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New forms of separation spaces in bitopology

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Abstract

The aim of this paper is to study how distinct points and a point and a closed set not containing that points are separated by non overlapping open neighborhoods, in a bitopological space. The separation is studied with respect to a new type of $(1,2)\alpha$-open set together with a continuous function. We named the new axioms as star-ultra T_1, star-ultra T_2, star-ultra regular and normal. The star-ultra regular spaces is studied in two different ways and are called as A-star-ultra regular and B-star-ultra regular spaces.

Keywords: star-ultra T_1, star-ultra T_2, star-ultra regular, star-ultra normal.
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1. Introduction and preliminaries

The separation axioms, as a group, became important in the study of metrisability: the question of which topological spaces can be given the structure of a metric space. Metric spaces satisfy all of the separation axioms; but in fact, studying spaces that satisfy only some axioms helps build up to the notion of full metrisability. The separation axioms that were first studied together in this way were the axioms for accessible spaces, Hausdorff spaces, regular spaces, and normal spaces. Kelly [2] in year 1963, introduced the bitopological spaces with a metric and a quasimetric.

Later (X, τ_1, τ_2) is defined as a bitopology, a non empty set together with two topologies τ_1 and τ_2. In 1981, Thivagar [4] introduced a new type of open sets in bitoplogy as (1,2) alpha sets and also he defined $(1,2)\alpha$-continuous function and its weaker and stronger forms between two bitopological spaces. Some Separation axioms say ultra-T_0, ultra-T_1, ultra-T_2, ultra-regular and some low separation axioms like ultra-R_0 were studied by Rajeswari [5] via $(1,2)\alpha$-open sets. In 2011, Chattopadhyay [1] introduced a concept known as star separation axioms using the continuous images of open sets and open sets.

In this paper, we extended the star separations to Bitopology and studied star-ultra T_1, star-ultra T_2, star-ultra regular, star-ultra normal. Also it is found that a ultra regular space is both A-star ultra regular and B-star ultra regular and are sustained by suitable examples.

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Definition 1.1. [4] Let A be a subset of a bitopological space (Y, σ_1, σ_2). Then A is said to be:

(1) $\sigma_1\sigma_2$-open if A $\in \sigma_1 \cup \sigma_2$,
(2) $\sigma_1\sigma_2$-closed if $A^c \in \sigma_1 \cup \sigma_2$,
(3) (1,2)α-open or ultra-open if A $\subseteq \sigma_1$-Int($\sigma_1\sigma_2$-Cl(σ_1-Int(A))), where σ_1-Int(A) is the interior of A with respect to the topology σ_1 and $\sigma_1\sigma_2$-Cl(A) is the intersection of all $\sigma_1\sigma_2$-closed sets containing A. Also A is said to be $(1,2)\alpha$-closed iff A^c is $(1,2)\alpha$-open.
(4) $Int_{(1,2)\alpha}$(A) is the union of all $(1,2)\alpha$-open sets contained in A.
(5) $Cl_{(1,2)\alpha}$(A) is the intersection of all $(1,2)\alpha$-closed sets containing A.

The set of all (1,2)α-open sets are denoted as (1,2)αO(X) and if this set forms a topology, then X is called as an ultra space.

Definition 1.2. [5] A bitopological space X is called an ultra T_0 space iff for any two distinct points x, y $\in$ X there exists a $(1,2)\alpha$-open set G containing x but not y.

Definition 1.3. [5] A bitopological space X is called an ultra T_1 space iff for every distinct points x, y $\in$ X, there exists a $(1,2)\alpha$-open set G containing x but not y and a $(1,2)\alpha$-open set H containing y but not x.

Definition 1.4. [3] A bitopological space X is called an ultra T_2 space iff for every distinct points x, y $\in$ X there exists a $(1,2)\alpha$-open sets G and H such that x $\in$ G and y $\in$ H.

2. Star-ultra $T_i (i = 1, 2)$ spaces

Definition 2.1. A bitopological space (X, $(1,2)\alpha_1$O(X)) is said to be star-ultra T_1 if there exists a bitopology $(1,2)\alpha_2$O(X) on X and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) $\rightarrow$ (X, $(1,2)\alpha_2$O(X)) such that for any two distinct points a, b in X there are open sets G and H in $(1,2)\alpha_2$O(X) satisfying the condition: a $\in f^{-1}$(G), b $\in$ H and b $\notin f^{-1}$(G), a $\notin$ H (or) b $\in f^{-1}$(G), a $\in$ H and a $\notin f^{-1}$(G), b $\notin$ H.

