$. Then I is an ideal of X and

$$\theta_I = \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,0),(3,2),(2,3)\}.$$

Theorem 1.14 If I be an ideal of KK-algebra X, then the relation θ_I is a congruence relation on X.

Proof Let I be an ideal of X and $x, y, z \in X$. By Theorem, $x*x = 0$ and assumption, $x*x \in I$. That is, $x\theta_I x$. Hence θ_I is reflexive. Next, suppose that $x\theta_I y$. It follows that $x*y \in I$ and $y*x \in I$. Then $y\theta_I x$, so θ_I is symmetric. Let $x\theta_I y$ and $y\theta_I z$. Then $x*y, y*x, y*z, z*y \in I$ and $(y*x)*((z*y)*(z*x)) = 0 \in I$. It follows that $(z * y) * (z * x) \in I$ and since $z * y \in I$, so $z * x \in I$. Similarly, $x * z \in I$. Thus θ_I is transitive. Assume that $u\theta_I v$ and $x\theta_I y$, for any $x, y, u, v \in X$, then $u * v, v * u, x * y, y * x \in I$ and by (KK-1), we see that $(u*v)*((v*x)*(u*x)) = 0$ and $(v*u)*((u*x)*(v*x)) = 0$. From assumption and I is an ideals of X, these imply that $(v*x)*(u*x) \in I$ and $(u*x)*(v*x) \in I$. This shows that $v * x\theta_I u * x$. Therefore θ_I is a congruence relation X. ■

2. Uniformity in KK-algebras

From now on X is a KK-algebra and $D \subseteq X$ is an ideal of KK-algebra. Let X be a non-empty set and U, V be any subset of $X \times X$. Define

$$U \circ V = \{(x, y) \in X \times X \mid \text{ for some } z \in X, (z, y) \in U \text{ and } (x, z) \in V\},$$

$$U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\},$$

$$\Delta = \{(x, x) \in X \times X \mid x \in X\}.$$

Definition 2.1 [3] By a Uniformity on X we shall mean a nonempty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:

- (U1) $\Delta \subseteq U$ for any $U \in \mathcal{K}$.
- (U2) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$.
- (U3) if $U \in \mathcal{K}$, then there exists a $V \in \mathcal{K}$ such that $V \circ V \subseteq U$.
- (U4) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$.
- (U5) if $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{K}$.

The pair $(X, \mathcal{K})$ is called a uniform structure (uniform space).

Theorem 2.2 Let X be a KK-algebra and Λ be an arbitrary family of ideals of the KK-algebra X such that it is closed under intersection. If $U_D = \{(x, y) \in X \times X \mid x\theta_D y\}$ and $\mathcal{K}^* = \{U_D \mid D \in \Lambda\}$. Then K^* satisfies the conditions (U1)-(U4).

Proof

- (U1) Since D is an ideal of X, $x\theta_D x$ for any $x \in X$, hence $\Delta \subseteq U_D$ for all $U_D \in \mathcal{K}^*$.
- (U2) For any $U_D \in \mathcal{K}^*$ we have $(x, y) \in (U_D)^{-1} \Leftrightarrow (y, x) \in U_D \Leftrightarrow y\theta_D x \Leftrightarrow x\theta_D y \Leftrightarrow (x, y) \in U_D$. Therefore $(U_D)^{-1} = U_D \in \mathcal{K}^*$.
- (U3) For any $U_D \in \mathcal{K}^*$, the transitivity of θ_D implies that $U_D \circ U_D \subseteq U_D$.
- (U4) For any $U_D, U_J \in \mathcal{K}^*$, we claim that $U_D \cap U_J = U_{D\cap J}$. Let $(x, y) \in U_D \cap U_J$. Then $x\theta_D y$ and $x\theta_J y$. Hence $x * y \in D, y * x \in D, x * y \in J, y * x \in J$. Then $x\theta_{D\cap J} y$ and hence $(x, y) \in U_{D\cap J}$.

  Conversely, let $(x, y) \in U_{D\cap J}$. Then $x\theta_{D\cap J} y$, hence $x * y \in D \cap J$ and $y * x \in D \cap J$. Then $x * y \in D, y * x \in D, x * y \in J, y * x \in J$. Therefore $x\theta_D y$ and $x\theta_J y$. Then $(x, y) \in U_D \cap U_J$. So $U_D \cap U_J = U_{D\cap J}$. Since D and J are in $\Lambda, D \cap J \in \Lambda$ thus $U_{D\cap J} \in \mathcal{K}^*$.

■

Theorem 2.3 Let $\mathcal{K} = \{U \subseteq X \times X \mid U_D \subseteq U \text{ for some } U_D \in \mathcal{K}^*\}$. Then $\mathcal{K}$ is a uniformity on X and the pair $(X, \mathcal{K})$ is a uniform structure.