Note 2.2. If $(X, (1,2)\alpha_1 O(X))$ is ultra T_1, then it is star-ultra T_1 only when $(1,2)\alpha_1 O(X) = (1,2)\alpha_2 O(X)$ and f, the identity function. The converse also need not be true and is justified in the following example.

Example 2.3. Let X = $\{a,b,c\}$, $\tau_1 = \{\phi, X, \{b\}\}$, $\tau_2 = \{\phi, X, \{b,c\}\}$. Then $(1,2)\alpha_1$O(X) = $\{\phi, X, \{b\}, \{b,c\}, \{a,b\}\}$ and $\sigma_1 = \{\phi, X, \{a\}\}$, $\sigma_2 = \{\phi, X, \{a,c\}\}$. Then $(1,2)\alpha_2$O(X) = $\{\phi, X, \{a\}, \{a,c\}, \{a,b\}\}$. Define f : (X, $(1,2)\alpha_1$O(X)) $\rightarrow$ (X, $(1,2)\alpha_2$O(X)) by f(a) = c, f(b) = a, f(c) = b. Then f is bijective and continuous.

Now, for the points a, b, choose G = $\{a\}$ and H = $\{a,c\}$. Then $f^{-1}(G) = \{b\}$. Therefore, b $\in f^{-1}$(G), a $\in$ H and a $\notin f^{-1}$(G), b $\notin$ H. For the points a, c, choose G = $\{a,b\}$ and H = $\{a\}$. Then $f^{-1}(G) = \{b,c\}$. Therefore, c $\in f^{-1}$(G), a $\in$ H and a $\notin f^{-1}$(G), c $\notin$ H. For the points b, c, choose G = $\{a\}$ and H = $\{a,c\}$. Then $f^{-1}(G) = \{b\}$. Therefore, b $\in f^{-1}$(G), c $\in$ H and c $\notin f^{-1}$(G), b $\notin$ H. Thus, (X, $(1,2)\alpha_1$O(X)) is star-ultra T_1 but not ultra T_1.

Note 2.4. A star-ultra T_1 space is ultra T_0.

For, f being continuous, $f^{-1}(G)$ is open in $(1,2)\alpha_1$O(X). But the converse need not be true and is justified in the following example.

Example 2.5. Let X = $\{a,b\}$, $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X\}$. Then $(1,2)\alpha_1$O(X) = $\{\phi, X, \{a\}\}$. Here, (X, $(1,2)\alpha_1$O(X)) is ultra T_0.

Claim: (X, $(1,2)\alpha_1$O(X)) is not star-ultra T_1. Let (X, $(1,2)\alpha_1$O(X)) is star-ultra T_1. Then there exists a bitopology (X, $(1,2)\alpha_2$O(X)) and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) $\rightarrow$ (X, $(1,2)\alpha_2$O(X)) such that for any two distinct points a, b in X, there are open sets G and H in $(1,2)\alpha_2$O(X) satisfying (i) a $\in f^{-1}$(G), b $\in$ H and b $\notin f^{-1}$(G), a $\notin$ H (or) (ii) b $\in f^{-1}$(G), a $\in$ H and a $\notin f^{-1}$(G), b $\notin$ H. Now, $f^{-1}(G)$ is open in $(1,2)\alpha_1$O(X) and hence $f^{-1}(G) = \{a\}$ or X. Since b $\notin f^{-1}$(G), a $\in f^{-1}$(G) it follows that $f^{-1}(G) = \{a\}$. So (i) holds. Now, G $\neq$ X,otherwise $f^{-1}(G)$ = X,which is not true. Therefore, G = $\{a\}$ or $\{b\}$.

Case (i): G = $\{a\}$. Since b ∈ H, a ∉ H, it follows that H = $\{b\}$. Thus, $(1,2)\alpha_2$O(X) is discrete topology. Now, f is continuous but $f(a) = a$ and $f(b) = b$ which is not open in $(1,2)\alpha_1$O(X). Since $f^{-1}(G) = \{a\}$ and G = $\{a\}$,it follows that f(a) = a and f(b) = a must hold which is a contradiction,since f is bijective.

Case(ii): G = $\{b\}$. Since $f^{-1}(G) = \{a\}$, f(a) = b. Since G and H are distinct, H = $\{a\}$ or X. But a ∉ H implies H ≠ X. So H = $\{a\}$. Since f is bijective and f(a) = b then f(b) = a. But $f^{-1}(H) = \{b\} \notin (1,2)\alpha_1$O(X),which is a contrdiction,since f is continuous.

Thus in both the cases we have contradiction. Hence, $(X, (1,2)\alpha_1 O(X))$ cannot be star-ultra T_1.

Note 2.6. A star-ultra T_1 axiom is lying between ultra T_0 and ultra T_1.

Definition 2.7. A bitopological space (X, $(1,2)\alpha_1$O(X)) is said to be star-ultra T_2 if there exists a bitopology $(1,2)\alpha_2$O(X) on X and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) such that for any two distinct points a, b in X there are open sets G and H in $(1,2)\alpha_2$O(X) satisfying the condition: a ∈ f^{-1}(G), b ∈ H, f^{-1}(G) ∩ H = ϕ (or) b ∈ f^{-1}(G), a ∈ H, f^{-1}(G) ∩ H = ϕ.

Note 2.8. If $(X, (1,2)\alpha_1 O(X))$ is ultra T_2, then it is star-ultra T_2 only when $(1,2)\alpha_1 O(X) = (1,2)\alpha_2 O(X)$ and f, the identity function. The converse also need not be true and is justified in the following example.

Example 2.9. Let X = $\{a,b,c\}$, $\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$, $\tau_2 = \{\phi, X\}$. Then $(1,2)\alpha_1$O(X) = $\{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$ and $\sigma_1 = \{\phi, X, \{a\}, \{c\}, \{a,c\}\}$, $\sigma_2 = \{\phi, X\}$. Then $(1,2)\alpha_2$O(X) = $\{\phi, X, \{a\}, \{c\}, \{a,c\}\}$. Define f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) by f(a) = a, f(b) = c, f(c) = b. Then f is bijective and continuous.

Now, for the points a, b, choose G = $\{c\}$ and H = $\{a\}$. Then $f^{-1}(G) = \{b\}$. Therefore, b ∈ f^{-1}(G), a ∈ H and f^{-1}(G) ∩ H = ϕ.

For the points a, c, choose G = $\{a,c\}$ and H = $\{c\}$. Then $f^{-1}(G) = \{a,b\}$. Therefore, $\{a,b\}$ ∈ f^{-1}(G), c ∈ H and f^{-1}(G) ∩ H = ϕ.

For the points b, c, choose G = $\{c\}$ and H = $\{a\}$. Then $f^{-1}(G) = \{b\}$. Therefore, b ∈ f^{-1}(G), c ∈ H and f^{-1}(G) ∩ H = ϕ.

Thus, (X, $(1,2)\alpha_1$O(X)) is star-ultra T_2 but it is not ultra T_2.

Note 2.10. A star-ultra T_2 space may not be ultra T_1.

In Example 2.9, (X, $(1,2)\alpha_1$O(X)) is star-ultra T_2 but it is not ultra T_1 with respect to $(1,2)\alpha_1 O(X)$. But the converse need not be true and is justified in the following example.

Example 2.11. Let X = R be a infinite set. $\tau_1 = \{\phi, R, R - \{1\}, R - \{2\}, R - \{1,2\}\}$, $\tau_2 = \{\phi, R\}$ be a co-finite topology on X. Then $(1,2)\alpha_1$O(X) = $\{\phi, R, R - \{1\}, R - \{2\}, R - \{1,2\}\}$. Here, it is an ultra T_1 space.

Claim: (X, $(1,2)\alpha_1$O(X)) is not star-ultra T_2. If possible,let (X, $(1,2)\alpha_1$O(X)) is star-ultra T_2. Then there exists a bitopology $(1,2)\alpha_2$O(X) on X and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) such that for any two distinct points a, b in X there are open sets G and H in $(1,2)\alpha_2$O(X) satisfying the condition:

(i) a ∈ f^{-1}(G), b ∈ H, f^{-1}(G) ∩ H = ϕ (or)
(ii) b ∈ f^{-1}(G), a ∈ H, f^{-1}(G) ∩ H = ϕ.

Let a, b in X and (i) hold. Then $f^{-1}(G) \in (1,2)\alpha_1$O(X) and X - $f^{-1}(G)$ is a finite set. Since $f^{-1}(G)$ ∩ H = ϕ, H ⊂ X - $f^{-1}(G)$, H is a finite set. Since H ∉ ϕ and $f^{-1}(H)$ is open and surjective $f^{-1}(H)$ is an infinite set with finite complements. But f is injective, $f^{-1}(H)$ cannot be infinite,which is a contradiction. Hence, (i) cannot hold. Similarly (ii) cannot hold. Hence, (X, $(1,2)\alpha_1$O(X)) is not star-ultra T_2.