Proof By Theorem 3.2, the collection $\mathcal{K}$ satisfy the conditions (U1)-(U4). It suffices to show that $\mathcal{K}$ satisfies (U5). Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$. Then there exists a $U_D \subseteq U \subseteq V$ which implies that $V \in \mathcal{K}$. ■

Let $x \in X$ and $U \in \mathcal{K}$. Define:

$$U[x] = \{y \in X \mid (x, y) \in U\}.$$

Theorem 2.4 If X is a KK-algebra, then

$$\mathcal{T} = \{G \subseteq X \mid \forall x \in G\ \exists U \in \mathcal{K}, U[x] \subseteq G\}$$

is a topology on X.

Proof It is clear that $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary union. Now we show that $\mathcal{T}$ is closed under finite intersection. Let $G, K \in \mathcal{T}$ and suppose $x \in G \cap K$. Then there exist U and $V \in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq K$. Let $W = U \cap V$, then $W \in \mathcal{K}$. Also $W[x] \subseteq U[x] \cap V[x]$ hence $W[x] \subseteq G \cap K$ and then $G \cap K \in \mathcal{T}$. Thus $\mathcal{T}$ is a topology on X. ■

Note that for any $x \in X$, $U[x]$ is an open neighborhood of x.

Definition 2.5 Let $(X, \mathcal{K})$ be a uniform structure, where X is a KK-algebra. Then the topology $\mathcal{T}$ is called the uniform topology on X induced by $\mathcal{K}$.

Proposition 2.6 Topological space $(X, \mathcal{T})$ is completely regular.

Proof See [3, Theorem 14.2.9]. ■

Example 2.7 Let $X = \{0, a, b, c\}$ and let $*$ be defined by the table

*	0	a	b	c
0	0	a	b	c
a	0	0	c	c
b	c	c	0	0
c	c	b	a	0

Thus, it can be easily shown that X is a $KK-$algebra and it is easy to show that $D_1 = \{0, a\}$, $D_2 = \{0, c\}, \{0\}$ and X are the only ideals of X.

We can see that

$$U_{D_1} = \Delta \cup \{(0, a), (a, 0), (b, c), (c, b)\},$$

$$U_{D_2} = \Delta \cup \{(0, c), (c, 0), (a, b), (b, a)\},$$

$$U_{\{0\}} = \Delta, U_X = X \times X.$$

Therefore $\mathcal{K}^* = \{U_{D_1}, U_{D_2}, U_{\{1\}}, U_X\}$ and $\mathcal{K} = \{U \subseteq X \times X \mid U_D \subseteq U \text{ for some } U_D \in \mathcal{K}^*\}$. If we take $U := U_{D_1}$, then $U[0] = U[a] = \{0, a\}, U[b] = U[c] = \{b, c\}$. Therefore $\mathcal{T} = \{G \subseteq X \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\} \supseteq \{X, \emptyset, \{0, a\}, \{b, c\}\}$. Since $\{X, \emptyset, \{0, a\}, \{b, c\}\}$ is a topology on X, the topology $\mathcal{T}$ on X induced by the ideal $D_1 = \{0, a\}$ relative to U_{D_1} is a finer topology than $\{X, \emptyset, \{0, a\}, \{b, c\}\}$. If we take $U := U_{D_2}$, then $U[0] = U[c] = \{0, c\}, U[a] = U[b] = \{a, b\}$. Therefore $\mathcal{T} = \{G \subseteq X \mid \forall x \in G, \exists U \in$

$\mathcal{K}, U[x] \subseteq G\} \supseteq \{X, \emptyset, \{a,b\}, \{0,c\}\}$. Since $\{X, \emptyset, \{0,c\}, \{a,b\}\}$ is a topology on X, the topology $\mathcal{T}$ on X induced by the ideal $D_2 = \{0,c\}$ relative to U_{D_2} is a finder topology than $\{X, \emptyset, \{a,b\}, \{0,c\}\}$. Let $D = \{1\}$. Then $U_D = \Delta$. If we take $U := U_D$, then $U[x] = \{x\}$, $\forall x \in X$ and we obtain $\mathcal{T} = 2^X$, the discrete topology. Moreover, if we take X as an ideal of X, then $U[x] = X, \forall x \in X$ and obtain $\mathcal{T} = \{\emptyset, X\}$, the indiscrete topology.

3. Topological property of space $(X, \mathcal{T})$

Let X be a KK-algebra and R, S subsets of X. Then we define $R * S$ as follows

$$R * S = \{x * y \mid x \in R, y \in S\}.$$

Let X be a KK-algebra and $\mathcal{T}$ be a topology on X. We say that the pair $(X, \mathcal{T})$ is a topological KK-algebra, if $*$ is continuous with respect to $\mathcal{T}$. The continuity of $*$ is equivalent to the following property.