Note 2.12. Every star-ultra T_2 is star-ultra T_1.

3. Star-ultra regular

Definition 3.1. A bitopological space X is called ultra-regular iff for each point x ∈ X and any $(1,2)\alpha$-closed set F in X where x ∉ F,there exists two disjoint $(1,2)\alpha$-open sets U and V in (X, $(1,2)\alpha$O(X)) such that x ∈ U, F ⊂ V, U ∩ V = ϕ.

Example 3.2. Let X = $\{a,b,c\}$, $\tau_1 = \{\phi, X, \{a\}, \{b,c\}\}$, $\tau_2 = \{\phi, X, \{a\}\}$. Then $(1,2)\alpha$O(X) = $\{\phi, X, \{a\}, \{b,c\}\}$ and $(1,2)\alpha$CL(X) = $\{\phi, X, \{a\}, \{b,c\}\}$. Here, X is an ultra-regular space.

Definition 3.3. A bitopological space (X, $(1,2)\alpha_1$O(X)) is said to be A-star-ultra regular if there exists a bitopology $(1,2)\alpha_2$O(X) on X and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) such that for any point x in X and any $(1,2)\alpha_1$-closed set F in (X, $(1,2)\alpha_1$O(X)) where x ∉ F, there are open sets G and H in $(1,2)\alpha_2$O(X) such that x ∈ f^{-1}(G), F ⊂ H and f^{-1}(G) ∩ H = ϕ.

Definition 3.4. A bitopological space (X, $(1,2)\alpha_1$O(X)) is said to be B-star-ultra regular if there exists a bitopology $(1,2)\alpha_2$O(X) on X and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) such that for any point x in X and any closed set F in (X, $(1,2)\alpha_1$O(X)) where x ∉ F, there are open sets G and H in $(1,2)\alpha_2$O(X) such that F ⊂ f^{-1}(G), x ∈ H and f^{-1}(G) ∩ H = ϕ.

Note 3.5. A ultra-regular space is A-star-ultra regular only when $(1,2)\alpha_1 O(X) = (1,2)\alpha_2 O(X)$ and f, the identity function. The converse also need not be true and is justified in the following example.

Example 3.6. Let X = $\{a,b,c\}$, $\tau_1 = \{\phi, X, \{a\}, \{a,b\}\}$, $\tau_2 = \{\phi, X, \{b\}\}$. Then $(1,2)\alpha_1$O(X) = $\{\phi, X, \{a\}, \{a,b\}\}$ and $\sigma_1 = \{\phi, X, \{c\}, \{b,c\}\}$, $\sigma 2 = \{\phi, X, \{b\}\}$. Then $(1,2)\alpha_2$O(X) = $\{\phi, X, \{c\}, \{b,c\}\}$. Define f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) by f(a) = c, f(b) = b, f(c) = a. Then f is bijective and continuous.

Now, a ∈ X and F = $\{b,c\}$, choose G = $\{c\}$ ∈ $(1,2)\alpha_2$O(X) and H = $\{b,c\}$ ∈ $(1,2)\alpha_2$O(X). Then a ∈ $f^{-1}(G)$, F ⊂ H and f^{-1}(G) ∩ H = ϕ.

For a ∈ X and a closed set F = $\{c\}$, choose G = $\{c\}$ ∈ $(1,2)\alpha_2$O(X) and H = $\{b,c\}$ ∈ $(1,2)\alpha_2$O(X), f^{-1}(G) ∩ H = ϕ.

For b ∈ X and a closed set F = $\{c\}$, choose G = $\{b,c\}$ ∈ $(1,2)\alpha_2$O(X) and H = $\{c\}$ ∈ $(1,2)\alpha_2$O(X), f^{-1}(G) ∩ H = ϕ.

Thus, (X, $(1,2)\alpha_1$O(X)) is A-star-ultra regular space. But it is not ultra-regular. Since $\{a\}$ and $\{b,c\}$ canot be separated by open sets in $(1,2)\alpha_1$O(X).

Note 3.7. A ultra-regular space is B-star-ultra regular.

Consider $(1,2)\alpha_1$O(X) = $(1,2)\alpha_2$O(X) and f, the identity function.

Theorem 3.8. A finite B-star-ultra regular space is ultra-regular.

Proof. If possible, let (X, $(1,2)\alpha_1$O(X)) be a finite B-star-ultra regular space which is not ultra-regular. Then,there exists a ∈ X, F ∈ $(1,2)\alpha_1$CL(X) such that a ∉ F and for every open set G_a(an open set containing a) and H_F (an open set containing F), a ∈ G_a, F ⊂ H_F such that $G_a \cap H_F \neq \phi$. Since (X, $(1,2)\alpha_1$O(X)) is B-star-ultra regular, for the bitopology $(1,2)\alpha_2$O(X) on X and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) satisfying the condition, so there exists open sets G and H in $(1,2)\alpha_2$O(X) such that F ⊂ f^{-1}(G), a ∈ H and f^{-1}(G) ∩ H = ϕ.

claim: F is not open in (X, $(1,2)\alpha_1$O(X)). Otherwise, a ∈ X - F, F ⊂ F and (X - F) ∩ F = ϕ, a contradiction. Since f is continuous, $f^{-1}(G)$ ∈ $(1,2)\alpha_1$O(X), and if $f^{-1}(G)$ - F ≠ ϕ. Let $a_1 \in f^{-1}(G)$ - F. Now, $f^{-1}(G) \cap cl_{(1,2)\alpha_1}(H) = \phi$. This implies $f^{-1}(G) \cap cl_{(1,2)\alpha_1}(\{a\}) = \phi \Rightarrow a_1 \notin cl_{(1,2)\alpha_1}(\{a\})$. Now,consider the closed set F ∪ $cl_{(1,2)\alpha_1}(\{a\})$ in (X, $(1,2)\alpha_1$O(X)), where $a_1 \notin$ F ∪ $cl_{(1,2)\alpha_1}(\{a\})$. Again,since X is B-star-ultra regular,there exists open sets G_1, H_1 in $(1,2)\alpha_2$O(X) such that F ∪ $cl_{(1,2)\alpha_1}(\{a\}) \subset f^{-1}(G_1)$, $a_1 \in H_1$, $f^{-1}(G_1) \cap H_1 = \phi$.

Now claim that F ∪ $cl_{(1,2)\alpha_1}(\{a\})$ ∉ $(1,2)\alpha_1$O(X). Otherwise, if [F ∪ $cl_{(1,2)\alpha_1}(\{a\})$] - $cl_{(1,2)\alpha_1}(\{a\})$ ∈ $(1,2)\alpha_1$O(X),then since F ∩ $cl_{(1,2)\alpha_1}(\{a\}) = \phi$, F ∈ $(1,2)\alpha_1$O(X), which is a contradiction. Hence, F ∪ $cl_{(1,2)\alpha_1}(\{a\})$ ∉ $(1,2)\alpha_1$O(X) ⇒ $f^{-1}(G_1)$ - [F ∪ $cl_{(1,2)\alpha_1}(\{a\})$]) ≠ ϕ. Let $a_2 \in f^{-1}(G_1$ - [F ∪ $cl_{(1,2)\alpha_1}(\{a\})$]). Now, $f^{-1}(G_1) \cap cl_{(1,2)\alpha_1}(H_1) = \phi$ ⇒

$f^{-1}(G_1) \cap cl_{(1,2)\alpha_1}(\{a_1\}) = \phi \Rightarrow a_2 \notin cl_{(1,2)\alpha_1}(\{a_1\})$. Also $a_2 \notin$ F $\cup$ $cl_{(1,2)\alpha_1}(\{a\})$. Now,consider the closed set F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\})$ in (X, $(1,2)\alpha_1$O(X)), where $a_2 \notin$ F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\})$. Again,since X is B-star-ultra regular,there exists open sets G_2, H_2 in $(1,2)\alpha_2$O(X) such that F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\}) \subset f^{-1}(G_2)$, $a_2 \in H_2$, $f^{-1}(G_2) \cap H_2 = \phi$.