(I) Let O be an open set and $x, y \in X$ such that $x * y \in O$. Then there exist open sets O_1 and O_2 such that $x \in O_1, y \in O_2$ and $O_1 * O_2 \subseteq O$.

Let X be a KK-algebra and $\mathcal{T}$ defined as in Theorem 2.3. Then with the above notations we have the following:

Theorem 3.1 The pair $(X, \mathcal{T})$ is a topological KK-algebra.

Proof Assume that $x * y \in R$ with $x, y \in X$ and R is an open subset of X. Then there exist $U \in \mathcal{K}$, $U[x * y] \subseteq R$ and an ideal D such that $U_D \subseteq U$. We prove that the following relation holds:

$$U_D[x] * U_D[y] \subseteq U[x * y]$$

Indeed for $h \in U_D[x]$ and $k \in U_D[y]$, we get $x\theta_D h$ and $y\theta_D k$. It follows that $x * y\theta_D h * k$, hence $(x * y, h * k) \in U_D \subseteq U$. Thus $h * k \in U_D[x * y] \subseteq U[x * y]$ and then $h * k \in R$. ■

Theorem 3.2 Let X be a set and $\mathcal{S} \subset \mathbb{P}(X \times X)$ be a family such that for every $U \in \mathcal{S}$ the following conditions hold:

(a) $\Delta \subseteq U$,
(b) U^{-1} contains a member of $\mathcal{S}$ and,
(c) there exists a $V \in \mathcal{S}$ such that $V \circ V \subseteq U$.

Then there exists a unique uniformity $\mathcal{U}$, for which $\mathcal{S}$ is a subbase.

Theorem 3.3 Let $\mathcal{B} = \{U_D \mid D \text{ is an ideal of } X\}$. Then $\mathcal{B}$ is a subbase for a uniformity of X. We denote this topology by $\mathcal{S}$.

Proof Since θ_D is a congruence relation, then it is clear that $\mathcal{B}$ satisfies axioms of Theorem 3.2. ■

We say that topology τ is finer than σ if $\sigma \subset \tau$ as subsets of the power set. Then we have:

Corollary 3.1 Topology $\mathcal{S}$ is finer than $\mathcal{T}$.

Theorem 3.4 Any ideal in the collection Λ is a clopen subset of X.

Proof Let D be an ideal of X in Λ and $y \in D^c$. Then $y \in U_D[y]$ and we get $D^c \subseteq \bigcup\{U_D[y] \mid y \in D^c\}$. We claim that, $U_D[y] \subseteq D^c$, for all $y \in D^c$. Let $z \in U_D[y]$. Then $z\theta_D y$. Hence $z * y \in D$. If $z \in D$ then $y \in D$, a contradiction. So $z \in D^c$ and we get $\bigcup\{U_D[y] \mid y \in D^c\}$ and since $U_D[y]$ is open for all $y \in X$, D is a closed subset. We show that $D = \bigcup\{U_D[y] \mid y \in D\}$. If $y \in D$ then $y \in U_D[y]$ and we get $D \subseteq \bigcup\{U_D[y] \mid y \in D\}$. Let $y \in D$, if $z \in U_D[y]$ then $z\theta_D y$ and so $y * z \in D$. Since $y \in D$, we have $z \in D$ and we get $\bigcup\{U_D[y] \mid y \in D\} \subseteq D$. So D is also an open subset of X. ■

Theorem 3.5 For any $x \in X$ and $D \in \Lambda, U_D[x]$ is a clopen subset of X.

Proof We show that $(U_D[x])^c$ is open. Let $y \in (U_D[x])^c$. Then $x * y \in D^c$ or $y * x \in D^c$. Let $y * x \in D^c$. Since D^c is open then there exists $U \in \mathcal{K}$ such that $U[y * x] \subseteq D^c$. From $y * x \in D^c$ we conclude that $U_D[x * y] \subseteq D^c$. Therefore $U_D[y] * U_D[x] \subseteq U_D[y * x] \subseteq D^c$. We prove that $U_D[y] \subseteq (U_D[x])^c$. Let $z \in U_D[y]$. Then $z * x \in (U_D[y] * U_D[x])$. So $z * x \in D^c$ then we get $z \in (U_D[x])^c$. Hence $U_D[x]$ is closed. It is clear that $U_D[x]$ is open. So $U_D[x]$ is clopen subset of X. ■

Corollary 3.2 The topological space $(X, \mathcal{T})$ is not connected space.