Now claim that F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\}) \notin (1,2)\alpha_1$O(X). Otherwise, if it is open in (X, $(1,2)\alpha_1$O(X)), then [F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\})$] - [$cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\})$] $\in (1,2)\alpha_1$O(X),then since F $\cap$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\}) = \phi$, F $\in (1,2)\alpha_1$O(X), which is a contradiction. Hence, F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\}) \notin (1,2)\alpha_1$O(X) $\Rightarrow f^{-1}(G_2)$ - [F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\})$]) $\neq \phi$. Let $a_3 \in f^{-1}(G_2$ - [F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\})$]). Now, $f^{-1}(G_2) \cap cl_{(1,2)\alpha_1}(H_2) = \phi \Rightarrow f^{-1}(G_2) \cap cl_{(1,2)\alpha_1}(\{a_2\}) = \phi \Rightarrow a_3 \notin cl_{(1,2)\alpha_1}(\{a_2\})$. Also $a_3 \notin$ F $\cup$ [$cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\})$]. Thus, $a_3 \notin$ F $\cup$ [$cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\})$ $a_3 \notin$ F $\cup$ [$cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_2\})$],where F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\}) \cup cl_{(1,2)\alpha_1}(\{a_2\})$ is a closed set in (X, $(1,2)\alpha_1$O(X)). By, B-star-ultra regular, F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\}) \cup cl_{(1,2)\alpha_1}(\{a_2\}) \in (1,2)\alpha_1$O(X). Proceeding like this after a finite number of steps,we must have F $\cup$ $cl_{(1,2)\alpha_1}(\{a\}) \cup cl_{(1,2)\alpha_1}(\{a_1\}) \cup cl_{(1,2)\alpha_1}(\{a_2\}) \cup \ldots \cup cl_{(1,2)\alpha_1}(\{a_n\}) \Rightarrow$ F is open in (X, $(1,2)\alpha_1$O(X)),which is a contradiction. □

Note 3.9. From the Theorem 3.8 and in Example 3.6, the space (X, $(1,2)\alpha_1$O(X)) is not B-star-ultra regular. Thus, the class of A-star-ultra regular space is different from the class of B-star-ultra regular spaces.

Theorem 3.10. A star-ultra T_1 and A-star-ultra regular space is star-ultra T_2.

Proof. Let (X, $(1,2)\alpha_1$O(X)) is star-ultra T_1. Then there exists a bitopology (X, $(1,2)\alpha_2$O(X)) on X and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) such that for any two distinct points a, b in X there are open sets G and H in $(1,2)\alpha_2$O(X) satisfying:

(i) a $\in f^{-1}$(G), b $\in$ H and b $\notin f^{-1}$(G), a $\notin$ H (or)
(ii) b $\in f^{-1}$(G), a $\in$ H and a $\notin f^{-1}$(G), b $\notin$ H.

Let a, b in X and a ≠ b. Suppose (i) holds. Then a $\notin$ X - $f^{-1}(G)$ = F(say),where F is closed in (X, $(1,2)\alpha_1$O(X)). Since, (X, $(1,2)\alpha_1$O(X)) is A-star-ultra regular there exists a bitopology $(1,2)\alpha_3$O(X) on X and a bijective continuous function g : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_3$O(X)) such that for any point p in X and any closed set V in (X, $(1,2)\alpha_1$O(X)) where p $\notin$ V, there are open sets M and W in $(1,2)\alpha_3$O(X) satisfying p $\in g^{-1}$(M), V $\subset$ W, g^{-1}(M) $\cap$ W = ϕ. Now a $\in$ X and F is closed in (X, $(1,2)\alpha_1$O(X)) and a $\notin$ F. Then there are open sets M and W in $(1,2)\alpha_3$O(X) satisfying a $\in g^{-1}$(M), F $\subset$ W, g^{-1}(M) $\cap$ W = ϕ. Now, b $\notin f^{-1}(G) \Rightarrow$ b $\in$ F. So, a $\in g^{-1}$(M), b $\in$ W, g^{-1}(M) $\cap$ W = ϕ. Hence, if (i) holds, then the condition for the space (X, $(1,2)\alpha_1$O(X)) to be star-ultra T_2 is satisfied. Simillarly, it is true when (2) holds. □

Theorem 3.11. A star-ultra T_1 and B-star-ultra regular space is star-ultra T_2.

Proof. Let (X, $(1,2)\alpha_1$O(X)) is star-ultra T_1. Then there exists a bitopology (X, $(1,2)\alpha_2$O(X)) on X and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) such that for any two distinct points a, b in X there are open sets G and H in $(1,2)\alpha_2$O(X) satisfying:

(i) a $\in f^{-1}$(G), b $\in$ H and b $\notin f^{-1}$(G), a $\notin$ H (or)
(ii) b $\in f^{-1}$(G), a $\in$ H and a $\notin f^{-1}$(G), b $\notin$ H.

Let a, b in X and a ≠ b. Suppose (i) holds. Then a $\notin$ X - $f^{-1}(G)$ = F(say),where F is closed in (X, $(1,2)\alpha_1$O(X)). Since, (X, $(1,2)\alpha_1$O(X)) is B-star-ultra regular there exists a bitopology $(1,2)\alpha_3$O(X) on X and a bijective continuous function g : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_3$O(X)) such that there are open sets M and W in $(1,2)\alpha_3$O(X) satisfying F $\subset g^{-1}$(M), a $\in$ W, g^{-1}(M) $\cap$ W = ϕ. Now, b $\notin f^{-1}(G) \Rightarrow$ b $\in$ F. So, b $\in g^{-1}$(M). Hence, we have a bitopology $(1,2)\alpha_3$O(X) on X and a bijective continuous function g : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_3$O(X)) such that for the points a, b, there are open sets M and W in $(1,2)\alpha_3$O(X) satisfying b $\in g^{-1}$(M), a $\in$ W, g^{-1}(M) $\cap$ W = ϕ. Hence, (X, $(1,2)\alpha_1$O(X)) is star-ultra T_2. □

Definition 3.12. A bitopological space X is said to be A-star-ultra T_3 (resp. B-star-ultra T_3) if it is star-ultra T_1 and A-star-ultra regular (resp. B-star-ultra regular).

Note 3.13. A A-star-ultra T_3 may not be ultra-T_2. In Example 2.6, the space (X, $(1,2)\alpha_1$O(X)) is star-ultra T_1 and A-star-ultra regular and hence star-ultra T_3. But (X, $(1,2)\alpha_1$O(X)) is not ultra T_2.

4. Star-ultra normal spaces

Definition 4.1. A bitopological space (X, $(1,2)\alpha_1$O(X)) is said to be star-ultra normal if there exists a bitopology $(1,2)\alpha_2$O(X) on X and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) such that for any two disjoint closed sets A and B in (X, $(1,2)\alpha_1$O(X)), there are open sets G and H in $(1,2)\alpha_2$O(X) such that A ⊂ f^{-1}(G), B ⊂ H, f^{-1}(G) ∩ H = ϕ.

Note 4.2. It is clear from the Definition 4.1 that, a star-ultra normal space is A-star-ultra regular.

Lemma 4.3. If (X, $(1,2)\alpha_1$O(X)) is B-star-ultra regular then for any two disjoint closed sets A and B in (X, $(1,2)\alpha_1$O(X)), A ∩ $cl_{(1,2)\alpha_2}(B) = \phi$ and B ∩ $cl_{(1,2)\alpha_2}(A) = \phi$, where $cl_{(1,2)\alpha_2}(B)$ denotes closure of B w. r. t. the B-star-ultra regular $(1,2)\alpha_2$O(X)-bitopology under consideration.

Proof. Let A and B be any two disjoint closed sets in (X, $(1,2)\alpha_1$O(X)). Let x ∈ A. Then x ∉ B. By the B-star -ultra regularity condition,there exists G, H belonging to the B-star-ultra regular $(1,2)\alpha_2$O(X)-bitopology on X such that B ⊂ f^{-1}(G), x ∈ H and f^{-1}(G) ∩ H = ϕ. Since H ∈ $(1,2)\alpha_2$O(X), $cl_{(1,2)\alpha_2}f^{-1}(G)$ ∩ H = ϕ and $cl_{(1,2)\alpha_2}(B) \subset cl_{(1,2)\alpha_2}f^{-1}(G) \Rightarrow cl_{(1,2)\alpha_2}(B)$ ∩ H = ϕ. Thus x ∈ A ⇒ x ∉ $cl_{(1,2)\alpha_2}(B)$. This implies that A ∩ $cl_{(1,2)\alpha_2}(B)$ = ϕ. Similarly, B ∩ $cl_{(1,2)\alpha_2}(A) = \phi$. □

Theorem 4.4. Let (X, $(1,2)\alpha_1$O(X)) be compact and T_1. Then the following statements are equivalent:

(i) (X, $(1,2)\alpha_1$O(X)) is A star-ultra regular.
(ii) (X, $(1,2)\alpha_1$O(X)) is star-ultra normal.
(iii) (X, $(1,2)\alpha_1$O(X)) is B-star-ultra regular.