Notation: We denote the uniform topology obtained by an arbitrary family Λ by T_Λ and if $\Lambda = \{D\}$, we denote $\mathcal{T}_\Lambda$ by $\mathcal{T}_D$.

Theorem 3.6 $\mathcal{T}_\Lambda = \mathcal{T}_J$ where $J = \bigcap\{D \mid D \in \Lambda\}$

Proof Let $\mathcal{K}$ and $\mathcal{K}^*$ be as in Theorems 2.1 and 2.2. Now consider $\Lambda_0 = \{J\}$, define $(\mathcal{K}_0)^* = \{U_J\}$ and $\mathcal{K}_0 = \{U \mid U_J \subseteq U\}$. Let $G \in \mathcal{T}_\Lambda$. So for all $x \in G$, there exist $U \in \mathcal{K}$ such that $U[x] \subseteq G$. From $J \subseteq D$ we get that $U_J \subseteq U_D$, for all ideals D of X. Since $U \in \mathcal{K}$, there exist $D \in \Lambda$ such that $U_D \subseteq U$. Hence $U_J[x] \subseteq U_D[x] \subseteq G$. Since $U_J \in \mathcal{K}_0, G \in \mathcal{T}_J$. So $\mathcal{T}_\Lambda \subseteq \mathcal{T}_J$.

Conversely, let $I \in \mathcal{T}_J$ then for all $x \in I$, there exist $U \in \mathcal{K}_0$ such that $U[x] \subseteq I$. So $U_J[x] \subseteq I$ and sine Λ is closed under intersection, $J \in \Lambda$. Then we get $U_J \in \mathcal{K}$ and so $I \in \mathcal{T}_\Lambda$. Thus $\mathcal{T}_J \subseteq \mathcal{T}_\Lambda$. ■

Corollary 3.3 Let D and J are ideals of KK-algebra X and $D \subseteq J$ then J is clopen in topological space $(X, \mathcal{T}_D)$.

Proof Let $\Lambda = \{D, J\}$. Then by Theorem 3.6, $\mathcal{T}_\Lambda = \mathcal{T}_D$ and therefore J is clopen in topological space $(X, \mathcal{T}_D)$. ■

Recall that a uniform space $(X, \mathcal{K})$ is totally bounded if for each $U \in \mathcal{K}$, there exists $x_1, x_2, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} U[x_i]$ and X is compact if any open cover of X has a finite subcover.

Theorem 3.7 Let D be an ideal of X. Then the following conditions are equivalent:

(1) Topological space $(X, \mathcal{T}_D)$ is compact.
(2) Topological space $(X, \mathcal{T}_D)$ is totally bounded.
(3) There exists $P = \{x_1, x_2, \ldots, x_n\} \subseteq X$ such that for all $a \in X$ there exists $x_i \in P (i = 1, 2, \ldots, n)$ with $a * x_i \in D$ and $x_i * a \in D$.

Proof (1) $\Rightarrow$ (2): It is clear by [4, Theorem 14.3.8].

(2) $\Rightarrow$ (3): Let $U_D \in \mathcal{K}$ since $(X, \mathcal{T}_D)$ is totally bounded, then there exists $x_1, x_2, \ldots x_n \in D$ such that $X = \bigcup_{i=1}^{n} U_D[x_i]$. If $a \in X$ then there exist x_i such that $a \in U_D[x_i]$, therefore $a * x_i \in D$ and $x_i * a \in D$.

(3) $\Rightarrow$ (1): For any $a \in X$, by hypothesis, there exists $x_i \in P$ with $a * x_i \in D$ and $x_i * a \in D$. Hence $a \in U_D[x_i]$, thus $X = \bigcup_{i=1}^{n} U_D[x_i]$. Now let $X = \bigcup_{\alpha \in \Omega} O_\alpha$ where each O_α is an open set of X, then for any $x_i \in X$ there exists $\alpha_i \in \Omega$ such that $x_i \in O_{\alpha_i}$, since O_{α_i} is an open set then $U_D[x_i] \subseteq O_{\alpha_i}$, so $X = \bigcup_{i=1}^{n} U_D[x_i] \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$, therefore $X = \bigcup_{i=1}^{n} O_{\alpha_i}$ which means that $(X, \mathcal{T}_D)$ is compact. ■

Theorem 3.8 If D is an ideal of X, then $U_D[x]$ is a compact set in topological space $(X, \mathcal{T}_D)$, for all $x \in X$.

Proof Let $U_D[x] \subseteq \bigcup_{\alpha \in \Omega} O_\alpha$, where each O_α is an open set of X. Since $x \in U_D[x]$, then there exists $\alpha \in \Omega$ such that $x \in O_\alpha$. Then $U_D[x] \subseteq O_\alpha$. Hence $U_D[x]$ is compact. ■

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