Proof. (i) ⇒ (ii): Let (X, $(1,2)\alpha_1$O(X)) is A-star-ultra regular. Then let A and B be any two disjoint closed sets in (X, $(1,2)\alpha_1$O(X)). Since (X, $(1,2)\alpha_1$O(X)) is A-star-ultra regular,there exists a bitopology $(1,2)\alpha_2$O(X) on X and a bijective continuous function f : (X, $(1,2)\alpha_1$O(X)) → (X, $(1,2)\alpha_2$O(X)) such that for any point a in X and any closed set F in (X, $(1,2)\alpha_1$O(X)) where a ∉ F, there are open sets G and H in $(1,2)\alpha_2$O(X) such that a ∈ f^{-1}(G), F ⊂ H, f^{-1}(G) ∩ H = ϕ. Then for each a ∈ A, there exists open sets G_a, H_a ∈ $(1,2)\alpha_2$O(X)) such that a ∈ f^{-1}(G), B ⊂ H_a, f^{-1}(G) ∩ $H_a = \phi$. Since f is continuous, $f^{-1}(G_a)$ ∈A is an open covering of A in (X, $(1,2)\alpha_1$O(X)). Since (X, $(1,2)\alpha_1$O(X)) is compact and A is closed in (X, $(1,2)\alpha_1$O(X)), A is compact in (X, $(1,2)\alpha_1$O(X)). So there exist a_1, a_2,. . . ,a_n ∈ A such that A ⊂ $f^{-1}(G_{a_1}) \cup f^{-1}(G_{a_2})$ . . . $f^{-1}(G_{a_n})$ = $f^{-1}(G_{a_1} \cup G_{a_1} \cup ... \cup G_{a_1}) = f^{-1}(G)$(say),where G = $G_{a_1} \cup G_{a_2} \cup ... \cup G_{a_n}$. Let H = $H_{a_1} \cap H_{a_2} \cap ... \cap H_{a_n}$. Then B ⊂ H, A ⊂ $f^{-1}(G)$, $f^{-1}(G)$ ∩ H = ϕ. Hence (X, $(1,2)\alpha_1$O(X)) is star-ultra normal.

(ii) ⇒ (iii): obvious.

(iii) ⇒ (ii): Let (X, $(1,2)\alpha_1$O(X)) is B-star-ultra regular.

Assume the bitopology $(1,2)\alpha_2$O(X) on X as B-star-ultra regular $(1,2)\alpha_2$O(X) on X under consideration. Let A and B be two disjoint closed sets in (X, $(1,2)\alpha_1$O(X)). By the Lemma 4.3, A ∩ $cl_{(1,2)\alpha_2}(B) = \phi$, where $cl_{(1,2)\alpha_2}(B)$ denotes the closure of B w. r. t. the B-star-ultra regular $(1,2)\alpha_2$O(X) on X under consideration. Let x ∈ $cl_{(1,2)\alpha_2}(B)$. Then x ∉ A. Then there exist G_x, H_x ∈ $(1,2)\alpha_2$O(X) such that A ⊂ $f^{-1}(G_x)$, x ∈ H_x and $f^{-1}(G_x) \cap H_x = \phi$. Consider U = $\{H_x : x \in cl_{(1,2)\alpha_2}(B)\}$. Since (X, $(1,2)\alpha_1$O(X)) is compact and f is continuous and surjective, (X, $(1,2)\alpha_2$O(X)) is compact. Then $cl_{(1,2)\alpha_2}(B)$ is compact in $(1,2)\alpha_2$O(X). Now U is an open covering of $cl_{(1,2)\alpha_2}(B)$ in $(1,2)\alpha_2$O(X). So there exists x_1, x_2,. . . x_n in $cl_{(1,2)\alpha_2}(B)$ such that $cl_{(1,2)\alpha_2}(B) \subset H_{x_1} \cup H_{x_2} \cup ... \cup H_{x_n}$. Let M = $H_{x_1} \cup H_{x_2} \cup ... \cup H_{x_n}$ and P = $G_{x_1} \cap G_{x_2} \cap ... \cap G_{x_n}$. Then M ∈ $(1,2)\alpha_2$O(X) and P ∈ $(1,2)\alpha_2$O(X). Now, $f^{-1}(G_{x_n}) = G_{x_1} \cap G_{x_2} \cap ... \cap G_{x_n} = f^{-1}(G_{x_1}) \cap f^{-1}(G_{x_2}) \cap ... \cap f^{-1}(G_{x_n})$ and A ⊂

$f^{-1}(G_{x_1}) \cap f^{-1}(G_{x_2}) \cap ... \cap f^{-1}(G_{x_n}) \Rightarrow$ A $\subset f^{-1}(P)$. Thus A $\subset f^{-1}(P)$ and B $\subset$ M. Also, $f^{-1}(P) \cap$ M $= \phi$. Hence (X, $(1,2)\alpha_1$O(X)) is star-ultra normal.

(ii) $\Rightarrow$ (i): obvious. □

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